Lecture notes on

General Relativity, Black Holes and Gravitational Waves

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Chapter 1

Introduction

General Relativity is the physical theory of gravity formulated by Einstein in 1915. It is based on the Equivalence Principle of Gravitation and Inertia, which establishes a fundamental connection between the gravitational field and the geometry of the spacetime, and on The Principle of General Covariance. General Relativity has changed quite dramatically our understanding of space and time, and the consequences of this theory, which we shall investigate in this course, disclose interesting and fascinating new phenomena, like for instance the existence of black holes and the generation of gravitational waves.

The language of General Relativity is that of tensor analysis, or, in a more modern formulation, the language of differential geometry. There is no way to understand the theory of gravity without knowing what is a manifold, or a tensor. Therefore we shall dedicate a few lectures to the the mathematical tools that are essential to describe the theory and its physical consequences. The first lecture, however, will be dedicated to answer the following questions:

1) why does the Newtonian theory become unappropriate to describe the gravitational field.

2) Why do we need a tensor to describe the gravitational field, and we why do we need to introduce the concept of manifold, metric, affine connections and other geometrical objects.

3) What is the role played by the equivalence principle in all that.

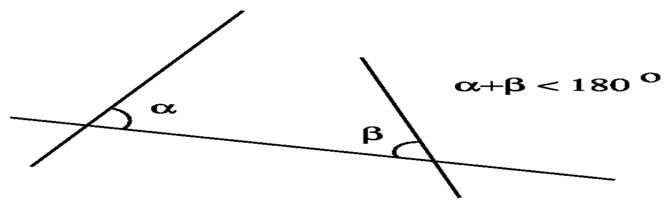
In the next lectures we shall rigorously define manifolds, vectors, tensors, and then, after introducing the principle of general covariance, we will formulate Einstein's equations.

But first of all, since as we have already anticipated that there is a connection between the gravitational field and the geometry of the spacetime, let us introduce non-euclidean geometries, which are in some sense the precursors of general relativity.

1.1 Non euclidean geometries

In the prerelativistic years the arena of physical theories was the flat space of euclidean geometry which is based on the five Euclide's postulates. Among them the fifth has been the object of a millennary dispute: for over 2000 years geometers tried to show, without succeeeding, that the fifth postulate is a consequence of the other four. The postulate states the following:

Consider two straight lines and a third straight line crossing the two. If the sum of the two internal angles (see figures) is smaller than 180^0 *, the two lines will meet at some point on the side of the internal angles.*



The solution to the problem is due to Gauss (1824, Germany), Bolyai (1832, Austria), and Lobachevski (1826, Russia), who independently discovered a geometry that satisfies all Euclide's postulates except the fifth. This geometry is what we may call, in modern terms, a *two dimensional space of constant negative curvature.* The analytic representation of this geometry was discovered by Felix Klein in 1870. He found that a point in this geometry is represented as a pair of real numbers (x^1, x^2) with

$$(x^1)^2 + (x^2)^2 < 1, \tag{1.1}$$

and the distance between two points x and X, $d(x, X)$, is defined as

$$d(x, X) = a\cosh^{-1}\left[\frac{1 - x^1X^1 - x^2X^2}{\sqrt{1-(x^1)^2-(x^2)^2}\sqrt{1-(X^1)^2-(X^2)^2}}\right], \tag{1.2}$$

where a is a lenghtscale. This space is infinite, because

$$d(x, X) \to \infty$$

when

$$(X^1)^2 + (X^2)^2 \to 1.$$

The logical independence of Euclide's fifth postulate was thus established.

In 1827 Gauss published the *Disquisitiones generales circa superficies curvas*, where for the first time he distinguished the **inner**, or **intrinsic** properties of a surface from the **outer**, or **extrinsic** properties. The first are those properties that can be measured by somebody living on the surface. The second are those properties deriving from embedding the surface in a higher-dimensional space. Gauss realized that the fundamental inner property is the distance between two points, defined as the shortest path between them on the surface.

For example a cylinder has the same inner properties of a plane. The reason is that it can be obtained by a flat piece of paper suitably rolled, without distorting its metric relations, i.e. without stretching or tearing. This means that the distance between any two points on the surface is the same as it was in the original piece of paper, and parallel lines remain parallel. Thus the **intrinsic geometry** of a cylinder is flat. This is not true in the case of a sphere, since a sphere cannot be mapped onto a plane without distortions: the inner properties of a sphere are different from those of a plane. It should be stressed that the intrinsic geometry of a surface considers only the relations between points on the surface.

However, since a cylinder is "round" in one direction, we think it is a curved surfaces. This is due to the fact that we consider the cylinder as a 2-dimensional surface in a 3-dimensional space, and we intuitively compare the curvature of the lines which are on the cylinder with straight lines in the flat 3-dimensional space. Thus, the extrinsic curvature relies on the notion of higher dimensional space. In the following, we shall be concerned only with the intrinsic properties of surfaces.

The distance between two points can be defined in a variety of ways, and consequently we can construct different metric spaces. Following Gauss, we shall select those metric spaces for which, given any sufficiently small region of space, it is possible to choose a system of coordinates (ξ^1, ξ^2) such that the *distance* between a point $P = (\xi^1, \xi^2)$, and the point $P'(\xi^1 + d\xi^1, \xi^2 + d\xi^2)$ satisfies Pythagoras' law

$$ds^2 = (d\xi^1)^2 + (d\xi^2)^2. \tag{1.3}$$

From now on, when we say the *distance between two points*, we mean the distance between two points that are infinitely close.

This property, i.e. the possibility of setting up a locally euclidean coordinate system, is a **local** property: it deals only with the inner metric relations for infinitesimal neighborhoods. Thus, unless the space is globally euclidean, the coordinates (ξ_1, ξ_2) have only a local meaning. Let us now consider some other coordinate system (x_1, x_2) . How do we express the *distance* between two points? If we explicitely evaluate $d\xi^1$ and $d\xi^2$ in terms of the new coordinates we find

$$\begin{aligned} \xi^1 &= \xi^1(x^1, x^2) \qquad \rightarrow d\xi^1 = \frac{\partial \xi^1}{\partial x^1} dx^1 + \frac{\partial \xi^1}{\partial x^2} dx^2 \\ \xi^2 &= \xi^2(x^1, x^2) \qquad \rightarrow d\xi^2 = \frac{\partial \xi^2}{\partial x^1} dx^1 + \frac{\partial \xi^2}{\partial x^2} dx^2 \end{aligned} \tag{1.4}$$

$$\begin{aligned} ds^2 &= \left[\left(\frac{\partial \xi^1}{\partial x^1} \right)^2 + \left(\frac{\partial \xi^2}{\partial x^1} \right)^2 \right] (dx^1)^2 + \left[\left(\frac{\partial \xi^1}{\partial x^2} \right)^2 + \left(\frac{\partial \xi^2}{\partial x^2} \right)^2 \right] (dx^2)^2 \\ &+ 2 \left[\left(\frac{\partial \xi^1}{\partial x^1} \right) \left(\frac{\partial \xi^1}{\partial x^2} \right) + \left(\frac{\partial \xi^2}{\partial x^1} \right) \left(\frac{\partial \xi^2}{\partial x^2} \right) \right] dx^1 dx^2 \\ &= g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + 2g_{12} dx^1 dx^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \end{aligned} \tag{1.5}$$

In the last line of eq. (1.5) we have defined the following quantities:

$$\begin{aligned} g_{11} &= \left[\left(\frac{\partial \xi^1}{\partial x^1} \right)^2 + \left(\frac{\partial \xi^2}{\partial x^1} \right)^2 \right] \\ g_{22} &= \left[\left(\frac{\partial \xi^1}{\partial x^2} \right)^2 + \left(\frac{\partial \xi^2}{\partial x^2} \right)^2 \right] \\ g_{12} &= \left[\left(\frac{\partial \xi^1}{\partial x^1} \right) \left(\frac{\partial \xi^1}{\partial x^2} \right) + \left(\frac{\partial \xi^2}{\partial x^1} \right) \left(\frac{\partial \xi^2}{\partial x^2} \right) \right], \end{aligned} \tag{1.6}$$

namely, we have defined the metric tensor $g_{\alpha\beta}$! i.e. the metric tensor is an object that allows us to compute the *distance* in any coordinate system. As it is clear from the preceeding equations, $g_{\alpha\beta}$ is a symmetric tensor, $(g_{\alpha\beta} = g_{\beta\alpha})$. *In this way the notion of metric associated to a space, emerges in a natural way.*

EINSTEIN's CONVENTION

In writing the last line of eq. (1.5) we have adopted the convenction that if there is a product of two quantities having the same index appearing once in the lower and once in the upper case ("dummy indices"), then summation is implied. For example, if the index α takes the values 1 and 2

$$v_\alpha V^\alpha = \sum_{i=1}^{2} v_i V^i = v_1 V^1 + v_2 V^2 \tag{1.7}$$

We shall adopt this convenction in the following.

EXAMPLE: HOW TO COMPUTE $g_{\mu\nu}$

Given the locally euclidean coordinate system (ξ_1, ξ_2) let us introduce polar coordinates $(r, \theta) = (x_1, x_2)$. Then

$$\xi_1 = r\cos\theta \qquad \to d\xi^1 = \cos\theta dr - r\sin\theta d\theta \tag{1.8}$$

$$\xi_2 = r\sin\theta \qquad \to d\xi^2 = \sin\theta dr + r\cos\theta d\theta \tag{1.9}$$

$$ds^2 = (d\xi^1)^2 + (d\xi^2)^2 = dr^2 + r^2 d\theta^2, \tag{1.10}$$

and therefore

$$g_{11} = 1, \qquad g_{22} = r^2, \qquad g_{12} = 0. \tag{1.11}$$

1.2 How does the metric tensor transform if we change the coordinate system

We shall now see how the metric tensor transforms under an arbitrary coordinate transformation. Let us assume that we know $g_{\alpha\beta}$ expressed in terms of the coordinate (x^1, x^2), and we want to change the reference to a new system $(x^{1'}, x^{2'})$. In section 1 we have shown that, for example, the component g_{11} is defined as (see eq. 1.7)

$$g_{11} = [(\frac{\partial\xi^1}{\partial x^1})^2 + (\frac{\partial\xi^2}{\partial x^1})^2], \tag{1.12}$$

where (ξ^1, ξ^2) are the coordinates of the locally euclidean reference frame, and (x^1, x^2) two arbitrary coordinates. If we now change from (x^1, x^2) to $(x^{1'}, x^{2'})$, where $x^1 = x^1(x^{1'}, x^{2'})$

, and $x^2 = x^2(x^{1'}, x^{2'})$, the metric tensor in the new coordinate frame $(x^{1'}, x^{2'})$ will be

$$\begin{aligned} g'_{11} \equiv g_{1'1'} &= [(\frac{\partial \xi^1}{\partial x^{1'}})^2 + (\frac{\partial \xi^2}{\partial x^{1'}})^2] \\ &= [(\frac{\partial \xi^1}{\partial x^1}\frac{\partial x^1}{\partial x^{1'}} + \frac{\partial \xi^1}{\partial x^2}\frac{\partial x^2}{\partial x^{1'}})^2 + [(\frac{\partial \xi^2}{\partial x^1}\frac{\partial x^1}{\partial x^{1'}} + \frac{\partial \xi^2}{\partial x^2}\frac{\partial x^2}{\partial x^{1'}})^2 \\ &= [(\frac{\partial \xi^1}{\partial x^1})^2 + (\frac{\partial \xi^2}{\partial x^1})^2](\frac{\partial x^1}{\partial x^{1'}})^2 + [(\frac{\partial \xi^1}{\partial x^2})^2 + (\frac{\partial \xi^2}{\partial x^2})^2](\frac{\partial x^2}{\partial x^{1'}})^2 \\ &+ 2(\frac{\partial \xi^1}{\partial x^1}\frac{\partial \xi^1}{\partial x^2} + \frac{\partial \xi^2}{\partial x^1}\frac{\partial \xi^2}{\partial x^2})(\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}}) \\ &= g_{11}(\frac{\partial x^1}{\partial x^{1'}})^2 + g_{22}(\frac{\partial x^2}{\partial x^{1'}})^2 + 2g_{12}(\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}}). \end{aligned} \tag{1.13}$$

In general we can write

$$g'_{\alpha\beta} = g_{\mu\nu}\frac{\partial x^\mu}{\partial x^{\alpha'}}\frac{\partial x^\nu}{\partial x^{\beta'}} \tag{1.14}$$

This is the manner in which a tensor transforms under an arbitrary coordinate transformation
(this point will be illustrated in more detail in following lectures).

Thus, given a space in which the *distance* can be expressed in terms of Pythagoras' law, if we make an arbitrary coordinate transformation the knowledge of $g_{\mu\nu}$ allows us to express the *distance* in the new reference system. The converse is also true: given a space in which

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta, \tag{1.15}$$

if this space belongs to the class defined by Gauss, at any given point it is always possible to choose a locally euclidean coordinate system (ξ^α) such that

$$ds^2 = (d\xi^1)^2 + (d\xi^2)^2. \tag{1.16}$$

This concept can be generalized to a space of arbitrary dimensions.
The metric tensor determines the intrinsic properties of a metric space.
We now want to define a function of $g_{\alpha\beta}$ and of its first and second derivatives, which depends on the inner properties of the surface, but does not depend on the particular coordinate system we choose. Gauss showed that in the case of two-dimensional surfaces this function can be determined, and it is called, after him, the *Gaussian curvature*, defined as

$$\begin{aligned} k(x^1, x^2) = \frac{1}{2g}&\left[2\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{11}}{\partial {x^2}^2} - \frac{\partial^2 g_{22}}{\partial {x^1}^2}\right] \\ -\frac{g_{22}}{4g^2}&\left[\left(\frac{\partial g_{11}}{\partial x^1}\right)\left(2\frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) - \left(\frac{\partial g_{11}}{\partial x^2}\right)^2\right] \\ +\frac{g_{12}}{4g^2}&\left[\left(\frac{\partial g_{11}}{\partial x^1}\right)\left(\frac{\partial g_{22}}{\partial x^2}\right) - 2\left(\frac{\partial g_{11}}{\partial x^2}\right)\left(\frac{\partial g_{22}}{\partial x^1}\right)\right. \\ &\left. + \left(2\frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right)\left(2\frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right)\right] \end{aligned} \tag{1.17}$$

$$-\frac{g_{11}}{4g^2}\left[\left(\frac{\partial g_{22}}{\partial x^2}\right)\left(2\frac{\partial g_{12}}{\partial x^1}-\frac{\partial g_{11}}{\partial x^2}\right)-\left(\frac{\partial g_{22}}{\partial x^1}\right)^2\right]$$

where g is the determinant of the 2-metric $g_{\alpha\beta}$

$$g = g_{11}g_{22} - g_{12}^2. \tag{1.18}$$

For example, given a spherical surface of radius a, with metric $ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2$, (polar coordinates) we find

$$k = \frac{1}{a^2}; \tag{1.19}$$

no matter how we choose the coordinates to describe the spherical surface, we shall always find that the gaussian curvature has this value. For the Gauss-Bolyai-Lobachewski geometry where

$$g_{11} = \frac{a^2\left[1-(x^2)^2\right]}{\left[1-(x^1)^2-(x^2)^2\right]^2}, \qquad g_{22} = \frac{a^2\left[1-(x^1)^2\right]}{\left[1-(x^1)^2-(x^2)^2\right]^2}, \qquad g_{12} = \frac{a^2 x^1 x^2}{\left[1-(x^1)^2-(x^2)^2\right]^2}, \tag{1.20}$$

we shall always find

$$k = -\frac{1}{a^2}; \tag{1.21}$$

if the space is flat, the gaussian curvature is $k = 0$. If we choose a different coordinate system, $g_{\alpha\beta}(x^1, x^2)$ will change but k will remain the same.

1.3 Summary

We have seen that it is possible to select a class of 2-dimensional spaces where it is possible to set up, in the neighborhoods of any point, a coordinate system (ξ^1, ξ^2) such that the *distance* between two close points is given by Pythagoras' law. Then we have defined the metric tensor $g_{\alpha\beta}$, which allows to compute the *distance* in an arbitrary coordinate system, and we have derived the law according to which $g_{\alpha\beta}$ transforms when we change reference. Finally, we have seen that there exists a scalar quantity, the gaussian curvature, which expressees the inner properties of a surface: it is a function of $g_{\alpha\beta}$ and of its first and second derivatives, and it is invariant under coordinate transformations.

These results can be extended to an arbitrary D-dimensional space. In particular, as we shall discuss in the following, we are interested in the case D=4, and we shall select those spaces, or better, those spacetimes, for which the *distance* is that prescribed by Special Relativity.

$$ds^2 = -(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2. \tag{1.22}$$

For the time being, let us only clarify the following point. In a D-dimensional space we need more than one function to describe the inner properties of a surface. Indeed, since g_{ij} is symmetic, there are only $D(D+1)/2$ independent components. In addition, we can choose D arbitrary coordinates, and impose D functional relations among them. Therefore the number of independent functions that describe the inner properties of the space will be

$$C = \frac{D(D+1)}{2} - D = \frac{D(D-1)}{2}. \tag{1.23}$$

If D=2, as we have seen, C=1. If D=4, C=6, therefore there will be 6 invariants to be defined for our 4-dimensional spacetime. The problem of finding these invariant quantities was studied by Riemann (1826-1866) and subsequently by Christoffel, LeviCivita, Ricci, Beltrami. We shall see in the following that Riemaniann geometries play a crucial role in the description of the gravitational field.

1.4 The Newtonian theory

In this section we shall discuss why the Newtonian theory of gravity became unappropriate to correctly describe the gravitational field. The Newtonian theory of gravity was published in 1685 in the "Philosophiae Naturalis Principia Mathematica", which contains an incredible variety of fundamental results and, among them, the cornerstones of classical physics:

1) *Newton's law*

$$\vec{F} = m_I \vec{a}, \tag{1.24}$$

2) *Newton's law of gravitation*

$$\vec{F}_G = m_G \vec{g}, \tag{1.25}$$

where

$$\vec{g} = -\frac{G \sum_i M_{Gi}(\vec{r} - \vec{r}_i')}{|\vec{r} - \vec{r}_i|^3} \tag{1.26}$$

depends on the position of the massive particle with respect to the other masses that generate the field, and it decreases as the inverse square of the distance $g \sim \frac{1}{r^2}$. The two laws combined together clearly show that a body falls with an acceleration given by

$$\vec{a} = \left(\frac{m_G}{m_I}\right) \vec{g}. \tag{1.27}$$

If $\frac{m_G}{m_I}$ is a constant independent of the body, the acceleration is the same for every infalling body, and independent of their mass. Galileo (1564-1642) had already experimentally discovered that this is, indeed, true, and Newton itself tested the equivalence principle studying the motion of pendulum of different composition and equal lenght, finding no difference in their periods. The validity of the equivalence principle was the core of Newton's arguments for the universality of his law of gravitation; indeed, after describing his experiments with different pendulum in the Principia he says:

But, without all doubt, the nature of gravity towards the planets is the same as towards the earth.

Since then a variety of experiments confirmed this crucial result. Among them Eotvos experiment in 1889 (accuracy of 1 part in 10^9), Dicke experiment in 1964 (1 part in 10^{11}), Braginsky in 1972 (1 part in 10^{12}) and more recently the Lunar-Laser Ranging experiments (1 part in 10^{13}). All experiments up to our days confirm *The Principle of Equivalence of* *the gravitational and the inertial mass.* Now before describing why at a certain point the Newtonian theory fails to be a satisfactory description of gravity, let me briefly describe the reasons of its great success, that remained untouched for more than 200 years.

In the Principia, Newton formulates the universal law of gravitation, he develops the theory of lunar motion and tides and that of planetary motion around the Sun, which are the most elegant and accomplished descriptions of these phenomena.

After Newton, the law of gravitation was used to investigate in more detail the solar system; its application to the study of the perturbations of Uranus' orbit around the Sun led, in 1846, Adams (England) and Le Verrier (France) to predict the existence of a new planet which was named Neptune. A few years later, the discovery of Neptun was a triumph of Newton's theory of gravitation.

However, already in 1845 Le Verrier had observed anomalies in the motion of Mercury. He found that the perihelium precession of $35''/100\ years$ exceeded the value due to the perturbation introduced by the other planets predicted by Newton's theory. In 1882 Newcomb confirmed this discrepancy, giving a higher value, of $43''/100\ year$. In order to explain this effect, scientists developed models that predicted the existence of some interplanetary matter, and in 1896 Seelinger showed that an ellipsoidal distribution of matter surrounding the Sun could explain the observed precession.

We know today that these models were wrong, and that the reason for the exceedingly high precession of Mercury's perihelium has a relativistic origin.

In any event, we can say that the Newtonian theory worked remarkably well to explain planetary motion, but already in 1845 the suspect that something did not work perfectly had some experimental evidence.

Let us turn now to a more philosophical aspect of the theory. The equations of Newtonian mechanics are invariant under Galileo's transformations

$$\begin{aligned} \vec{x}' &= R_0\vec{x} + \vec{v}t + \vec{d}_0 \\ t' &= t + \tau \end{aligned} \tag{1.28}$$

where R_0 is the orthogonal, constant matrix expressing how the second frame is rotated with respect to the first (its elements depend on the three Euler angles), $\vec{v}$ is the relative velocity of the two frames, and $\vec{d}_0$ the initial distance between the two origins. The ten parameters (3 Euler angles, 3 components for $\vec{v}$ and $\vec{d}$, + the time shift τ) identify the Galileo group.

The invariance of the equations with respect to Galileo's transformations implies the existence of **inertial frames**, where the laws of Mechanics hold. What then determines which frames are inertial frames? For Newton, the answer is that there exists an absolute space, and the result of the famous experiment of the rotating vessel is a proof of its existence [1]: inertial frames are those in uniform relative motion with respect to the absolute space.

[1]The vessel experiment: a vessel is filled with water and rotates with a given angular velocity about the symmetry axis. After some time the surface of the water assumes the typical shape of a paraboloid, being in equilibrium under the action of the gravity force, the centrifugal force and the fluid forces. Now suppose that the masses in the entire universe would rigidly rotate with respect to the vessel at the same angular velocity: in this case, for Newton the water surface would remain at rest and would not bend, because the vessel is not moving with respect to the absolute space and therefore no centrifugal force acts on it.

However this idea was rejected by Leibniz who claimed that there is no philosophical need for such a notion, and the debate on this issue continued during the next centuries. One of the major opponents was Mach, who argued that if the masses in the entire universe would rigidly rotate with respect to the vessel, the water surface would bend in exactely the same way as when the vessel was rotating with respect to them. This is because the inertia is a measure of the gravitational interaction between a body and the matter content of the rest of the Universe.

The problems I have described (the discrepancy in the advance of perihelium and the postulate absolute space) are however only small clouds: the Newtonian theory remains *The* theory of gravity until the end of the ninentheenth century. The big storm approaches with the formulation of the theory of electrodynamics presented by Maxwell in 1864. Maxwell's equations establish that the velocity of light is an universal constant. It was soon understood that these equations are not invariant under Galileo's transformations; indeed, according to eqs. (1.28), if the velocity of light is c in a given coordinate frame, it cannot be c in a second frame moving with respect to the first with assigned velocity $\vec{v}$. To justify this discrepancy, Maxwell formulated the hypothesis that light does not really propagate in vacuum: electromagnetic waves are carried by a medium, the *luminiferous ether*, and the equations are invariant only with respect to a set of galilean inertial frames that are at rest with respect to the ether. However in 1887 Michelson and Morley showed that the velocity of light is the same, within 5km/s (today the accuracy is less than 1km/s), along the directions of the Earth's orbital motion, and transverse to it. How this result can be justified? One possibility was to say the Earth is at rest with respect to the ether; but this hypothesis was totally unsatisfactory, since it would have been a coming back to an antropocentric picture of the world. Another possibility was that the ether simply does not exist, and one has to accept the fact that the speed of light is the same in any direction, and whatever is the velocity of the source. This was of course the only reasonable explanation. But now the problem was to find the coordinate transformation with respect to which Maxwell's equations are invariant. The problem was solved by Einstein in 1905; he showed that Galileo's transformations have to be replaced by the Lorentz transformations

$$x^{\alpha\prime} = L^{\alpha}{}_{\gamma} x^{\gamma}, \tag{1.29}$$

where $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, and

$$L^0_0 = \gamma, \quad L^0{}_j = L^j_0 = \frac{\gamma}{c} v_j, \qquad L^i{}_j = \delta^i{}_j + \frac{\gamma - 1}{v^2} v_i v_j. \quad i, j = 1, 3 \tag{1.30}$$

and v^i are the components of the velocity of the boost.

As it was immediately realised, however, while Maxwell's equations are invariant with respect to Lorentz transformations, Newton's equations were not, and consequently one should face the problem of how to modify the equations of mechanics and gravity in such a way that they become invariant with respect to Lorentz transformations. It is at this point that Einstein made his fundamental observation.

1.5 The role of the Equivalence Principle in the formulation of the new theory of gravity

Let us consider the motion of a non relativistic particle moving in a **constant** gravitational field. Be $\vec{F}_k$ some other forces acting on the particle. According to Newtonian mechanics, the equation of motion are

$$m_I \frac{d^2\vec{x}}{dt^2} = m_G\vec{g} + \sum_k \vec{F}_k \tag{1.31}$$

Let us now jump on an elevator which is freely falling in the same gravitational field, i.e. let us make the following coordinate transformation

$$\vec{x}' = \vec{x} - \frac{1}{2}\vec{g}t^2, \qquad t' = t. \tag{1.32}$$

In this new reference frame eq. (1.31) becomes

$$m_I \left[\frac{d^2\vec{x}'}{dt^2} + \vec{g}\right] = m_G\vec{g} + \sum_k \vec{F}_k. \tag{1.33}$$

Since by the Equivalence Principle $m_I = m_G$, and since this is true for **any** particle, this equation becomes

$$m_I \frac{d^2\vec{x}'}{dt^2} = \sum_k \vec{F}_k. \tag{1.34}$$

Let us compare eq. (1.31) and eq. (1.34). It is clear that that an observer O′ who is in the elevator, i.e. in free fall in the gravitational field, sees the same laws of physics as the initial observer O, but he does not feel the gravitational field. **This result follows from the equivalence, experimentally tested, of the inertial and gravitational mass**. If m_I would be different from m_G, or better, if their ratio would not be constant and the same for all bodies, this would not be true, because we could not simplify the term in $\vec{g}$ in eq. (1.33)! It is also apparent that if $\vec{g}$ would not be constant eq. (1.34) would contain additional terms containing the derivatives of $\vec{g}$. However, we can always consider an interval of time so short that $\vec{g}$ can be considered as constant and eq. (1.34) holds. Consider a particle at rest in this frame and no force $\vec{F}_k$ acting on it. Under this assumption, according to eq. (1.34) it will remain at rest forever. Therefore we can define this reference as a **locally inertial frame**. If the gravitational field is constant and unifom everywhere, the coordinate transformation (1.32) defines a locally inertial frame that covers the whole spacetime. If this is not the case, we can set up a locally inertial frame only in the neighborhood of any given point.

The points discussed above are crucial to the theory of gravity, and deserve a further explanation. Gravity is distinguished from all other forces because all bodies, given the same initial velocity, follow the same trajectory in a gravitational field, regardless of their internal constitution. This is not the case, for example, for electromagnetic forces, which act on charged but not on neutral bodies, and in any event the trajectories of charged particles depend on the ratio between charge and mass, which is not the same for all particles. Similarly, other forces, like the strong and weak interactions, affect different particles differently.

It is this distinctive feature of gravity that makes it possible to describe the effects of gravity in terms of curved geometry, as we shall see in the following.

Let us now state the Principle of Equivalence. There are two formulations:

The strong Principle of Equivalence

In an arbitrary gravitational field, at any given spacetime point, we can choose a locally inertial reference frame such that, in a sufficiently small region surrounding that point, all physical laws take the same form they would take in absence of gravity, namely the form prescribed by Special Relativity.

There is also a weaker version of this principle

The weak Principle of Equivalence

Same as before, but it refers to the laws of motion of freely falling bodies, instead of all physical laws.

The preceeding formulations of the equivalence principle resembles very much to the axiom that Gauss chose as a basis for non-euclidean geometries, namely: *at any given point in space, there exist a locally euclidean reference frame such that, in a sufficiently small region surrounding that point, the distance between two points is given by the law of Pythagoras.*

The Equivalence Principle states that in a locally inertial frame all laws of physics must coincide, locally, with those of Special Relativity, and consequently in this frame the *distance* between two points must coincide with Minkowsky's expression

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2. \tag{1.35}$$

We therefore expect that the equations of gravity will look very similar to those of Riemaniann geometry. In particular, as Gauss defined the inner properties of curved surfaces in terms of the derivatives $\frac{\partial \xi^\alpha}{\partial x^\mu}$ (which in turn defined the metric, see eqs. (1.5) and (1.7)), where ξ^α are the "locally euclidean coordinates" and x^μ are arbitrary coordinates, in a similar way we expect that the effects of a gravitational field will be described in terms of the derivatives $\frac{\partial \xi^\alpha}{\partial x^\mu}$ where now ξ^α are the "locally inertial coordinates", and x^μ are arbitrary coordinates. All this will follow from the equivalence principle. Up to now we have only established that, as a consequence of the Equivalence Principle there exist a connection between the gravitational field and the metric tensor. But which connection?

1.6 The geodesic equations as a consequence of the Principle of Equivalence

Let us start exploring what are the consequences of the Principle of Equivalence. We want to find the equations of motion of a particle that moves under the exclusive action of a gravitational field (i.e. it is in free fall), when this motion is observed in an arbitrary reference frame. We shall now work in a four-dimensional spacetime with coordinates $(x^0 = ct, x^1, x^2, x^3)$.

First we start analysing the motion in a locally inertial frame, the one in free fall with the particle. According to the Principle of Equivalence, in this frame the *distance* between two neighboring points is

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu, \tag{1.36}$$

where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ is the metric tensor of the flat, Minkowsky spacetime. If τ is the particle proper time, and if it is chosen as time coordinate, for what we said before the equations of motion are

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0. \tag{1.37}$$

We now change to a frame where the coordinates are labelled $x^\alpha = x^\alpha(\xi^\alpha)$, i.e. we assign a transformation law which allows to express the new coordinates as functions of the old ones. In a following lecture we shall clarify and make rigorous all concepts that we are now using, such us metric tensor, coordinate transformations etc. In the new frame the *distance* is

$$ds^2 = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu = g_{\mu\nu}dx^\mu dx^\nu, \tag{1.38}$$

where we have defined the metric tensor $g_{\mu\nu}$ as

$$g_{\mu\nu} = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}. \tag{1.39}$$

This formula is the 4-dimensional generalization of the 2-dimensional gaussian formula (see eq. (1.5)). In the new frame the equation of motion of the particle (1.37) becomes:

$$\frac{d^2x^\alpha}{d\tau^2} + \left[\frac{\partial x^\alpha}{\partial\xi^\lambda}\frac{\partial^2\xi^\lambda}{\partial x^\mu\partial x^\nu}\right]\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] = 0. \tag{1.40}$$

(see the detailed calculations in appendix A). If we now define the following quantities

$$\Gamma^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial\xi^\lambda}\frac{\partial^2\xi^\lambda}{\partial x^\mu\partial x^\nu}, \tag{1.41}$$

eq. (1.40) become

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] = 0. \tag{1.42}$$

The quantities (1.41) are called the **affine connections**, or **Christoffel's symbols**, the properties of which we shall investigate in a following lecture. Equation (1.42) is the **geodesic equation**, i.e. the equation of motion of a freely falling particle when observed in an arbitrary coordinate frame. Let us analyse this equation. We have seen that if we are in a locally inertial frame, where, by the Equivalence Principle, we are able to eliminate the gravitational force, the equations of motion would be that of a free particle (eq. 1.37). If we change to another frame we feel the gravitational field (and in addition all apparent forces like centrifugal, Coriolis, and dragging forces). In this new frame the geodesic equation becomes eq. (1.42) and the additional term

$$\Gamma^\alpha_{\mu\nu}\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] \tag{1.43}$$

expresses the gravitational force per unit mass that acts on the particle. If we were in Newtonian mechanics, this term would be $\vec{g}$ (plus the additional apparent accelerations, but let us assume for the time being that we choose a frame where they vanish), and $\vec{g}$ is the gradient of the gravitational potential. What does that mean? The affine connection $\Gamma^{\alpha}_{\mu\nu}$ contains the second derivatives of (ξ^{α}). Since the metric tensor (1.39) contains the first derivatives of (ξ^{α}) (see eq. (1.39)), it is clear that $\Gamma^{\alpha}{}_{\mu\nu}$ will contain first derivatives of $g_{\mu\nu}$. This can be shown explicitely, and in a next lecture we will show that

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} \right\}. \tag{1.44}$$

Thus, in analogy with the Newtonian law, we can say that the affine connections are the generalization of the Newtonian gravitational field, and that the metric tensor is the generalization of the Newtonian gravitational potential.

I would like to stress that this is a *physical* analogy, based on the study of the motion of freely falling particles compared with the Newtonian equations of motion.

1.7 Summary

We have seen that once we introduce the Principle of Equivalence, the notion of metric and affine connections emerge in a natural way to describe the effects of a gravitational field on the motion of falling bodies. It should be stressed that the metric tensor $g_{\mu\nu}$ represents the gravitational potential, as it follows from the geodesic equations. But in addition it is a geometrical entity, since, through the notion of *distance* , it characterizes the spacetime geometry. This double role, physical and geometrical of the metric tensor, is a direct consequence of the Principle of Equivalence, as I hope it is now clear.

Now we can answer the question " why do we need a tensor to describe a gravitational field": the answer is in the Equivalence Principle.

1.8 Locally inertial frames

We shall now show that if we know $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ (i.e. $g_{\mu\nu}$ and its first derivatives) at a point X, we can determine a locally inertial frame $\xi^{\alpha}(x)$ in the neighborhood of X in the following way. Multiply $\Gamma^{\beta}_{\mu\nu}$ by $\frac{\partial \xi^{\beta}}{\partial x^{\lambda}}$

$$\frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \Gamma^{\lambda}{}_{\mu\nu} = \frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} = \tag{1.45}$$
$$\delta^{\beta}_{\alpha} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} = \frac{\partial^2 \xi^{\beta}}{\partial x^{\mu} \partial x^{\nu}},$$

i.e.

$$\frac{\partial^2 \xi^{\beta}}{\partial x^{\mu} \partial x^{\nu}} = \frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \Gamma^{\lambda}{}_{\mu\nu}. \tag{1.46}$$

This equation can be solved by a series expansion near X

$$\begin{aligned}\xi^\beta(x) &= \xi^\beta(X) + [\frac{\partial \xi^\beta(x)}{\partial x^\lambda}]_{x=X}(x^\lambda - X^\lambda) \\ &+ \frac{1}{2}[\frac{\partial \xi^\beta(x)}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu}]_{x=X}(x^\mu - X^\mu)(x^\nu - X^\nu) + ... \\ &= a^\beta + b^\beta_\lambda(x^\lambda - X^\lambda) + \frac{1}{2}b^\beta_\lambda \Gamma^\lambda_{\mu\nu}(x^\mu - X^\mu)(x^\nu - X^\nu) + ...\end{aligned} \tag{1.47}$$

On the other hand we know by eq. (1.39) that

$$g_{\mu\nu}(X) = \eta_{\alpha\beta}\frac{\partial \xi^\alpha(x)}{\partial x^\mu}|_{x=X}\frac{\partial \xi^\beta(x)}{\partial x^\nu}|_{x=X} = \eta_{\alpha\beta}b^\alpha_\mu b^\beta_\nu, \tag{1.48}$$

and from this equation we compute b^β_μ. Thus, given $g_{\mu\nu}$ and $\Gamma^\alpha_{\mu\nu}$ at a given point X we can determine the local inertial frame to order $(x-X)^2$ by using eq. (1.47). This equation defines the coordinate system except for the ambiguity in the constants a^μ. In addition we have still the freedom to make an inhomogeneous Lorentz transformation, and the new frame will still be locally inertial, as it is shown in appendix B.

1.9 Appendix 1A

Given the equation of motion of a free particle

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0, \qquad (A1)$$

let us make a coordinate transformation to an arbitrary system x^α

$$\xi^\alpha = \xi^\alpha(x^\gamma), \qquad \rightarrow \qquad \frac{d\xi^\alpha}{d\tau} = \frac{\partial \xi^\alpha}{\partial x^\gamma}\frac{dx^\gamma}{d\tau}, \qquad (A2)$$

eq. (A1) becomes

$$\frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\gamma}\frac{dx^\gamma}{d\tau}\right) = \frac{d^2x^\gamma}{d\tau^2}\frac{\partial \xi^\alpha}{\partial x^\gamma} + \frac{\partial^2 \xi^\alpha}{\partial x^\beta \partial x^\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0. \quad (A3)$$

Multiply eq. (A3) by $\frac{\partial x^\sigma}{\partial \xi^\alpha}$ remembering that

$$\frac{\partial \xi^\alpha}{\partial x^\gamma}\frac{\partial x^\sigma}{\partial \xi^\alpha} = \frac{\partial x^\sigma}{\partial x^\gamma} = \delta^\sigma_\gamma,$$

where δ^σ_γ is the Kronecker symbol (= 1 if $\sigma = \gamma$ 0 otherwise), we find

$$\frac{d^2x^\gamma}{d\tau^2}\delta^\sigma_\gamma + \frac{\partial x^\sigma}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\beta \partial x^\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0, \quad (A4)$$

which finally becomes

$$\frac{d^2x^\sigma}{d\tau^2} + [\frac{\partial x^\sigma}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\beta \partial x^\gamma}]\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0, \quad (A5)$$

which is eq. (1.40).

1.10 Appendix 1B

Given a locally inertial frame ξ^α

$$ds^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu. \qquad (B1)$$

let us consider the Lorentz transformation

$$\xi^i = L^i{}_{j'} \xi^{j'}, \qquad (B2)$$

where

$$L^i_j = \delta^i_j + v^i v_j \frac{\gamma - 1}{v^2}, \qquad L^0_j = \frac{\gamma v_j}{c}, \qquad L^0_0 = \gamma, \qquad \gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}. \qquad (B3)$$

The *distance* will now be

$$ds^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu = \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial \xi^{i'}} \frac{\partial \xi^\nu}{\partial \xi^{j'}} d\xi^{i'} d\xi^{j'}.; \qquad (B4)$$

Since

$$\frac{\partial \xi^\mu}{\partial \xi^{i'}} = L^\mu{}_\beta \delta^\beta{}_{i'} = L^\mu{}_{i'}, \qquad (B5)$$

it follows that

$$ds^2 = \eta_{\mu\nu} L^\mu{}_{i'} L^\nu{}_{j'} d\xi^{i'} d\xi^{j'}. \qquad (B6)$$

Since L^i_j is a Lorentz transformation,

$$\eta_{\mu\nu} L^\mu{}_{i'} L^\nu{}_{j'} = \eta_{i'j'},$$

consequently the new frame is still a locally inertial frame.

$$ds^2 = \eta_{\mu'\nu'} d\xi^{\mu'} d\xi^{\nu'}. \qquad (B7)$$

Chapter 2

Topological Spaces, Mapping, Manifolds

In chapter 1 we have shown that the Principle of Equivalence allows to establish a relation between the metric tensor and the gravitational field. We used vectors and tensors, we made coordinate transformations, but we did not define the geometrical objects we were introducing, and we did not discuss whether we are entitled to use these notions. We shall now define in a more rigorous way what is the type of space we are working in, what is a coordinate transformation, a vector, a tensor. Then we shall introduce the metric tensor and the affine connections as geometrical objects and, after defining the covariant derivative, we shall finally be able to introduce the Riemann tensor. This work is preliminary to the derivation of Einstein's equations.

2.1 Topological spaces

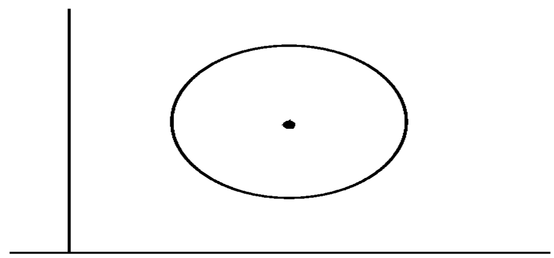
In general relativity we shall deal with *topological spaces*. The word topology has two distinct meanings: local topology (to which we are mainly interested), and global topology, which involves the study of the large scale features of a space.

Before introducing the general definition of a topological space, let us recall some properties of R^n, which is a particular case of topological space; this will help us in the understanding of the general definition of topological spaces.

Given a point $y = (y^1, y^2, ...y^n) \in \mathbf{R}^n$, a neighborhood of y is the collection of points x such that

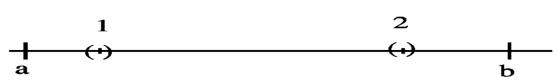
$$|x - y| \equiv \sqrt{\sum_{i=1}^{n}(x^i - y^i)^2} < r, \tag{2.1}$$

where r is a real number. (This is sometimes called an 'open ball').



A set of points $\mathbf{S} \in \mathbf{R}^n$ is *open* if every point $x \in \mathbf{S}$ has a neighborhood entirely contained in **S**. This implies that an open set does not include the points on the boundary of the set. For instance, an open ball is an open set; a closed ball, defined by $|x-y| \leq r$, is not an open set, because the points of the boundary, i.e. $|x-y| = r$, do not admit a neighborhood contained in the set.

Intuitively we have an idea that this is a **continuum space**, namely that there are points of R^n arbitrarily close to any given point, that the line joining two points can be subdivided into arbitrarily many pieces which also join points of R^n. A non continuous space is, for example, a lattice. A formal characterization of a continuum space is the *Haussdorff criterion*: any two points of a continuum space have neighborhoods which do not intersect.



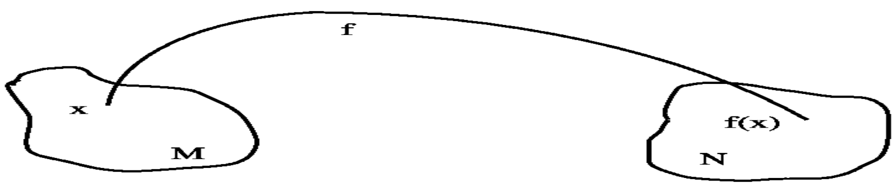
The open sets of R^n satisfy the following properties:
(1) if $\mathbf{O}_1$ and $\mathbf{O}_2$ are open sets, so it is their intersection.
(2) the union of any collection (possibly infinite in number) of open sets is open.

Let us now consider a general set **T**. Furthermore, we consider a collection of subsets of **T**, say $\mathbf{O}=\{\mathbf{O}_i\}$, and call them *open sets*. We say that the couple $(\mathbf{T},\mathbf{O})$ formed by the set and the collection of subsets is a **topological space** if it satisfies the properties (1) and (2) above.

We remark that the space **T** is not necessarily $\mathbf{R}^n$: it can be any kind of set; the only specification we give is the collection of subsets **O**, which are by definition the open sets, and that satisfy the properties (1), (2). In particular, in a topological space the notion of distance is a structure which has not been introduced: all definitions only require the notion of open sets.

2.2 Mapping

A map f from a space **M** to a space **N** is a rule which associates with an element x of **M**, a unique element $y = f(x)$ of **N**

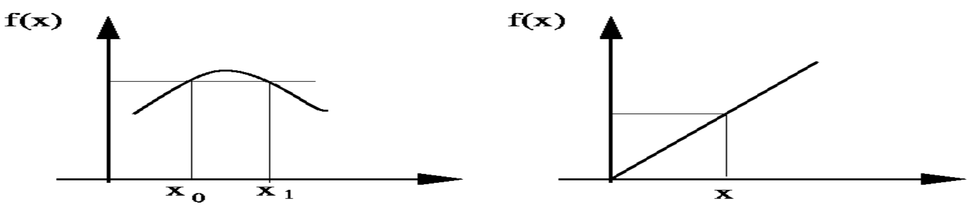


M and **N** need not to be different. For example, the simplest maps are ordinary real-valued functions on **R**

$$\text{EXAMPLE} \qquad y = x^3, \qquad x \in \mathbf{R}, \qquad \text{and} \qquad y \in \mathbf{R}. \tag{2.2}$$

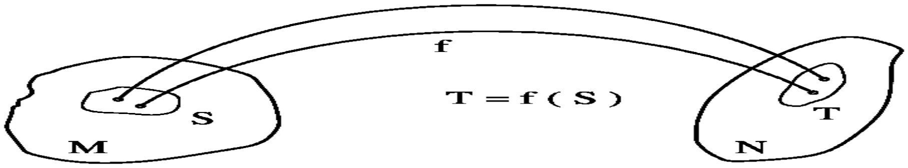
In this case **M** and **N** coincide.

A map gives a unique $f(x)$ for every x, but not necessarily a unique x for every $f(x)$. EXAMPLE



map **many to one** map **one to one**

If f maps **M** to **N** then for any set **S** in **M** we have an *image* in **N**, i.e. the set **T** of all points mapped by f from **S** in **N**



Conversely the set **S** is the *inverse image* of **T**

$$\mathbf{S} = f^{-1}(\mathbf{T}). \tag{2.3}$$

Inverse mapping is possible only in the case of one-to-one mapping. The statement "f maps **M** to **N**" is indicated as

$$f : \mathbf{M} \rightarrow \mathbf{N}. \tag{2.4}$$

f maps a particular element $x \in \mathbf{M}$ to $y \in \mathbf{N}$ is indicated as

$$f : x \quad | \rightarrow y \tag{2.5}$$

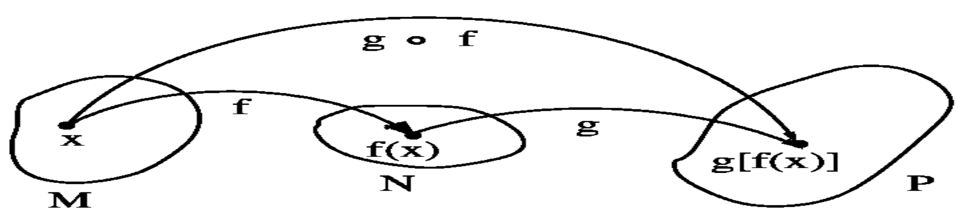
the image of a point x is $f(x)$.

2.3 Composition of maps

Given two maps $f : \mathbf{M} \rightarrow \mathbf{N}$ and $g : \mathbf{N} \rightarrow \mathbf{P}$, there exists a map $g \circ f$ that maps **M** to **P**

$$g \circ f : \mathbf{M} \rightarrow \mathbf{P}. \tag{2.6}$$

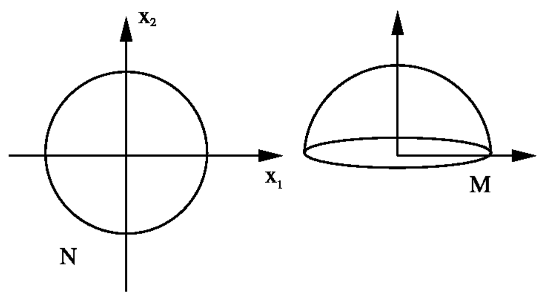
This means: take a point $x \in \mathbf{M}$ and find the image $f(x) \in \mathbf{N}$, then use g to map this point to a point $g\left(f(x)\right) \in \mathbf{P}$



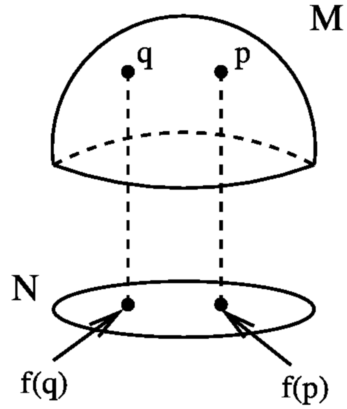
$$\text{EXAMPLE} \quad \begin{array}{llll} f: & x & | \rightarrow y & y = x^3 \\ g: & y & | \rightarrow z & z = y^2 \\ g \circ f: & x & | \rightarrow z & z = x^6 \end{array} \tag{2.7}$$

Map into: If a map is defined for all ponts of a manifold **M**, it is a mapping from **M into N**.
Map onto: If, in addition, every point of **N** has an inverse image (but not necessarily a unique one), it is a map from **M onto N**.

EXAMPLE: be **N** the unit open disc in $\mathbf{R}^2$, i.e. the set of all points in $\mathbf{R}^2$ such that the distance from the center is less than one, $d(0, x) < 1$. Be **M** the surface of an emisphere $\theta < \frac{\pi}{2}$ belonging to the unit sphere.



There exists a one-to one mapping f from $\mathbf{M}$ onto $\mathbf{N}$.



2.4 Continuous mapping

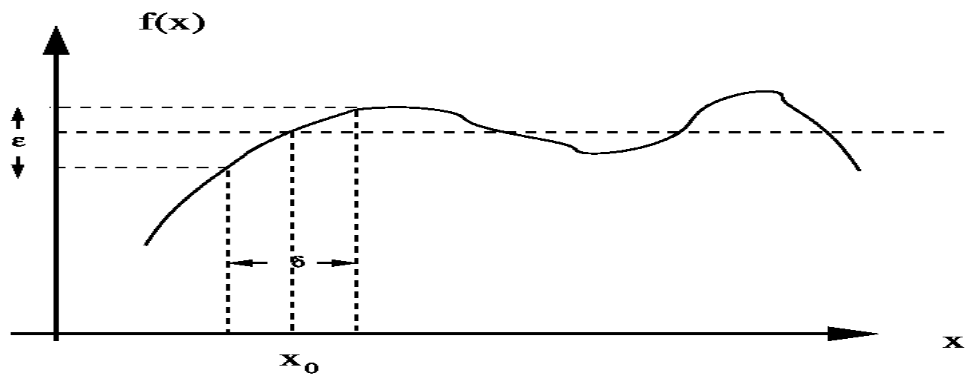
A map $f : \mathbf{M} \to \mathbf{N}$ is continuous at $x \in \mathbf{M}$ if any open set of $\mathbf{N}$ containing $f(x)$ contains the image of an open set of $\mathbf{M}$. $\mathbf{M}$ and $\mathbf{N}$ must be topological spaces, otherwise the notion of continuity has no meaning.

This definition is related to the familiar notion of continuous functions. Suppose that f is a real-valued function of one real variable. That is f is a map of $\mathbf{R}$ to $\mathbf{R}$

$$f : \mathbf{R} \to \mathbf{R}. \tag{2.8}$$

In the elementary calculus we say that f is continuous at a point $x0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(x0)| < \epsilon, \qquad \forall x \quad \text{such that} \quad |x - x0| < \delta. \tag{2.9}$$



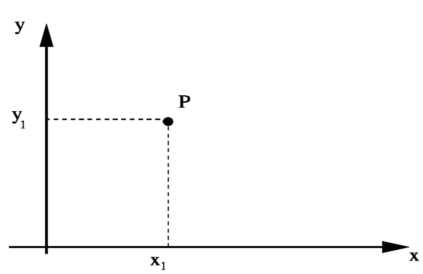
Let us translate this definition in terms of open sets. From the figure it is apparent that any open set containing $f(x0)$, i.e. $|f(x) - f(x0)| < r$ with r arbitrary, contains an image of an open set of **M** . This is true at least in the domain of definition of f. This definition is more general than that of continuous functions, because it is based on the notion of open sets, and not on the notion of distance.

2.5 Manifolds and differentiable manifolds

The notion of manifold is crucial to define a coordinate system.

A **manifold M** *is a topological space, which satisfies the Haussdorff criterion, and such that each point of* **M** *has an open neighborhood which has a continuous 1-1 map* **onto** *an open set of R^n. n is the dimension of the manifold.*

In this definition we have used the concepts defined in the preceeding pages: the space must be topological, continuous, and we want to associate an n-tuple of real numbers, i.e. *a set of coordinates* to each point. For example, when we consider the diagram



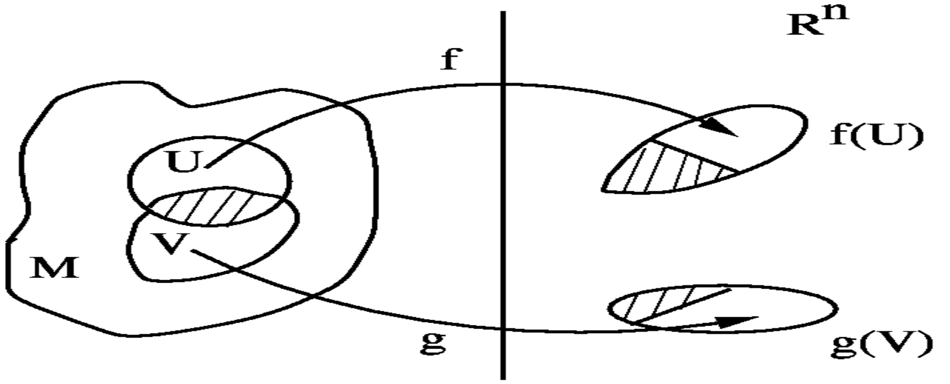
we are just using the notion of manifold: we take a point P, and map it to the point $(x^1, y^1) \in \mathbf{R}^2$. And this operation can be done for any open neighborhood of P. It should be stressed that the definition of manifold involves open sets and not the whole of **M** and R^n, because we do not want to restrict the global topology of **M** . Moreover, at this stage we only require the map to be 1-1. We have not yet introduced any geometrical notion as lenght, angles etc. At this level we only require that the local topology of **M** is the same as that of R^n. *A manifold is a space with this topology.*

DEFINITION OF COORDINATE SYSTEMS

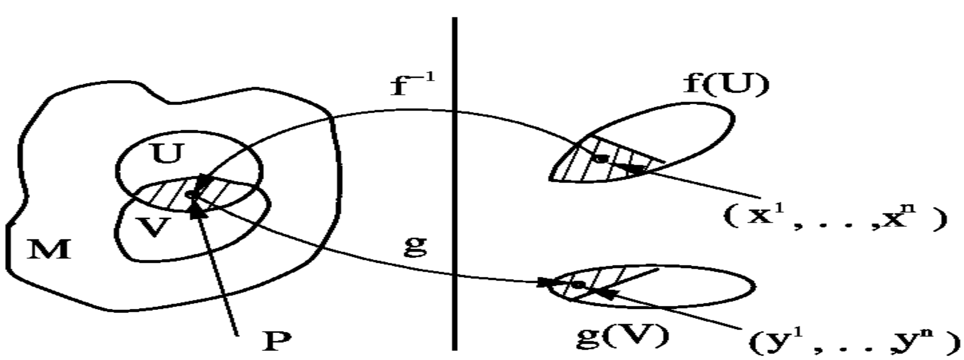
A coordinate system, or a **chart**, is a pair consisting of an open set of **M** and its map to an open set of R^n. The open set does not necessarily include all **M** , thus there will be other open sets with the associated maps, and each point of **M** must lie in at least one of such open sets.

AND NOW WE WANT TO MAKE A COORDINATE TRANSFORMATION.

Let us consider, for example, the following situation: **U** and **V** are two overlapping open sets of **M** with two distinct maps onto R^n



The overlapping region is open (since it is the intesection of two open sets), and is given two different coordinate systems by the two maps, thus there must exist some equation relating the two. We want to find it.



Pick a point in the image of the overlapping region belonging to $f(\mathbf{U})$, say the point $(x^1, ...x^n)$. The map f has an inverse f^{-1} which brings to the point P. Now from P, by using the map g, we go to the image of P belonging to $g(\mathbf{V})$, i.e. to the point $(y^1, ...y^n)$ in R^n

$$g \circ f^{-1} : \mathbf{R}^n \to \mathbf{R}^n. \tag{2.10}$$

The result of this operation is a functional relation between the two sets of coordinates:

$$\begin{cases} y^1 = y^1(x^1, ...x^n) \\ . \\ . \\ y^n = y^n(x^1, ...x^n), \end{cases} \tag{2.11}$$

If the partial derivatives of order $\leq k$ of all the functions $\{y^i\}$ with respect to all $\{x^i\}$ exist and are continuous, then the charts $(\mathbf{U}, f)$ and $(\mathbf{V}, g)$ are said to be C^k related. If it is possible to construct a system of charts such that each point of **M** belongs at least to one of the open sets, and every chart is C^k related to every other one it overlaps with, then the manifold is said to be a C^k manifold. If k=1, it is called a **differentiable manifold**.

The notion of differentiable manifold is crucial, because it allows to add "structure" to the manifold, i.e. one can define vectors, tensors, differential forms, Lie derivatives etc.

In order to complete our definition of a coordinate transformation we still need another element. Eqs. (2.11) can be written as

$$y^i = f^i(x^1, ...x^n), \qquad i = 1, ...n, \tag{2.12}$$

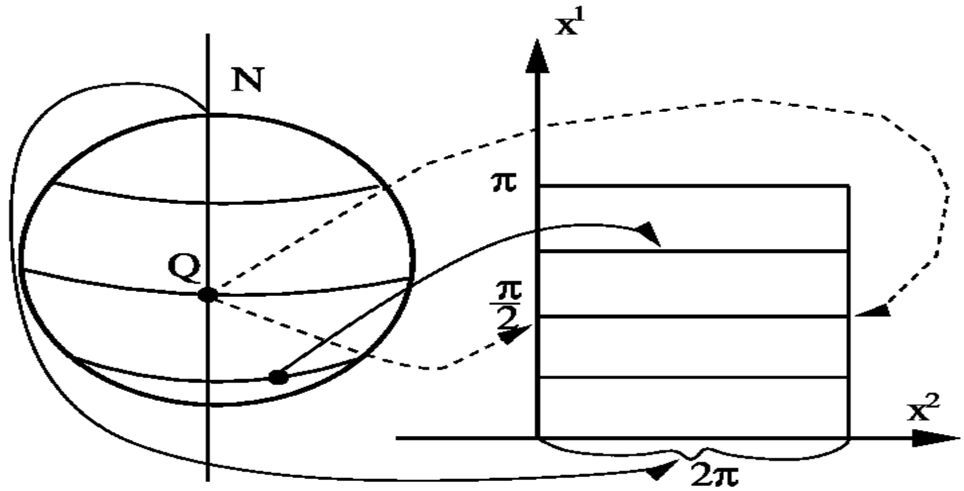
where f^i are C^k differentiable. Be **J** the jacobian of the transformation

$$\mathbf{J} = \frac{\partial(f^1, ...f^n)}{\partial(x^1, ...x^n)} = det \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & . & . & . & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & . & . & . & \frac{\partial f^2}{\partial x^n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial f^n}{\partial x^1} & \frac{\partial f^n}{\partial x^2} & . & . & . & \frac{\partial f^n}{\partial x^n} \end{pmatrix} \tag{2.13}$$

If **J** is non zero at some point P, then the inverse function theorem ensures that the map f is 1-1 and **onto** in some neighborhood of P. If **J** is zero at some point P the transformation is singular.

AN EXAMPLE OF MANIFOLD.

Consider the 2-sphere (also called $\mathbf{S}^2$). It is defined as the set of all points in $\mathbf{R}^3$ such that $(x^1)^2 + (x^2)^2 + (x^3)^2 = const.$ Suppose that we want to map the *whole* sphere to $\mathbf{R}^2$ by using a single chart. For example let us use spherical coordinates $\theta \equiv x^1$, and $\varphi \equiv x^2$. The sphere appears to be mapped onto the rectangle $0 \leq x^1 \leq \pi$, $0 \leq x^2 \leq 2\pi$



(note that this manifold has no boundary). But now consider the north pole $\theta = 0$: this point is mapped to the entire line

$$x^1 = 0, \quad 0 \le x^2 \le 2\pi. \tag{2.14}$$

Thus there is no map at all.

In addition all points of the emicircle $\varphi = 0$ are mapped in two places

$$x^2 = 0, \qquad \text{and} \qquad x^2 = 2\pi. \tag{2.15}$$

Again there is no map at all. In order to avoid these problems, we must restrict the map to open regions

$$0 < x^1 < \pi, \quad 0 < x^2 < 2\pi. \tag{2.16}$$

The two poles and the semicircle $\varphi = 0$ are left out. Then we may consider a second map, again in spherical coordinates but "rotated" in such a way that the line $\varphi = 0$ would coincide with the equator of the old system. Then every point of the sphere would be covered by one of the two charts, and in principle one should be able to find the coordinate transformation for the overlapping region. It is interesting to note that

1) this mapping does not preserve angles and lenghts.

2) there exist manifolds that cannot be covered by a single chart, i.e. by a single coordinate system.

Chapter 3

Vectors and One-forms

3.1 The traditional definition of a vector

Let us consider an N-dimensional manifold, and a generic coordinate transformation

$$x^{\alpha'} = x^{\alpha'}(x^{\mu}), \qquad \alpha', \mu = 1, \dots, N\,. \tag{3.1}$$

A comment on notation

Here and in the following, we shall use indices with and without primes to refer to different coordinate frames.

Strictly speaking, eq. (3.1) should be written as

$$x'^{\alpha'} = x'^{\alpha'}(x^{\mu}), \qquad \alpha', \mu = 1, \dots, N\,, \tag{3.2}$$

because the coordinate with (say) $\alpha' = 1$ belongs to the new frame, and is then different from the coordinate with $\mu = 1$, belonging to the old frame. However, for brevity of notation, we will omit the primes in the coordinates, keeping only the primes in the indices.

A *contravariant* vector

$$\vec{V} \to_0 \{V^{\mu}\}, \quad \mu = 1, 2, \dots N, \tag{3.3}$$

where the symbol $\to_0$ indicates that $\vec{V}$ has components $\{V^{\mu}\}$ with respect to a given frame O, is a collection of N numbers which transform under the coordinate transformation (3.1) as follows:

$$V^{\mu'} = \sum_{\alpha=1,\dots,N} \frac{\partial x^{\mu'}}{\partial x^{\alpha}} V^{\alpha} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} V^{\alpha}. \tag{3.4}$$

Notice that in writing the last term we have used Einstein's convenction. $V^{\mu'}$ are the components of the vector in the new frame. If we now define the $N \times N$ matrix

$$(\Lambda^{\alpha'}{}_{\beta}) = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^{1}} & \frac{\partial x^{1'}}{\partial x^{2}} & \dots \\ . & . & \dots \\ . & . & \dots \\ . & . & \dots \\ \frac{\partial x^{N'}}{\partial x^{1}} & \frac{\partial x^{N'}}{\partial x^{2}} & \dots \end{pmatrix}, \tag{3.5}$$

the transformation law can be written in the general form

$$V^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} V^{\beta}. \tag{3.6}$$

In addition, *covariant vectors* are defined as objects that transform according to the following rule

$$A_{\mu'} = \frac{\partial x^{\beta}}{\partial x^{\mu'}} A_{\beta} = \Lambda^{\beta}{}_{\mu'} A_{\beta}, \tag{3.7}$$

where $\Lambda^{\beta}{}_{\mu'}$ is the inverse matrix of $\Lambda^{\beta'}{}_{\mu}$. However, a vector is a geometrical object. In fact it is an oriented segment that joins two points of a given space. We can associate to this object the components with respect to an assigned reference frame; when we change frame the vector components change, but the vector itself does not change. We shall now give a more adequate definition.

3.2 A geometrical definition

In order to define a vector as a geometrical object we need to introduce the notions of *paths* and *curves*.

PATH

A path is a connected series of points in the plane (or in any arbitrary N-dimensional manifold)

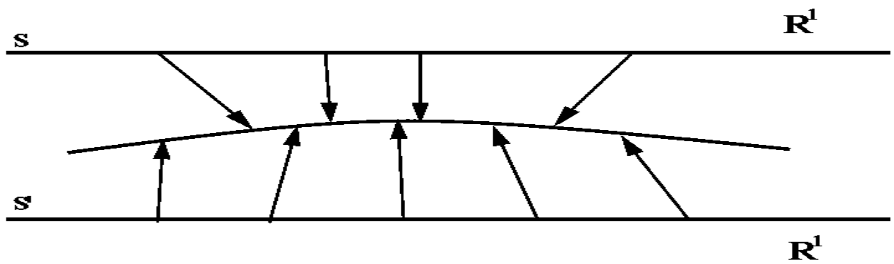
CURVE

A curve is a path with a real number associated with each point of the path, i.e. it is a mapping of an interval of $\mathbf{R}^1$ into a path in the plane (or in the N-dimensional manifold). The number is called the *parameter*. For example

$$\text{curve} \quad : \{x^1 = f(s), x^2 = g(s), a \leq s \leq b\}, \tag{3.8}$$

means that each point of the path has coordinates that can be expressed as functions of s. The path is called the image of the curve in the plane (or in the manifold). What happens if we change the parameter? If $s' = s'(s)$ we shall get a new curve

$$\{x^1 = f'(s'), x^2 = g'(s'), a' \leq s' \leq b'\}, \tag{3.9}$$



where f', g' are new functions of s'. This is a *new curve*, but with the same *image*. Thus there are an infinite number of curves corresponding to the same path.

FOR EXAMPLE: The position of a bullet shot by a gun in the 2-dimensional plane (x,z) is a PATH; when we associate the parameter t (time) at each point of the trajectory, we define a CURVE; if we change the parameter, say for instance the curvilinear abscissa, we define a new curve.

VECTORS

A vector is a geometrical object defined as the tangent vector to a given curve at a point P.

The set of numbers $\{\frac{dx^i}{ds}\} = (\frac{dx^1}{ds}, \frac{dx^2}{ds})$ are the components of a vector tangent to the curve. (In fact if $\{dx^i\}$ are infinitesimal displacements *along* the curve, dividing them by ds only changes the scale but not the direction of the displacement). Every curve has a unique tangent vector

$$\vec{V} \rightarrow \{\frac{dx^i}{ds}\}. \tag{3.10}$$

One must be careful and not to confuse the curve with the path. In fact a path has, at any given point, an infinite number of tangent vectors, all parallel, but with different lenght. The lenght depends on the parameter s that we choose to label the points of the path, and consequently it is different for different curves having the same image. **A curve has a unique tangent vector, since the path and the parameter are given.**

It should be reminded that a vector is tangent to an infinite number of different curves , for two different reasons. The first is that there are curves that are tangent to one another in P, and therefore have the same tangent vector:

The second is that a path can be reparametrized in such a way that its tangent vector remains the same.

We shall now derive how does a vector transform if we change the coordinate system, and put for example $x^{1'} = x^{1'}(x^1, x^2), x^{2'} = x^{2'}(x^1, x^2)$. The parameter s is unaffected, thus

$$\begin{cases} \frac{dx^{1'}}{ds} = \frac{\partial x^{1'}}{\partial x^1}\frac{dx^1}{ds} + \frac{\partial x^{1'}}{\partial x^2}\frac{dx^2}{ds} \\ \frac{dx^{2'}}{ds} = \frac{\partial x^{2'}}{\partial x^1}\frac{dx^1}{ds} + \frac{\partial x^{2'}}{\partial x^2}\frac{dx^2}{ds} \end{cases} \qquad \begin{pmatrix} \frac{dx^{1'}}{ds} \\ \frac{dx^{2'}}{dx^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \end{pmatrix} \cdot \begin{pmatrix} \frac{dx^1}{ds} \\ \frac{dx^2}{ds} \end{pmatrix}$$

As expected, this is the same transformation as (3.6) that was used to define a contravariant vector

$$V^{\mu'} = \Lambda^{\mu'}{}_{\beta} V^{\beta}. \tag{3.11}$$

3.3 The directional derivative along a curve form a vector space at P

In order to understand the meaning of the statement contained in the heading of this section, let us consider a curve, parametrized with an assigned parameter λ, and a differentiable function $\Phi(x^1, ...x^N)$, in a general N-dimensional manifold. The directional derivative of Φ along the curve will be

$$\frac{d\Phi}{d\lambda} = \frac{\partial \Phi}{\partial x^1}\frac{dx^1}{d\lambda} + ... + \frac{\partial \Phi}{\partial x^N}\frac{dx^N}{d\lambda} = \frac{\partial \Phi}{\partial x^i}\frac{dx^i}{d\lambda}, \qquad i = 1, ...N. \tag{3.12}$$

Since the function Φ is totally arbitrary, we can rewrite this expression as

$$\frac{d}{d\lambda} = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}, \tag{3.13}$$

where $\frac{d}{d\lambda}$ is now the operator of directional derivative, while $\{\frac{dx^i}{d\lambda}\}$ are the components of the tangent vector.

Let us consider two curves $x^i = x^i(\lambda)$ and $x^i = x^i(\mu)$ passing through the same point P, and write the two directional derivatives along the two curves

$$\frac{d}{d\lambda} = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}, \qquad \frac{d}{d\mu} = \frac{dx^i}{d\mu}\frac{\partial}{\partial x^i}. \tag{3.14}$$

$\{\frac{dx^i}{d\mu}\}$ are the components of the vector tangent to the second curve. Let us also consider a real number a.

- We define the *sum* of the two directional derivatives as the directional derivative

$$\frac{d}{d\lambda} + \frac{d}{d\mu} \equiv \left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i}. \tag{3.15}$$

  The numbers $\left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)$ are the components of a new vector, which is certainly tangent to some curve through P. Thus there must exist a curve with a parameter, say, s, such that at P

$$\frac{d}{ds} = \left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i} = \frac{d}{d\lambda} + \frac{d}{d\mu}. \tag{3.16}$$

- We define the *product* of the directional derivative $\partial/\partial\lambda$ with the real number a as the directional derivative
$$a\frac{d}{d\lambda} \equiv \left(a\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i}. \tag{3.17}$$
The numbers $\left(a\frac{dx^i}{d\lambda}\right)$ are the components of a new vector, which is certainly tangent to some curve through P. Thus there must exist a curve with a parameter, say, s', such that at P
$$\frac{d}{ds'} = \left(a\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i} = a\frac{d}{d\lambda}\,. \tag{3.18}$$

In this way we have defined two operations on the space of the directional derivatives along the curves passing through a point P: the sum of two directional derivatives, and the multiplication of a directional derivative with a real number.

We remind the mathematical definition of a vector space[1].

A vector space is a set V on which two operations are defined:

1. *Vector addition*
$$(\vec{v},\vec{w}) \to \vec{v}+\vec{w} \tag{3.19}$$
2. *Multiplication by a real number:*
$$(a,\vec{v}) \to a\vec{v} \tag{3.20}$$

(where $\vec{v},\vec{w} \in V$, $a \in I\!R$), which satisfy the following properties:

- *Associativity and commutativity of vector addition*
$$\begin{aligned}\vec{v}+(\vec{w}+\vec{u}) &= (\vec{v}+\vec{w})+\vec{u}\\ \vec{v}+\vec{w} &= \vec{w}+\vec{v}\,. \qquad \forall\,\vec{v},\vec{w},\vec{u} \in V\,.\end{aligned} \tag{3.21}$$
- *Existence of a zero vector, i.e. of an element $\vec{0} \in V$ such that*
$$\vec{v}+\vec{0} = \vec{v} \qquad \forall\,\vec{v} \in V\,.$$
- *Existence of the opposite element: for every $\vec{w} \in V$ there exists an element $\vec{v} \in V$ such that*
$$\vec{v}+\vec{w} = \vec{0}\,.$$
- *Associativity and distributivity of multiplication by real numbers:*
$$\begin{aligned}a(b\vec{v}) &= (ab)\vec{v}\\ a(\vec{v}+\vec{w}) &= a\vec{v}+a\vec{w}\\ (a+b)\vec{v} &= a\vec{v}+b\vec{v} \qquad \forall \vec{v} \in V\,, \forall\,a,b \in I\!R\,.\end{aligned} \tag{3.22}$$

[1] To be precise, what we are defining here is a *real* vector space, but we will omit this specification, because in this book only real vector spaces will be considered.

- *Finally, the real number* 1 *must act as an identity on vectors:*

$$1\,\vec{v} = \vec{v} \qquad \forall\,\vec{v}\,. \tag{3.23}$$

Coming back to directional derivatives (taken at a given point P of the manifold), it is easy to verify that the operations of addition and multiplication by a real number defined in (3.15),(3.17) respectively, satisfy the above properties. For instance:

- Commutativity of the addition:

$$\frac{d}{d\lambda} + \frac{d}{d\mu} = \left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i} = \left(\frac{dx^i}{d\mu} + \frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i} = \frac{d}{d\mu} + \frac{d}{d\lambda}\,. \tag{3.24}$$

- Associativity of multiplication by real numbers:

$$\begin{aligned} a\left(b\frac{d}{d\lambda}\right) &= a\left(\left(b\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i}\right) \\ &= \left(a\left(b\frac{dx^i}{d\lambda}\right)\right)\frac{\partial}{\partial x^i} = \left(ab\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i} \\ &= ab\frac{d}{d\lambda}\,. \end{aligned} \tag{3.25}$$

- Distributivity of multiplication by real numbers:

$$\begin{aligned} a\left(\frac{d}{d\lambda} + \frac{d}{d\mu}\right) &= a\left(\left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i}\right) \\ &= \left(a\left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)\right)\frac{\partial}{\partial x^i} = \left(a\frac{dx^i}{d\lambda} + a\frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i} \\ &= \left(a\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i} + \left(a\frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i} = a\frac{d}{d\lambda} + a\frac{d}{d\mu}\,. \end{aligned} \tag{3.26}$$

- The zero element is the vector tangent to the curve $x^\mu \equiv const.$, which is simply the point P.

- The opposite of the vector $\vec{v}$ tangent to a given curve is obtained by changing sign to the parametrization

$$\lambda \to -\lambda\,. \tag{3.27}$$

The proof of the remaining properties is analogous.
Therefore, the set of directional derivatives is a vector space.

In any coordinate system there are special curves, the coordinates lines (think for example to the grid of cartesian coordinates). The directional derivatives along these lines are

$$\frac{d}{dx^i} = \frac{dx^k}{dx^i}\frac{\partial}{\partial x^k} = \delta^k_i\frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^i}\,,$$

Eq. (3.13) shows that the generic directional derivative $\frac{d}{d\lambda}$ can always be expressed as a linear combination of $\frac{\partial}{\partial x^i}$. It follows that $\frac{d}{dx^i} \equiv \frac{\partial}{\partial x^i}$ are a basis for this vector space, and $\{\frac{dx^i}{d\lambda}\}$ are the components of $\frac{d}{d\lambda}$ on this basis. But $\{\frac{dx^i}{d\lambda}\}$ are also the components of a tangent vector at P. Therefore *the space of all tangent vectors and the space of all derivatives along curves at P are in 1-1 correspondence.* For this reason we can say that $\frac{d}{d\lambda}$ is the vector tangent to the curve $x^i(\lambda)$.
TO SUMMARIZE: the vectors tangent to the coordinate lines in a point P, i.e. the directional derivatives in P along these lines in a coordinate system $(x^1, ...x^N)$, have the following components

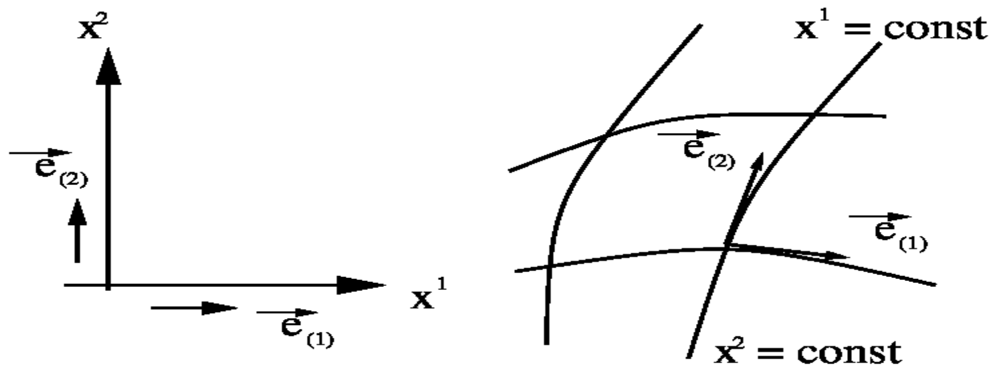
$$\frac{\vec{\partial}}{\partial x^1} = (1, 0, ...0), \qquad \frac{\partial}{\partial x^2} = (0, 1, ...0), \qquad ...., \frac{\partial}{\partial x^N} = (0, 0, ...1).$$

If we use the $\left\{\frac{\vec{\partial}}{\partial x^i}\right\}$ as a basis for vectors, the vector $\frac{\vec{d}}{d\lambda}$, tangent to the curve $x^i(\lambda)$, with respect to this basis has components $\{\frac{dx^i}{d\lambda}\}$.

Vectors do not lie in **M**, *but in the tangent space to* **M**, *called* $\mathbf{T_P}$For example in the two-dimensional case analysed above the tangent plane was the plane itself, but if the manifold is a sphere, since we cannot define a vector as an "arrow" on the sphere, we need to define the tangent space, i.e. the plane tangent to the sphere at each point. For more general manifolds it is not easy to visualize $\mathbf{T_P}$. In any event $\mathbf{T_P}$*has the same dimensions as the manifold* **M**.

3.4 Coordinate bases

Any collection of n linearly independent vectors of $\mathbf{T_P}$is a basis for $\mathbf{T_P}$. However, a natural basis is provided by the vectors that are tangent to the coordinate lines, i.e. $\left\{\vec{e}_{(i)}\right\} \equiv \left\{\frac{\vec{\partial}}{\partial x^{(i)}}\right\}$; this is *the coordinate basis.*
IMPORTANT:
To hereafter, we shall enclose within () the indices that indicate which vector of the basis we are choosing, not to be confused with the index which indicates the vector components. For instance $e^1_{(2)}$ indicates the component 1 of the basis vector $\vec{e}_{(2)}$.



Any vector $\vec{A}$at a point P, can be expressed as a linear combination of the basis vectors

$$\vec{A} = A^i \vec{e}_{(i)}, \tag{3.28}$$

(Remember Einstein's convention: $\sum_i A^i \vec{e}_{(i)} \equiv A^i \vec{e}_{(i)}$) where the numbers $\{A^i\}$ are the components of $\vec{A}$with respect to the chosen basis.

If we make a coordinate transformation to a new set of coordinates $(x^{1'}, x^{2'}, ...x^{n'})$, there will be a new coordinate basis: $\{\vec{e}_{(i')}\} \equiv \left\{\frac{\vec{\partial}}{\partial x^{(i')}}\right\}$.
We now want to find the relation between the new and the old basis, i.e we want to express each new vector $\vec{e}_{(j')}$ as a linear combination of the old ones $\{\vec{e}_{(j)}\}$. In the new basis, the vector $\vec{A}$ will be written as

$$\vec{A} = A^{j'} \vec{e}_{(j')}, \tag{3.29}$$

where $\{A^{j'}\}$ are the components of $\vec{A}$with respect to the basis $\left\{\vec{e}_{(j')}\right\}$. But the vector $\vec{A}$is the same in any basis, therefore

$$A^i \vec{e}_{(i)} = A^{i'} \vec{e}_{(i')}. \tag{3.30}$$

From eq. (3.11) we know how to express $A^{i'}$ as functions of the components in the old basis, and substituting these expressions into eq. (3.30) we find

$$A^i \vec{e}_{(i)} = \Lambda^{i'}{}_k A^k \vec{e}_{(i')}. \tag{3.31}$$

By relabelling the dummy indices this equation can be written as

$$\left[\Lambda^{i'}{}_k \vec{e}_{(i')} - \vec{e}_{(k)}\right] A^k = 0, \tag{3.32}$$

i.e.

$$\vec{e}_{(k)} = \Lambda^{i'}{}_k \vec{e}_{(i')}. \tag{3.33}$$

Multiplying both members by $\Lambda^k{}_{j'}$ and remembering that

$$\Lambda^k{}_{j'}\Lambda^{i'}{}_k = \frac{\partial x^k}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^k} = \frac{\partial x^{i'}}{\partial x^{j'}} = \delta^{i'}_{j'} \tag{3.34}$$

we find the transformation we were looking for

$$\vec{e}_{(j')} = \Lambda^k{}_{j'}\vec{e}_{(k)}. \tag{3.35}$$

Summarizing:

$$\begin{cases} \vec{e}_{(k)} = \Lambda^{i'}{}_k\vec{e}_{(i')}, \\ \vec{e}_{(i')} = \Lambda^k{}_{i'}\vec{e}_{(k)}. \end{cases} \tag{3.36}$$

We are now in a position to compute the new basis vectors in terms of the old ones.

EXAMPLE

Consider the 4-dimensional flat spacetime of Special Relativity, but let us restrict to the (x-y) plane, where we choose the coordinates $(ct, x, y) \equiv (x^0, x^1, x^2)$. The coordinate basis is the set of vectors

$$\begin{aligned} \frac{\vec{\partial}}{\partial x^{(0)}} &= \vec{e}_{(0)} \to \quad (1,0,0) \\ \frac{\vec{\partial}}{\partial x^{(1)}} &= \vec{e}_{(1)} \to \quad (0,1,0) \\ \frac{\vec{\partial}}{\partial x^{(2)}} &= \vec{e}_{(2)} \to \quad (0,0,1), \end{aligned} \tag{3.37}$$

or, in a compact form

$$e^{\beta}_{(\alpha)} = \delta^{\beta}_{\alpha}. \tag{3.38}$$

(The superscript β now indicates the β-component of the α-th vector). In this basis any vector $\vec{A}$ can be written as

$$\vec{A} = A^0\vec{e}_{(0)} + A^1\vec{e}_{(1)} + A^2\vec{e}_{(2)} = A^{\alpha}\vec{e}_{(\alpha)}, \qquad \alpha = 0,..2 \tag{3.39}$$

where $\{A^{\alpha}\} = (A^0, A^1, A^2)$ are the components of $\vec{A}$ with respect to this basis. Let us consider the following coordinate transformation

$$\begin{cases} (x^0, x, y) \to (x^0, r, \theta) \\ x^0 = x^{0'} \\ x^1 = r\cos\theta \\ x^2 = r\sin\theta, \end{cases} \tag{3.40}$$

i.e. $x^{1'} = r, x^{2'} = \theta$. The new coordinate basis is

$$\frac{\vec{\partial}}{\partial x^{(0')}} = \vec{e}_{(0')}, \qquad \frac{\vec{\partial}}{\partial r} \equiv \frac{\vec{\partial}}{\partial x^{(1')}} = \vec{e}_{(1')}, \qquad \frac{\vec{\partial}}{\partial \theta} \equiv \frac{\vec{\partial}}{\partial x^{(2')}} = \vec{e}_{(2')}. \tag{3.41}$$

From eq. (3.35) we find

$$\vec{e}_{(0')} = \Lambda^{\alpha}{}_{0'}\vec{e}_{(\alpha)}, \qquad \Lambda^{\alpha}{}_{0'} = \frac{\partial x^{\alpha}}{\partial x^{0'}}. \tag{3.42}$$

In the example we are considering only $\Lambda^{0}{}_{0'} \neq 0$ and it is equal to 1. It follows that

$$\vec{e}_{(0)} \equiv \frac{\vec{\partial}}{\partial x^{(0')}} = \vec{e}_{(0')}. \tag{3.43}$$

In addition

$$\vec{e}_{(1')} = \Lambda^{\alpha}{}_{1'}\vec{e}_{(\alpha)}, \tag{3.44}$$

and since

$$\Lambda^{0}{}_{1'} = \frac{\partial x^0}{\partial r} = 0, \qquad \Lambda^{1}{}_{1'} = \frac{\partial x^1}{\partial r} = \cos\theta, \qquad \Lambda^{2}{}_{1'} = \frac{\partial x^2}{\partial r} = \sin\theta, \tag{3.45}$$

$$\vec{e}_{(1')} \equiv \frac{\vec{\partial}}{\partial r} = \cos\theta\vec{e}_{(1)} + \sin\theta\vec{e}_{(2)}. \tag{3.46}$$

Similarly

$$\vec{e}_{(2')} = \Lambda^{\alpha}{}_{2'}\vec{e}_{(\alpha)}, \tag{3.47}$$

and since

$$\Lambda^{0}{}_{2'} = 0, \qquad \Lambda^{1}{}_{2'} = \frac{\partial x^1}{\partial \theta} = -r\sin\theta, \qquad \Lambda^{2}{}_{2'} = \frac{\partial x^2}{\partial \theta} = r\cos\theta, \tag{3.48}$$

hence

$$\vec{e}_{(2')} \equiv \frac{\vec{\partial}}{\partial \theta} = -r\sin\theta\vec{e}_{(1)} + r\cos\theta\vec{e}_{(2)}. \tag{3.49}$$

Summarizing,

$$\begin{cases} \vec{e}_{(0')} = \vec{e}_{(0)} \\ \vec{e}_{(1')} = \vec{e}_{(r)} = \cos\theta\vec{e}_{(1)} + \sin\theta\vec{e}_{(2)} \\ \vec{e}_{(2')} = \vec{e}_{(\theta)} = -r\sin\theta\vec{e}_{(1)} + r\cos\theta\vec{e}_{(2)} \ . \end{cases} \tag{3.50}$$

It should be noted that we do not need to choose necessarily a coordinate basis. We may choose a set of independent basis vectors that are not tangent to the coordinate lines. In this case the matrix which allows to transform from one basis to another has to be assigned and will not be $\Lambda^{\alpha}{}_{\beta'}$ as in eq. (3.35).

3.5 One-forms

A one-form is a linear, real valued function of vectors. This means the following: a one-form (or 1-form) $\tilde{q}$ at the point P takes the vector $\vec{V}$at P and associates a number to it, which we call $\tilde{q}(\vec{V})$. To hereafter a " $\tilde{}$ " will indicate 1-forms, as an arrow "→" indicates vectors.

By definition, a one-form is linear. This means that, for every couple of vectors $\vec{V}, \vec{W}$, for every couple of real numbers a, b, for every one-form $\tilde{q}$,

$$\tilde{q}(a\vec{V} + b\vec{W}) = a\tilde{q}(\vec{V}) + b\tilde{q}(\vec{W}). \tag{3.51}$$

We define two operations acting on the space of one-forms:

- Multiplication by real numbers: given a one-form $\tilde{q}$ and a real number a, we define the new one-form $a\tilde{q}$ such that, for every vector $\vec{V}$,

$$(a\tilde{q})(\vec{V}) = a[\tilde{q}(\vec{V})]\,. \tag{3.52}$$

- Addition: given two one-forms $\tilde{q}, \tilde{\sigma}$, we define the new one-form $\tilde{q} + \tilde{\sigma}$ such that, for every vector $\vec{V}$,

$$[\tilde{q} + \tilde{\sigma}](\vec{V}) = \tilde{q}(\vec{V}) + \tilde{\sigma}(\vec{V}). \tag{3.53}$$

One-forms satisfy the axioms (3.21-3.23). Let us show this for some of the axioms.

- Commutativity of addition. Given two one-forms $\tilde{q}, \tilde{\sigma}$, we have that, for every vector field $\vec{V}$,

$$(\tilde{q} + \tilde{\sigma})\left(\vec{V}\right) = \tilde{q}\left(\vec{V}\right) + \tilde{\sigma}\left(\vec{V}\right) = \tilde{\sigma}\left(\vec{V}\right) + \tilde{q}\left(\vec{V}\right) = (\tilde{\sigma} + \tilde{q})\left(\vec{V}\right)\,. \tag{3.54}$$

- Distributivity of multiplication with real numbers. Given two one-forms $\tilde{q}, \tilde{\sigma}$ and a real number a, we have that, for every vector field $\vec{V}$,

$$\begin{aligned} [a\left(\tilde{q} + \tilde{\sigma}\right)]\left(\vec{V}\right) &= a\left[\left(\tilde{q} + \tilde{\sigma}\right)\right]\left(\vec{V}\right) = a\left[\tilde{q}\left(\vec{V}\right) + \tilde{\sigma}\left(\vec{V}\right)\right] \\ &= a\left[\tilde{q}\left(\vec{V}\right)\right] + a\left[\tilde{\sigma}\left(\vec{V}\right)\right] = (a\tilde{q})\left(\vec{V}\right) + (a\tilde{\sigma})\left(\vec{V}\right) \\ &= \left[(a\tilde{q}) + (a\tilde{\sigma})\right]\left(\vec{V}\right) \end{aligned} \tag{3.55}$$

  then, being this true for every $\vec{V}$,

$$a\left(\tilde{q} + \tilde{\sigma}\right) = (a\tilde{q}) + (a\tilde{\sigma})\,. \tag{3.56}$$

- Existence of the zero element. The zero one-form $\tilde{0}$ is the one-form such that, for every $\vec{V}$,

$$\tilde{0}(\vec{V}) = 0\,. \tag{3.57}$$

The other axioms can be proved in a similar way.
Therefore, one-forms form a vector space, which is called the **dual** vector space to $\mathbf{T_P}$, and it is indicated as $\mathbf{T^*_P}$; this is also called the cotangent space in P.

$\mathbf{T^*_P}$ is the space of the maps (the 1-forms) that associate to any given vector a number, i.e. that map $\mathbf{T_P}$ on $\mathbf{R}^1$. The reason why $\mathbf{T^*_P}$ is called dual to $\mathbf{T_P}$ is that vectors also can be regarded as linear, real valued functions of one-forms: a vector $\vec{V}$ takes a 1-form $\tilde{q}$ and associates a number to it, which we call $\vec{V}(\tilde{q})$, and

$$\tilde{q}(\vec{V}) \equiv \vec{V}(\tilde{q}), \tag{3.58}$$

in the sense that the two "operations" give as a result the same number. This point will be further clarified in the following. Once we choose a basis for vectors, say $\{\vec{e}_{(i)}, i = 1, \dots, N\}$, we can introduce a dual basis for one-forms defined as follows:
the dual basis $\{\tilde{\omega}^{(i)}, i = 1, \dots, N\}$, takes any vector $\vec{V}$ in $\mathbf{T_P}$ and produces its components

$$\tilde{\omega}^{(i)}(\vec{V}) = V^i. \tag{3.59}$$

It should be remembered that an index in parenthesis does not refer to a component, but selects the -th one-form (or vector) of the basis. Thus the i-th basis one-form applied to $\vec{V}$gives as a result a number, which is the component V^i of the vector $\vec{V}$. As expected, this operation is linear in the argument

$$\tilde{\omega}^{(i)}(\vec{V}+\vec{W}) = V^i + W^i, \tag{3.60}$$

since $\vec{V}+\vec{W}$is a vector whose i-th component is $V^i + W^i$. In particular, if the argument of a one-form is $\vec{e}_{(j)}$, i.e. one of the basis vectors of the tangent space at the point P, since only the j-th component of $\vec{e}_{(j)}$ is different from zero and equal to 1, we have

$$\tilde{\omega}^{(i)}(\vec{e}_{(j)}) = \delta^i_j. \tag{3.61}$$

We now want to answer the questions:

1. Who tells us that $\{\tilde{\omega}^{(i)}\}$ form a basis for one-forms?
2. Can we define the components of a 1-form as we define the components of a vector?

---

1. Consider any one-form $\tilde{q}$ acting on an arbitrary vector $\vec{V}$. By expressing $\vec{V}$as a linear combination of the basis vectors $\vec{e}_{(j)}$, and using the linearity of one-forms we can write

$$\begin{aligned} \tilde{q}(\vec{V}) = &\quad \tilde{q}(V^j \vec{e}_{(j)}) = V^j \tilde{q}(\vec{e}_{(j)}) = \\ = &\quad \tilde{\omega}^{(j)}(\vec{V})\tilde{q}(\vec{e}_{(j)}), \end{aligned} \tag{3.62}$$

   where the last equality follows from eq.(3.59). This equation holds for any vector $\vec{V}$therefore we can write

$$\tilde{q} = \tilde{\omega}^{(j)}\ \tilde{q}(\vec{e}_{(j)}); \tag{3.63}$$

   since $\tilde{q}(\vec{e}_{(j)})$ are real numbers, this equation shows that any one-form $\tilde{q}$ can be written as a linear combination of the $\{\tilde{\omega}^{(j)}\}$; consequently $\{\tilde{\omega}^{(j)}\}$ form a basis for one-forms.

2. We now define *the components of* $\tilde{q}$ *on the basis* $\{\tilde{\omega}^{(i)}\}$ as

$$q_j = \tilde{q}(\vec{e}_{(j)}) \tag{3.64}$$

   and consequently we can write

$$\tilde{q} = q_j\ \tilde{\omega}^{(j)}. \tag{3.65}$$

Consider an open region **U** of the manifold **M**, and choose a coordinate system $\{x^i\}$. We have seen that this defines a *natural coordinate basis for vectors* $\vec{e}_{(i)} \equiv \{\frac{\vec{\partial}}{\partial x^{(i)}}\}$. Furthermore, it also defines a *natural coordinate basis for one-forms* (dual to the natural basis for vectors), often indicated as $\{\tilde{dx}^{(i)}\}$, whose components are

$$\tilde{\omega}^{(i)}{}_j \equiv \tilde{dx}^{(i)}{}_j = \tilde{dx}^{(i)}\left(\frac{\vec{\partial}}{\partial x^{(j)}}\right) = \delta^i_j.$$

And now the most important thing. From eq. (3.65) it follows that for any vector $\vec{V}$

$$\tilde{q}(\vec{V}) = q_j \ \tilde{\omega}^{(j)}(\vec{V}). \tag{3.66}$$

Since $\tilde{\omega}^{(j)}(\vec{V}) = V^j$, we find

$$\tilde{q}(\vec{V}) = q_j V^j. \tag{3.67}$$

This operation is called **contraction** and tells us how to compute the number which results from the application of $\tilde{q}$ on $\vec{V}$(or viceversa), once we know the components of $\tilde{q}$ and $\vec{V}$.

From eq. (3.67) we can now better understand why vectors and one-forms are dual of each other. In fact, if q_j and V^j are respectively the components of the one-form $\tilde{q}$ and of the vector $\vec{V}$

$$\tilde{q}(\vec{V}) = q_j V^j = q_1 V^1 + \ldots + q_N V^N; \tag{3.68}$$

The right-hand side of this equation can be considered as a linear combination of the components of $\vec{V}$ with coefficients q_j, or alternatively, as a linear combination of the components of $\tilde{q}$ with coefficients V^j, and this follows from the linearity of the previous expression. Therefore, we can define vectors as those linear functions that, when applied to one-forms, produce a number.

Let us now make a coordinate transformation $x^{k'} = x^{k'}(x^i)$ and let us consider the following questions.

1. How do the components of one-forms transform?
2. Will the new coordinate basis for one-forms be a linear combination of the old ones, and if so, and which combination?

---

1. By definition

$$q_j = \tilde{q}(\vec{e}_{(j)}). \tag{3.69}$$

If we change coordinates, we will have a new set of basis vectors $\{\vec{e}_{(j')}\}$, and we have seen that they are related to the old ones by

$$\vec{e}_{(i')} = \Lambda^k{}_{i'} \vec{e}_{(k)}, \tag{3.70}$$

where $\Lambda^k{}_{i'} = \frac{\partial x^k}{\partial x^{i'}}$. The new components of $\tilde{q}$ will be

$$q_{j'} = \tilde{q}(\vec{e}_{(j')}) = \tilde{q}[\Lambda^k{}_{j'} \vec{e}_{(k)}] = \Lambda^k{}_{j'} \tilde{q}(\vec{e}_{(k)}) = \Lambda^k{}_{j'} q_k, \tag{3.71}$$

hence

$$q_{j'} = \Lambda^k{}_{j'} q_k. \tag{3.72}$$

If we compare this result with eq. (3.7) we immediately recognize that this is the way covariant vectors transform, thus *covariant vectors are one-forms.*

2. We now want to check whether the new basis one-forms can be expressed as a linear combination of the old ones. We shall proceed along the same lines of section 3.4. From eq. (3.65) we see that

$$\tilde{q} = q_j \; \tilde{\omega}^{(j)} = q_{k'} \; \tilde{\omega}^{(k')}, \tag{3.73}$$

(sum removed according to Einstein's convenction), where $\{\tilde{\omega}^{(k')}\}$ are the new basis one-forms. But

$$q_{k'} = \Lambda^i{}_{k'} q_i, \tag{3.74}$$

therefore

$$q_j \tilde{\omega}^{(j)} = \Lambda^i{}_{k'} q_i \tilde{\omega}^{(k')}. \tag{3.75}$$

This equation can be rewritten as

$$[\Lambda^i{}_{k'} \tilde{\omega}^{(k')} - \tilde{\omega}^{(i)}] q_i = 0, \tag{3.76}$$

hence

$$\tilde{\omega}^{(i)} = \Lambda^i{}_{k'} \tilde{\omega}^{(k')}. \tag{3.77}$$

The matrix $\Lambda^i{}_{j'}$ is inverse of $\Lambda^{k'}{}_i$. Thus

$$\Lambda^{k'}{}_j \Lambda^j{}_{i'} = \delta^{k'}_{i'}, \qquad \text{or} \qquad \Lambda^{k'}{}_j \Lambda^i{}_{k'} = \delta^i_j. \tag{3.78}$$

Multiplying both sides of eq. (3.77) by $\Lambda^{j'}{}_i$ we find

$$\Lambda^{j'}{}_i \; \tilde{\omega}^{(i)} = \Lambda^{j'}{}_i \Lambda^i{}_{k'} \; \tilde{\omega}^{(k')} = \delta^{j'}_{k'} \; \tilde{\omega}^{(k')}, \tag{3.79}$$

hence

$$\tilde{\omega}^{(j')} = \Lambda^{j'}{}_i \; \tilde{\omega}^{(i)}, \tag{3.80}$$

Summarizing, the transformation laws for the basis one-forms are

$$\begin{cases} \tilde{\omega}^{(i)} = \Lambda^i{}_{k'} \; \tilde{\omega}^{(k')} \\ \tilde{\omega}^{(k')} = \Lambda^{k'}{}_j \; \tilde{\omega}^{(j)} \end{cases} \tag{3.81}$$

EXAMPLE

Let us consider the same coordinate transformation analyzed in section 3.4. We start with Minkowskian coordinates (x^0, x^1, x^2). The coordinate basis for vectors is $\{\frac{\vec{\partial}}{\partial x^\alpha}\}$ and the dual basis for one-forms is $\{\tilde{d}x^\alpha\}$

$$\tilde{d}x^{(0)} \to (1, 0, 0) \tag{3.82}$$
$$\tilde{d}x^{(1)} \to (0, 1, 0) \tag{3.83}$$
$$\tilde{d}x^{(2)} \to (0, 0, 1) \tag{3.84}$$

If we now change to polar coordinates $(x^{0'} = x^0, x^{1'} = r, x^{2'} = \theta)$, according to eq. (3.80) we find

$$\tilde{\omega}^{(0')} = \Lambda^{0'}{}_\alpha \tilde{d}x^{(\alpha)}. \tag{3.85}$$

Since $\Lambda^{0'}{}_{\alpha} = \frac{\partial x^{0'}}{\partial x^{\alpha}}$, only $\Lambda^{0'}{}_{0} = 1 \neq 0$, thus

$$\tilde{\omega}^{(0')} = \tilde{d}x^{(0)}. \tag{3.86}$$

Similarly

$$\tilde{\omega}^{(1')} = \Lambda^{1'}{}_{\alpha}\tilde{d}x^{(\alpha)} = \frac{\partial x^{1'}}{\partial x^{\alpha}}\tilde{d}x^{(\alpha)} = \frac{\partial x^{1'}}{\partial x^{1}}\tilde{d}x^{(1)} + \frac{\partial x^{1'}}{\partial x^{2}}\tilde{d}x^{(2)}. \tag{3.87}$$

Since

$$\frac{\partial x^{1'}}{\partial x^{1}} = \frac{x^{1}}{x^{1'}} = \cos\theta, \qquad \text{and} \qquad \frac{\partial x^{1'}}{\partial x^{2}} = \frac{x^{2}}{x^{1'}} = \sin\theta \tag{3.88}$$

it follows that

$$\tilde{\omega}^{(1')} = \cos\theta\tilde{d}x^{1} + \sin\theta\tilde{d}x^{2}. \tag{3.89}$$

Moreover

$$\tilde{\omega}^{(2')} = \frac{\partial x^{2'}}{\partial x^{1}}\tilde{d}x^{(1)} + \frac{\partial x^{2'}}{\partial x^{2}}\tilde{d}x^{(2)}, \tag{3.90}$$

hence

$$\tilde{\omega}^{(2')} = -\frac{1}{r}\sin\theta\tilde{d}x^{(1)} + \frac{1}{r}\cos\theta\tilde{d}x^{(2)}. \tag{3.91}$$

Summarizing,

$$\begin{cases} \tilde{\omega}^{(0')} = \tilde{\omega}^{(0)} \\ \tilde{\omega}^{(1')} = \cos\theta\tilde{\omega}^{(1)} + \sin\theta\tilde{\omega}^{(2)} \\ \tilde{\omega}^{(2')} = -\frac{1}{r}\sin\theta\tilde{\omega}^{(1)} + \frac{1}{r}\cos\theta\tilde{\omega}^{(2)} \ . \end{cases} \tag{3.92}$$

AN EXAMPLE OF ONE-FORM.
Consider a scalar field $\Phi(x^1, ...x^N)$. The gradient of a scalar field is

$$\tilde{\Phi} \rightarrow (\frac{\partial \Phi}{\partial x^1}, ..., \frac{\partial \Phi}{\partial x^N}). \tag{3.93}$$

It is easy to see, for example, that the components transform according to eq. (3.72), in fact

$$\tilde{\Phi}_j = \frac{\partial \Phi}{\partial x^j}, \qquad \text{and} \qquad \tilde{\Phi}_{j'} = \frac{\partial \Phi}{\partial x^{j'}} = \frac{\partial \Phi}{\partial x^k} \cdot \frac{\partial x^k}{\partial x^{j'}}; \tag{3.94}$$

since $\Lambda^{k}{}_{j'} = \frac{\partial x^k}{\partial x^{j'}}$, it follows that

$$\tilde{\Phi}_{j'} = \Lambda^{k}{}_{j'}\tilde{\Phi}_k, \tag{3.95}$$

same as eq. (3.72). Thus the gradient of a scalar field is a one-form.

3.6 Vector fields and one-form fields

The vectors and one-forms are defined *on a point* P of the manifold, and belong to the vector spaces $\mathbf{T_P}$and $\mathbf{T^*_P}$, respectively, which also refer to a specific point P of the manifold; to make this explicit, we could also denote a vector in P as $\vec{V}_{\mathbf{P}}$, a one-form in P as $\tilde{W}_{\mathbf{P}}$. We shall now define *vector fields* and *one-form fields*.

Given an open set $\mathbf{S}$ of a differentiable manifold $\mathbf{M}$, we define the vector spaces

$$\mathbf{T_S} \equiv \bigcup_{P \in \mathbf{S}} \mathbf{T_P}$$
$$\mathbf{T^*_S} \equiv \bigcup_{P \in \mathbf{S}} \mathbf{T^*_P}\,,$$

i.e., the union of the tangent spaces on the points $P \in\mathbf{S}$, and the union of the cotangent spaces on the points $P \in\mathbf{S}$.

A vector field $\vec{V}$ is a mapping

$$\begin{aligned} \vec{V} \,:\, \mathbf{S} &\rightarrow \mathbf{T_S} \\ P &\rightarrow \vec{V}_\mathbf{P} \end{aligned}$$

which associates, to every point $P \in\mathbf{S}$, a vector $\vec{V}_\mathbf{P}$ defined on the tangent space in P, $\mathbf{T_P}$. A one-form field $\tilde{W}$ is a mapping

$$\begin{aligned} \tilde{W} \,:\, \mathbf{S} &\rightarrow \mathbf{T^*_S} \\ P &\rightarrow \tilde{W}_\mathbf{P} \end{aligned}$$

which associates, to every point $P \in\mathbf{S}$, a one-form $\vec{W}_\mathbf{P}$ defined on the cotangent space in P, $\mathbf{T^*_P}$. If a coordinate system (a chart) $\{x^\mu\}$ is defined on $\mathbf{S}$, we can indicate the vector field and the one-form field as $\vec{V}(x)$, $\tilde{W}(x)$.

In the following, we will mainly consider vector fields and one-form fields; however, for brevity of notation, we will often refer to them simply as vectors and one-forms.

Chapter 4

Tensors

4.1 Geometrical definition of a Tensor

The definition of a tensor is a generalization of the definition of one-forms.
Consider a point P of an n-dimensional manifold M. A tensor of type $\begin{pmatrix} N \\ N' \end{pmatrix}$ at P is defined to be *a linear, real valued function, which takes as arguments* N *one-forms and* N' *vectors and associates a number to them.*
For example if F is a $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ tensor this means that

$$F(\tilde{\omega}, \tilde{\sigma}, \vec{V}, \vec{W})$$

is a number and the linearity implies that

$$F(a\tilde{\omega} + b\tilde{g}, \tilde{\sigma}, \vec{V}, \vec{W}) = aF(\tilde{\omega}, \tilde{\sigma}, \vec{V}, \vec{W}) + bF(\tilde{g}, \tilde{\sigma}, \vec{V}, \vec{W})$$

and

$$F(\tilde{\omega}, \tilde{g}, a\vec{V}_1 + b\vec{V}_2, \vec{W}) = aF(\tilde{\omega}, \tilde{g}, \vec{V}_1, \vec{W}) + bF(\tilde{\omega}, \tilde{g}, \vec{V}_2, \vec{W})$$

and similarly for the other arguments.
This definition of tensors is rather abstract, but we shall see how to make it concrete with specific examples.
The order in which the arguments are placed is important, as it is true for any function of real variables. For example if

$$f(x, y) = 4x^3 + 5y \qquad \text{, then} \qquad f(1, 5) \neq f(5, 1). \tag{4.1}$$

In the same way

$$F(\tilde{\omega}, \tilde{g}, \vec{V}, \vec{W}) \neq F(\tilde{g}, \tilde{\omega}, \vec{V}, \vec{W}). \tag{4.2}$$

EXAMPLES

A $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor is a function that takes a vector as argument, and produces a number.
This is precisely what one-forms do (on the other hand this *is* the definition of one-forms).

Thus, a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor is a one-form.

$$\tilde{q}(\vec{V}) = \sum_\alpha q_\alpha V^\alpha \equiv q_\alpha V^\alpha. \tag{4.3}$$

A $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ tensor is a function that takes a one-form as an argument, and produces a number.
Thus a $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ tensor is a vector

$$\vec{V}(\tilde{q}) = q_\alpha V^\alpha. \tag{4.4}$$

Let us now consider a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor. It is a function that takes 2 vectors and associates a number to them.
Let us first define the tensor components: generalizing the definition (3.64) for the components of a one-form, they are the numbers that are obtained when the $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor is applied to the basis vectors:

$$F_{\alpha\beta} = F(\vec{e}_{(\alpha)}, \vec{e}_{(\beta)}); \tag{4.5}$$

since there are n basis vectors, $F_{\alpha\beta}$ will be an $n \times n$ matrix.
If we now take as arguments of F two arbitrary vectors $\vec{A}$ and $\vec{B}$ we find

$$\begin{aligned} F(\vec{A}, \vec{B}) &= F(A^\alpha \vec{e}_{(\alpha)}, B^\beta \vec{e}_{(\beta)}) = \\ &= A^\alpha B^\beta F(\vec{e}_{(\alpha)}, \vec{e}_{(\beta)}) = \\ &= F_{\alpha\beta} A^\alpha B^\beta. \end{aligned} \tag{4.6}$$

It should be stressed that in going from the first to the second line of eq. (4.6) we have used the property that tensors are linear functions of the arguments.

It is now clear what is the number that F associates to the two vectors: the number is $F_{\alpha\beta} A^\alpha B^\beta$.
We shall now construct a basis for $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensors as we did for one-forms.
We want to write

$$F = F_{\alpha\beta} \omega^{(\alpha)(\beta)} \tag{4.7}$$

where $\omega^{(\alpha)(\beta)}$ are the basis $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensors.
If the arguments of F are two arbitrary vectors $\vec{A}$ and $\vec{B}$, eq. (4.7) gives

$$F(\vec{A}, \vec{B}) = F_{\alpha\beta} \omega^{(\alpha)(\beta)}(\vec{A}, \vec{B}). \tag{4.8}$$

On the other hand, since $A^\alpha = \tilde{\omega}^{(\alpha)}(\vec{A})$ and $B^\beta = \tilde{\omega}^{(\beta)}(\vec{B})$, eq. (4.6) gives

$$F(\vec{A}, \vec{B}) = F_{\alpha\beta} \tilde{\omega}^{(\alpha)}(\vec{A}) \tilde{\omega}^\beta(\vec{B}), \tag{4.9}$$

and, by equating eqs. (4.8) and (4.9) we find

$$\omega^{(\alpha)(\beta)}(\vec{A}, \vec{B}) = \tilde{\omega}^{(\alpha)}(\vec{A}) \tilde{\omega}^{(\beta)}(\vec{B}).$$

The previous equation holds for **any** two vectors $\vec{A}$ and $\vec{B}$, consequently we write

$$\omega^{(\alpha)(\beta)} = \tilde{\omega}^{(\alpha)} \otimes \tilde{\omega}^{(\beta)}, \tag{4.10}$$

where the symbol $\otimes$ indicates the "outer product" of the two basis one-forms, and means precisely that if $\omega^{(\alpha)(\beta)}$ is applied to the vectors $\vec{A}$ and $\vec{B}$, the result is a number, which coincides with the number produced by the application of $\tilde{\omega}^{(\alpha)}$ to $\vec{A}$, times that produced by the application of $\tilde{\omega}^{(\beta)}$ to $\vec{B}$ (the order is important!).

Thus the basis for $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensors can be constructed by taking the outer product of the basis one-forms. Finally, we can write

$$F = F_{\alpha\beta}\tilde{\omega}^{(\alpha)} \otimes \tilde{\omega}^{(\beta)}. \tag{4.11}$$

It is now clear that we can construct any sort of tensors using the procedure that we have developed in the previous pages. Thus for example a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor T is a function that associates to two one-forms $\tilde{\alpha}$ and $\tilde{\sigma}$ a number, $T(\tilde{\alpha}, \tilde{\sigma})$.

The components of this tensor are found by applying T to the basis one-forms

$$T^{\mu\nu} = T(\tilde{\omega}^{(\mu)}, \tilde{\omega}^{(\nu)}), \tag{4.12}$$

and the number produced when T is applied to any two one-forms $\tilde{\alpha}$, $\tilde{\sigma}$ will be

$$T(\tilde{\alpha}, \tilde{\sigma}) = T(\alpha_\mu \tilde{\omega}^{(\mu)}, \sigma_\nu \tilde{\omega}^{(\nu)}) = \alpha_\mu \sigma_\nu T(\tilde{\omega}^{(\mu)}, \tilde{\omega}^{(\nu)}) = \alpha_\mu \sigma_\nu T^{\mu\nu}, \tag{4.13}$$

where again use has been made of the linearity of tensors with respect to their arguments. By following the same procedure used to find the basis for a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor, it is easy to show that the basis appropriate for a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor will be

$$e_{(\alpha)(\beta)} = \vec{e}_\alpha \otimes \vec{e}_\beta, \tag{4.14}$$

and consequently

$$T = T^{\alpha\beta}\vec{e}_\alpha \otimes \vec{e}_\beta. \tag{4.15}$$

Exercise: prove that the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor $\vec{V} \otimes \tilde{\sigma}$ has components $V^\mu \sigma_\nu$ and find the basis for $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensors.

Now we ask the following question: how do the components of a tensor transform if we make a coordinate transformation?
We start with a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor

$$F = F_{\alpha\beta}\tilde{\omega}^{(\alpha)} \otimes \tilde{\omega}^{(\beta)} \tag{4.16}$$

If we change coordinates, we shall have a new set of basis one forms $\{\tilde{\omega}^{(\mu')}\}$ which are related to the old ones by the equations

$$\tilde{\omega}^{(\alpha)} = \Lambda^{\alpha}{}_{\mu'}\tilde{\omega}^{(\mu')} \quad , \quad \tilde{\omega}^{(\mu')} = \Lambda^{\mu'}{}_{\alpha}\tilde{\omega}^{(\alpha)} \tag{4.17}$$

In the new basis the tensor $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ will be

$$F = F_{\alpha'\beta'}\tilde{\omega}^{(\alpha')} \otimes \tilde{\omega}^{(\beta')}. \tag{4.18}$$

By equating (4.16) and (4.18)

$$F_{\alpha'\beta'}\tilde{\omega}^{(\alpha')} \otimes \tilde{\omega}^{(\beta')} = F_{\alpha\beta}\tilde{\omega}^{(\alpha)} \otimes \tilde{\omega}^{(\beta)}.$$

Replacing $\tilde{\omega}^{(\alpha)}$ and $\tilde{\omega}^{(\beta)}$ by using the first of eqs. 4.17

$$F_{\alpha'\beta'}\tilde{\omega}^{(\alpha')} \otimes \tilde{\omega}^{(\beta')} = F_{\alpha\beta}\Lambda^{\alpha}{}_{\mu'}\tilde{\omega}^{(\mu')} \otimes \Lambda^{\beta}{}_{\nu'}\tilde{\omega}^{(\nu')} = F_{\alpha\beta}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}\tilde{\omega}^{(\mu')} \otimes \tilde{\omega}^{(\nu')},$$

or by relabelling the dummy indices

$$F_{\mu'\nu'}\tilde{\omega}^{(\mu')} \otimes \tilde{\omega}^{(\nu')} = F_{\alpha\beta}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}\tilde{\omega}^{(\mu')} \otimes \tilde{\omega}^{(\nu')},$$

and finally

$$F_{\mu'\nu'} = F_{\alpha\beta}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}, \tag{4.19}$$

or, by writing explicitely the elements of the matrix $\Lambda^{\alpha}{}_{\mu'}$

$$F_{\mu'\nu'} = F_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x^{\mu'}}\frac{\partial x^{\beta}}{\partial x^{\nu'}}, \tag{4.20}$$

where $\{x^{\mu'}\}$ are the new coordinates.
In a similar way, by using eqs. 3.33 and 3.35 we would find that

$$T^{\mu'\nu'} = T^{\alpha\beta}\Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}, \tag{4.21}$$

and

$$T^{\mu'}{}_{\nu'} = T^{\alpha}{}_{\beta}\Lambda^{\mu'}{}_{\alpha}\Lambda^{\beta}{}_{\nu'} \tag{4.22}$$

IMPORTANT

The following point should be stressed: the notion of tensor we have introduced is independent of which coordinates, i.e. which basis, we use.
In fact the number that an $\begin{pmatrix} N \\ N' \end{pmatrix}$ tensor associates to N one-forms and N' vectors does not depend on the particular basis we choose.
This is the reason why, for example, we can equate eqs. (4.16) and (4.18).

The operations that we are allowed to make with tensors are the following.

- *Multiplication by a real number*

  Given a tensor T of type $\begin{pmatrix} N \\ N' \end{pmatrix}$ and a real number a, we define the tensor, of the same type,
  $$W = aT\,.$$
  Let the components of T, in a given frame, be $\{T^{\alpha\dots}{}_{\beta\dots}\}$. The components of W are
  $$W^{\alpha\dots}{}_{\beta\dots} = aT^{\alpha\dots}{}_{\beta\dots}\,.$$

- *Addition of tensors*

  Given two tensors T, G of the same type $\begin{pmatrix} N \\ N' \end{pmatrix}$, we define the tensor, of the same type,
  $$W = T + G\,.$$
  Let the components of T, G, in a given frame, be $\{T^{\alpha\dots}{}_{\beta\dots}\}$, $\{G^{\alpha\dots}{}_{\beta\dots}\}$. The components of W in that frame are
  $$W^{\alpha\dots}{}_{\beta\dots} = T^{\alpha\dots}{}_{\beta\dots} + G^{\alpha\dots}{}_{\beta\dots}\,.$$

- *Outer product*

  Given two tensors T, G of types $\begin{pmatrix} N_1 \\ N_1' \end{pmatrix}$, $\begin{pmatrix} N_2 \\ N_2' \end{pmatrix}$, respectively. We define the tensor, of type $\begin{pmatrix} N_1 + N_2 \\ N_1' + N_2' \end{pmatrix}$,
  $$W = T \otimes G\,.$$
  Let the components of T, G, in a given frame, be $\{T^{\alpha\dots}{}_{\beta\dots}\}$, $\{G^{\gamma\dots}{}_{\delta\dots}\}$. The components of W in that frame are
  $$W^{\alpha\dots\gamma\dots}{}_{\beta\dots\delta\dots} = T^{\alpha\dots}{}_{\beta\dots} G^{\gamma\dots}{}_{\delta\dots}\,.$$
  For instance, if both T, G are of type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$,
  $$W_{\alpha\beta\gamma\delta} = T_{\alpha\beta} G_{\gamma\delta}\,.$$

- *Contraction*

  Given a tensor T of type $\begin{pmatrix} N \\ N' \end{pmatrix}$, with components $\{T^{\alpha_1\alpha_2\dots\alpha_N}{}_{\beta_1\beta_2\dots\beta_{N'}}\}$ in a given frame. We define a new tensor W of type $\begin{pmatrix} N-1 \\ N'-1 \end{pmatrix}$, the components of which are obtained by contraction of one contravariant (i.e. upper) and one covariant (i.e. lower) index of T, i.e.
  $$W^{\dots\alpha_{i-1}\ \alpha_{i+1}\dots}{}_{\dots\beta_{j-1}\ \beta_{j+1}\dots} = T^{\dots\alpha_{i-1}\ \sigma\ \alpha_{i+1}\dots}{}_{\dots\beta_{j-1}\ \sigma\ \beta_{j+1}\dots}\,.$$
  For instance, if T is of type $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and we choose to contract the first contravariant index with the second covariant index
  $$W^{\beta}{}_{\sigma\delta} = T^{\alpha\beta}{}_{\sigma\alpha\delta} = T^{0\beta}{}_{\sigma 0\delta} + T^{1\beta}{}_{\sigma 1\delta} + T^{2\beta}{}_{\sigma 2\delta} + \dots$$

and W is a $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

These are called *tensor operations* and an equation involving tensor components and tensor operations is a *tensor equation.*

Finally, we remark that since a tensor T has been defined as an application from vectors and one-forms, it is defined on the product of a certain number of copies of the tangent and the cotangent spaces *on a point* P, $\mathbf{T_p}$, $\mathbf{T^*_p}$. Then, we can define *tensor fields*, i.e., a tensor for each point P of an open subset of the manifold; in a given coordinate system $\{x^\mu\}$, we can write a tensor field as $T(x)$. For brevity of notation, in the following we will often refer to a tensor field simply as a tensor.

4.2 Symmetries

A $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor F is **Symmetric** if

$$F(\vec{A},\vec{B}) = F(\vec{B},\vec{A}) \qquad \forall \vec{A},\vec{B}. \tag{4.23}$$

As a consequence of eq. (4.6) we see that if the tensor is symmetric

$$F_{\alpha\beta}A^\alpha B^\beta = F_{\alpha\beta}B^\alpha A^\beta, \tag{4.24}$$

and, by relabelling the indices on the RHS

$$F_{\alpha\beta}A^\alpha B^\beta = F_{\beta\alpha}B^\beta A^\alpha, \tag{4.25}$$

i.e.

$$F_{\alpha\beta} = F_{\beta\alpha} \tag{4.26}$$

i.e. if a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor is symmetric the matrix representing its components is symmetric. Given any $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor F we can always construct from it a symmetric tensor $F_{(s)}$

$$F^{(s)}(\vec{A},\vec{B}) = \frac{1}{2}[F(\vec{A},\vec{B}) + F(\vec{B},\vec{A})]. \tag{4.27}$$

In fact $\forall \vec{A},\vec{B}$

$$\frac{1}{2}[F(\vec{A},\vec{B}) + F(\vec{B},\vec{A})] = \frac{1}{2}[F(\vec{B},\vec{A}) + F(\vec{A},\vec{B})].$$

Moreover

$$\begin{aligned} F^{(s)}(\vec{A},\vec{B}) &= F^{(s)}_{\alpha\beta}A^\alpha B^\beta = \frac{1}{2}[F_{\alpha\beta}A^\alpha B^\beta + F_{\alpha\beta}B^\alpha A^\beta] = \frac{1}{2}[F_{\alpha\beta}A^\alpha B^\beta + F_{\beta\alpha}B^\beta A^\alpha] \\ &= \frac{1}{2}[F_{\alpha\beta} + F_{\beta\alpha}]A^\alpha B^\beta, \end{aligned}$$

and consequently the components of the symmetric tensor are

$$F^{(s)}_{\alpha\beta} = \frac{1}{2}[F_{\alpha\beta} + F_{\beta\alpha}]. \tag{4.28}$$

The components of a symmetric tensor are often indicated as

$$F_{(\alpha\beta)} = \frac{1}{2}[F_{\alpha\beta} + F_{\beta\alpha}]. \tag{4.29}$$

A $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor F is **antisymmetric** if

$$F(\vec{A}, \vec{B}) = -F(\vec{B}, \vec{A}) \qquad \forall \vec{A}, \vec{B}, \qquad \text{i.e.} \qquad F_{\alpha\beta} = -F_{\beta\alpha}. \tag{4.30}$$

Again from any $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor we can construct an antisymmetric tensor $F^{(a)}$ defined as

$$F^{(a)}(\vec{A}, \vec{B}) = \frac{1}{2}[F(\vec{A}, \vec{B}) - F(\vec{B}, \vec{A})].$$

Proceeding as before, we find that its components are

$$F^{(a)}_{\alpha\beta} = \frac{1}{2}[F_{\alpha\beta} - F_{\beta\alpha}],$$

also indicated as

$$F_{[\alpha\beta]} = \frac{1}{2}[F_{\alpha\beta} - F_{\beta\alpha}]. \tag{4.31}$$

It is clear that any tensor $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ can be written as the sum of its symmetric and antisymmetric part

$$h[\vec{A}, \vec{B}] = \frac{1}{2}[h(\vec{A}, \vec{B}) + h(\vec{B}, \vec{A})] + \frac{1}{2}[h(\vec{A}, \vec{B}) - h(\vec{B}, \vec{A})]$$

4.3 The metric Tensor

In chapter 1 we have seen that the metric tensor has a central role in the relativistic theory of gravity. In this section we shall discuss its geometrical meaning.

Definition: *the metric tensor* **g** *is a* $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ *tensor that, having two arbitrary vectors* $\vec{A}$ *and* $\vec{B}$ *as arguments, associates to them a real number that is the* **inner product** *(or scalar product)* $\vec{A} \cdot \vec{B}$

$$g(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B}. \tag{4.32}$$

The scalar product is usually defined to be a linear function of two vectors that satisfies the following properties

$$\begin{aligned} &\vec{U} \cdot \vec{V} = \vec{V} \cdot \vec{U} \\ &(a\vec{U}) \cdot \vec{V} = a(\vec{U} \cdot \vec{V}) \\ &(\vec{U} + \vec{V}) \cdot \vec{W} = \vec{U} \cdot \vec{W} + \vec{V} \cdot \vec{W} \end{aligned} \tag{4.33}$$

From the first eq. (4.33) it follows that **g** is a symmetric tensor. In fact

$$\vec{U}\cdot\vec{V} = g(\vec{U},\vec{V}) = \vec{V}\cdot\vec{U} = g(\vec{V},\vec{U}), \quad \to \quad g(\vec{U},\vec{V}) = g(\vec{V},\vec{U}). \tag{4.34}$$

The second and third eqs. (4.33) imply that **g** is a linear functions of the arguments, a condition which is automatically satisfied since **g** is a tensor.

As usual the components of the metric tensor are obtained by replacing $\vec{A}$ and $\vec{B}$ with the basis vectors

$$g_{\alpha\beta} = g(\vec{e}_{(\alpha)},\vec{e}_{(\beta)}) = \vec{e}_{(\alpha)}\cdot\vec{e}_{(\beta)}\,. \tag{4.35}$$

Thus the metric tensor allows to compute the scalar product of two vectors in any space and whatever coordinates we use:

$$\vec{A}\cdot\vec{B} = g(\vec{A},\vec{B}) = g(A^\alpha\vec{e}_{(\alpha)}, B^\beta\vec{e}_{(\beta)}) = A^\alpha B^\beta g(\vec{e}_{(\alpha)},\vec{e}_{(\beta)}) = A^\alpha B^\beta g_{\alpha\beta}. \tag{4.36}$$

EXAMPLES

1)
The metric of four dimensional Minkowski spacetime, in Minkowskian coordinates $x^\alpha = (ct,x,y,z)$ is

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \equiv \eta_{\alpha\beta}$$

i.e.

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -c^2dt^2 + dx^2 + dy^2 + dz^2\,. \tag{4.37}$$

This implies that the basis vectors in the coordinate basis

$$\begin{aligned} \vec{e}_{(0)} &= \vec{e}_{(ct)} \to (1,0,0,0) \\ \vec{e}_{(1)} &= \vec{e}_{(x)} \to (0,1,0,0) \\ \vec{e}_{(2)} &= \vec{e}_{(y)} \to (0,0,1,0) \\ \vec{e}_{(3)} &= \vec{e}_{(z)} \to (0,0,0,1) \end{aligned}$$

are, in this case, mutually orthogonal:

$$\vec{e}_{(\alpha)}\cdot\vec{e}_{(\beta)} = g_{\alpha\beta} = 0 \qquad \text{if} \qquad \alpha \neq \beta\,.$$

In addition, since

$$g_{11} = g_{22} = g_{33} = 1, \qquad \text{and} \qquad g_{00} = -1\,,$$

the basis vectors are unit vectors, $\vec{e}_{(0)}$ is a timelike vector, and $\vec{e}_{(i)}$ $(i = 1,2,3)$ are spacelike vectors:

$$\vec{e}_{(k)}\cdot\vec{e}_{(k)} = 1 \qquad \text{if} \qquad k = 1,\ldots,3\,,$$

$$\vec{e}_{(0)} \cdot \vec{e}_{(0)} = -1\,.$$

From now on we shall indicate as $\eta_{\alpha\beta}$ the components of the metric tensor of the Minkowski spacetime when expressed in cartesian coordinates.

2)
Let us now consider the metric of Minkowski spacetime in three dimensions, i.e. we suppress the coordinate z:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \equiv \eta_{\alpha\beta} \tag{4.38}$$

with $\alpha, \beta = 0, \ldots, 2$. The vectors of the coordinate basis have components

$$\begin{aligned} \vec{e}_{(0)} &\rightarrow (1,0,0) \\ \vec{e}_{(1)} &\rightarrow (0,1,0) \\ \vec{e}_{(2)} &\rightarrow (0,0,1)\,. \end{aligned}$$

We now change to polar coordinates

$$x^0 = x^{0'}, \qquad x^1 = r\cos\theta, \qquad x^2 = r\sin\theta. \tag{4.39}$$

The vectors of the coordinate basis in the new coordinate system have been computed in Sec. 3.4, and are

$$\vec{e}_{(0')} = \vec{e}_{(0)} \tag{4.40}$$
$$\vec{e}_{(1')} = \vec{e}_{(r)} = \cos\theta\vec{e}_{(1)} + \sin\theta\vec{e}_{(2)} \tag{4.41}$$
$$\vec{e}_{(2')} = \vec{e}_{(\theta)} = -r\sin\theta\vec{e}_{(1)} + r\cos\theta\vec{e}_{(2)}.$$

We can determine the metric tensor in the new frame by computing the scalar product of the vectors of this frame:

$$\begin{aligned} g_{0'0'} &= \vec{e}_{(0')} \cdot \vec{e}_{(0')} = \vec{e}_{(0)} \cdot \vec{e}_{(0)} = -1 \\ g_{0'i'} &= 0 \quad i' = 1,2 \\ g_{1'1'} &= \vec{e}_{(1')} \cdot \vec{e}_{(1')} = (\cos\theta\vec{e}_{(1)} + \sin\theta\vec{e}_{(2)}) \cdot (\cos\theta\vec{e}_{(1)} + \sin\theta\vec{e}_{(2)}) = \cos^2\theta + \sin^2\theta = 1 \\ g_{2'2'} &= \vec{e}_{(2')} \cdot \vec{e}_{(2')} = r^2\sin^2\theta + r^2\cos^2\theta = r^2 \\ g_{1'2'} &= -r\cos\theta\sin\theta + r\cos\theta\sin\theta = 0 \end{aligned}$$

i.e.

$$g_{\alpha'\beta'} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & r^2 \end{pmatrix} \tag{4.42}$$

i.e.

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = g_{\alpha'\beta'}dx^{\alpha'}dx^{\beta'} = -dt^2 + dr^2 + r^2 d\theta^2\,. \tag{4.43}$$

We note that although the metric tensor is the same, its components in the two coordinate frames, (4.38) and (4.42), are different, since $g_{2'2'} = \vec{e}_{(2')} \cdot \vec{e}_{(2')} = r^2 \neq 1$. Thus, $\vec{e}_{(2')}$ is not an unit vector. In general the basis vectors are not required to have unitary norm, even in a coordinate frame.

Usually, to determine the components of the metric tensor in a new frame, one does not use the procedure above, based on the computation of the scalar products. One rather employs the transformation law

$$g_{\mu'\nu'} = \Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'} g_{\alpha\beta}$$

which, in this case, has the form

$$g_{\mu'\nu'} = \Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'} \eta_{\alpha\beta}\,.$$

Since $\eta_{\alpha\beta}$ is diagonal, we only need to consider the components with $\alpha = \beta$.

$$g_{0'0'} = \Lambda^{\alpha}{}_{0'}\Lambda^{\beta}{}_{0'}\eta_{\alpha\beta} = \left(\frac{\partial x^0}{\partial x^{0'}}\right)^2 \eta_{00} = 1\cdot(-1) = -1$$

$$g_{0'i'} = \Lambda^{\alpha}{}_{0'}\Lambda^{\beta}{}_{i'}\eta_{\alpha\beta} = \frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^0}{\partial x^{i'}}\eta_{00} + \frac{\partial x^1}{\partial x^{0'}}\frac{\partial x^1}{\partial x^{i'}}\eta_{11} + \frac{\partial x^2}{\partial x^{0'}}\frac{\partial x^2}{\partial x^{i'}}\eta_{22} = 0 \qquad i' = 1,2$$

because $\frac{\partial x^0}{\partial x^{i'}} = \frac{\partial x^1}{\partial x^{0'}} = \frac{\partial x^2}{\partial x^{0'}} = 0$.

$$\begin{aligned} g_{1'1'} &= \Lambda^{\alpha}{}_{1'}\Lambda^{\beta}{}_{1'}\eta_{\alpha\beta} = (\Lambda^{0}{}_{1'})^2\eta_{00} + (\Lambda^{1}{}_{1'})^2\eta_{11} + (\Lambda^{2}{}_{1'})^2\eta_{22} = \\ &= \left(\frac{\partial x^0}{\partial x^{1'}}\right)^2\cdot(-1) + \left(\frac{\partial x^1}{\partial x^{1'}}\right)^2\cdot 1 + \left(\frac{\partial x^2}{\partial x^{1'}}\right)^2\cdot 1 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 \end{aligned}$$

$$g_{1'1'} = \cos^2\theta + \sin^2\theta = 1$$

Proceeding in this way we find the metric in the frame $(x^{0'}, x^{1'}, x^{2'}) = (ct, r, \theta)$, i.e. (4.42).

4.3.1 The metric tensor allows to compute the distance between two points

Let us consider, for example, a three-dimensional space.

$$(x^0, x^1, x^2) \equiv (ct, x, y)$$

The distance between two points infinitesimally close, $P(x^0, x^1, x^2)$ and $P'(x^0 + dx^0, x^1 + dx^1, x^2 + dx^2)$, is

$$\vec{ds} = dx^0\vec{e}_{(0)} + dx^1\vec{e}_{(1)} + dx^2\vec{e}_{(2)} = dx^{\alpha}\vec{e}_{(\alpha)} \qquad (4.44)$$

where $\vec{e}_{(\alpha)}$ are the basis vectors. ds^2 is the norm of the vector $\vec{ds}$, i.e. the square of the distance between P and P':

$$\begin{aligned} ds^2 &= \vec{ds}\cdot\vec{ds} = (dx^0\vec{e}_{(0)} + dx^1\vec{e}_{(1)} + dx^2\vec{e}_{(2)})\cdot(dx^0\vec{e}_{(0)} + dx^1\vec{e}_{(1)} + dx^2\vec{e}_{(2)}) \\ &= (dx^0)^2(\vec{e}_{(0)}\cdot\vec{e}_{(0)}) + dx^1dx^0(\vec{e}_{(1)}\cdot\vec{e}_{(0)}) + dx^2dx^0(\vec{e}_{(2)}\cdot\vec{e}_{(0)}) + \\ &+ dx^0dx^1(\vec{e}_{(0)}\cdot\vec{e}_{(1)}) + (dx^1)^2(\vec{e}_{(1)}\cdot\vec{e}_{(1)}) + dx^2dx^1(\vec{e}_{(2)}\cdot\vec{e}_{(1)}) + \\ &+ dx^0dx^2(\vec{e}_{(0)}\cdot\vec{e}_{(2)}) + dx^2dx^1(\vec{e}_{(2)}\cdot\vec{e}_{(1)}) + (dx^2)^2(\vec{e}_{(2)}\cdot\vec{e}_{(2)}) \end{aligned}$$

By definition of the metric tensor

$$(\vec{e}_{(i)} \cdot \vec{e}_{(j)}) = g(\vec{e}_{(i)}, \vec{e}_{(j)}) = g_{ij},$$

therefore

$$ds^2 = g(\vec{ds}, \vec{ds}) = (dx^0)^2 g_{00} + 2dx^0 dx^1 g_{01} + 2dx^0 dx^2 g_{02} + 2dx^1 dx^2 g_{12} + (dx^1)^2 g_{11} + (dx^2)^2 g_{22} \tag{4.45}$$

where we have used the fact that $g_{\alpha\beta} = g_{\beta\alpha}$.
This calculation is simplified if we use the following notation

$$\begin{aligned} ds^2 = g(\vec{ds}, \vec{ds}) &= g(\sum_{\alpha=0}^{2} dx^\alpha \vec{e}_{(\alpha)}, \sum_{\beta=0}^{2} dx^\beta \vec{e}_{(\beta)}) \equiv g(dx^\alpha \vec{e}_{(\alpha)}, dx^\beta \vec{e}_{(\beta)}) = \\ &= dx^\alpha dx^\beta g(\vec{e}_{(\alpha)}, \vec{e}_{(\beta)}) = g_{\alpha\beta} dx^\alpha dx^\beta \end{aligned} \tag{4.46}$$

with $\alpha, \beta = 0, \ldots, 2$.
This way of writing is completely equivalent to eq. (4.45). For example, if the space is Minkowski spacetime $g_{\alpha\beta} = \eta_{\alpha\beta} = diag(-1, 1, 1)$, and eq. (4.46) gives

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2, \tag{4.47}$$

as expected.
If we now change to a coordinate system $(x^{0'}, x^{1'}, x^{2'})$, the distance $\vec{PP'}$ will be $ds^{'2} = ds^2$, i.e.

$$\begin{aligned} g(\vec{ds'}, \vec{ds'}) &= \vec{ds'} \cdot \vec{ds'} = ds'^2 = ds^2 = \\ &= g(dx^{\alpha'} \vec{e}_{(\alpha')}, dx^{\beta'} \vec{e}_{(\beta')}) = dx^{\alpha'} dx^{\beta'} g(\vec{e}_{(\alpha')}, \vec{e}_{(\beta')}), \end{aligned}$$

where $\{\vec{e}_{(\alpha')}\}$ are the new basis vectors. Therefore

$$ds^2 = g_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'} \tag{4.48}$$

where now $g_{\alpha'\beta'}$ are the components of the metric tensor in the new basis. For example, if we change from carthesian to polar coordinates $(x^{0'}, x^{1'}, x^{2'}) \equiv (ct, r, \theta)$,

$$ds^2 = (dx^{0'})^2 g_{0'0'} + (dx^{1'})^2 g_{1'1'} + (dx^{2'})^2 g_{2'2'} = -(dx^0)^2 + dr^2 + r^2 d\theta^2. \tag{4.49}$$

Thus if we know the components of the metric tensor in any reference frame, we can compute the distance between two points infinitesimally close, ds^2.

The "infinitesimal" interpretation of ds^2 we have discussed above is useful to understand the role of the metric in measuring distances. In order to compute *finite* distances, we need to proceed as follows. Let us consider a curve, i.e. a path $\mathcal{C}$ and a map

$$\begin{aligned} [a, b] \subset I\!R &\rightarrow \mathcal{C} \\ \lambda &\rightarrow P(\lambda) \end{aligned} \tag{4.50}$$

which, in a given coordinate system $\{x^\mu\}$, corresponds to the real functions

$$\lambda \rightarrow \{x^\mu(\lambda)\}\,. \tag{4.51}$$

We can define the lenght of the path $\mathcal{C}$ as

$$\Delta s = \int_a^b d\lambda \frac{ds}{d\lambda} = \int_a^b d\lambda \sqrt{\left| g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right|}. \tag{4.52}$$

This definition corresponds, in infinitesimal form, to $ds = \sqrt{|ds^2|} = \sqrt{|g_{\mu\nu} dx^\mu dx^\nu|}$.

In other words, if we have a curve, characterized, in a given coordinate system, by the functions $\{x^\mu(\lambda)\}$, and then by the tangent vector

$$U^\mu = \frac{dx^\mu}{d\lambda} \,,$$

the measure element on the curve $ds/d\lambda$ (which, integrated in $d\lambda$, gives the lenght of the path) is

$$\frac{ds}{d\lambda} = \sqrt{\left| g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right|} = \sqrt{|g_{\mu\nu} U^\mu U^\nu|} \,. \tag{4.53}$$

This can be expressed in a coordinate-independent way:

$$\frac{ds}{d\lambda} = \sqrt{\left| g(\vec{U}, \vec{U}) \right|} \,. \tag{4.54}$$

Note that if we change coordinate system, $\{x^\mu\} \to \{x^{\alpha'}\}$, the quantity (4.54) does not change. Furthermore, if we change the parametrization of the curve,

$$\lambda \to \lambda' = \lambda'(\lambda) \,,$$

the new measure element is

$$\frac{ds}{d\lambda'} = \sqrt{\left| g_{\mu\nu} \frac{dx^\mu}{d\lambda'} \frac{dx^\nu}{d\lambda'} \right|} = \sqrt{\left| g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{d\lambda}{d\lambda'} \frac{dx^\nu}{d\lambda} \frac{d\lambda}{d\lambda'} \right|} = \frac{ds}{d\lambda} \frac{d\lambda}{d\lambda'} \tag{4.55}$$

and

$$\Delta s = \int_a^b d\lambda \frac{ds}{d\lambda} = \int_a^b \left(d\lambda' \frac{d\lambda}{d\lambda'} \right) \frac{ds}{d\lambda} = \int_{a'}^{b'} d\lambda' \frac{ds}{d\lambda'} \,. \tag{4.56}$$

Therefore, Δs does not depend on the parametrization, and is a charateristic of the path, given the metric, not of the curve.

4.3.2 The metric tensor maps vectors into one-forms

As we have seen, the metric tensor is a linear function of two vectors: this means that it takes two vectors and associates a number to them. The number is their scalar product.
But now suppose that we write $g(\ ,\vec{V})$, namely we leave the first slot empty. What is this? We know that if we fill the first slot with a generic vector $\vec{A}$ we will get a number, thus $g(\ ,\vec{V})$ must be a linear function of a generic vector that we can put in the empty slot, and that associates a number to this vector.
But this is the definition of one-forms! Thus $g(\ ,\vec{V})$ is a one-form.

In addition, it is a particular one-form because it depends on $\vec{V}$: if we change $\vec{V}$, the one-form will be different. Let us indicate this one-form as

$$g(\ \ ,\vec{V}) = \tilde{V}\,. \tag{4.57}$$

By definition the components of $\tilde{V}$ are

$$V_\alpha = \tilde{V}(\vec{e}_{(\alpha)}) = g(\vec{e}_{(\alpha)}, \vec{V}) = g(\vec{e}_{(\alpha)}, V^\beta \vec{e}_{(\beta)}) = V^\beta g(\vec{e}_{(\alpha)}, \vec{e}_{(\beta)}) = V^\beta g_{\alpha\beta},$$

hence

$$V_\alpha = g_{\alpha\beta} V^\beta. \tag{4.58}$$

Thus the tensor g associates to any vector $\vec{V}$ a one-form $\tilde{V}$, dual of $\vec{V}$, whose components can be computed if we know $g_{\alpha\beta}$ and V^α.

In addition, if we multiply eq. (4.58) by $g^{\alpha\gamma}$, where $g^{\alpha\gamma}$ is the matrix inverse to $g_{\alpha\gamma}$

$$g_{\alpha\gamma} g^{\gamma\beta} = \delta^\beta_\alpha, \tag{4.59}$$

we find

$$g^{\alpha\gamma} V_\alpha = g^{\alpha\gamma} g_{\alpha\beta} V^\beta = \delta^\gamma_\beta V^\beta = V^\gamma,$$

i.e.

$$V^\gamma = g^{\alpha\gamma} V_\alpha, \tag{4.60}$$

Consequently the metric tensor also maps one-forms into vectors . In a similar way the metric tensor can map a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor in a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor

$$A^\alpha{}_\beta = g_{\beta\gamma} A^{\alpha\gamma},$$

or in a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor

$$A_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} A^{\mu\nu},$$

or viceversa

$$A^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} A_{\mu\nu}.$$

These maps are called index raising and lowering.

Summarizing, the metric tensor

1) allows to compute the inner product of two vectors $g(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B}$, and consequently the norm of a vector $g(\vec{A}, \vec{A}) = \vec{A} \cdot \vec{A} = A^2$.

2)As a consequence it allows to compute the distance between two points $ds^2 = g(\vec{ds}, \vec{ds}) = g_{\alpha\beta} dx^\alpha dx^\beta$.

3) It maps one-forms into vectors and viceversa.

4) It allows to raise and lower indices.

Chapter 5

Affine Connections and Parallel Transport

In chapter 1 we showed that there are two quantities that describe the effects of a gravitational field on moving bodies by virtue of the Equivalence Principle: the metric tensor and the affine connections. In chapter 4 we discussed the geometrical properties of the metric tensor. In this chapter we shall define the affine connections as the quantities that allow to compute the derivative of a vector in an arbitrary space, and we shall show that they coincide with the Γ 's introduced in chapter 1.

5.1 The covariant derivative of vectors

Let us consider a vector (field) $\vec{V} = V^\mu \vec{e}_{(\mu)}$. The derivative of $\vec{V}$ is

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e}_{(\alpha)} + V^\alpha \frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta}. \tag{5.1}$$

The first term on the right-hand side is a linear combination of the basis vectors, therefore it is a vector and we know how to compute it. The second term involves the derivative of the basis vectors, for which we need to compute the quantities $\vec{e}_{(\alpha)}(\mathbf{p}') - \vec{e}_{(\alpha)}(\mathbf{p})$, i.e. to subtract vectors which are applied in different points of the manifold M. Note that the vectors $\vec{e}_{(\alpha)}(\mathbf{p})$ and $\vec{e}_{(\alpha)}(\mathbf{p}')$ belong to the tangent space to M, respectively, in $\mathbf{p}$ and $\mathbf{p}'$, and that $\mathbf{T_p} \neq \mathbf{T_{p'}}$. Thus, to define the derivative of a vector field on a manifold, we need to specify a rule to compare vectors belonging to different tangent spaces; such a rule is called a **connection**.

Let us start considering Minkowski's spacetime, where it is possible to define a *global* coordinate system (ct, x, y, z) which covers the entire spacetime; at any given point $\mathbf{p}$ of the manifold there exists the coordinate basis $\vec{e}_{M(\alpha)}(\mathbf{p})$ which belongs to the tangent space $\mathbf{T_p}$. In this case a simple rule to compare vectors on different tangent spaces is to impose that each basis vector in a point $\mathbf{p}$ of the manifold is equal to the corresponding basis vector in any other point $\mathbf{p}'$, i.e.

$$\vec{e}_{M(\alpha)}(\mathbf{p}) = \vec{e}_{M(\alpha)}(\mathbf{p}') \, . \tag{5.2}$$

This rule is the affine connection in Minkowski's spacetime. Note that, with this choice the basis vectors of the Minkowskian frame are, by definition, constant:

$$\frac{\partial \vec{e}_{M(\alpha)}}{\partial x^\beta} = \vec{0}\,. \tag{5.3}$$

Let us now consider a general spacetime. The equivalence principle tells us that at any point of the manifold we can choose a locally inertial frame, in which the laws of physics are (locally) those of Special Relativity. Thus, the natural choice for the affine connection in a general spacetime is the following: we impose that in a locally inertial frame the basis vectors are constant. We shall now show that, using this rule, we will be able to compute the derivative of a vector 5.1 at a given point **p** of the manifold.

Let us make a coordinate transformation to the local inertial frame in **p**, introducing the new basis vectors $\vec{e}_{M(\alpha')}$, related to the old basis vectors $\vec{e}_{(\alpha)}$ by the equation

$$\vec{e}_{(\alpha)} = \Lambda^{\alpha'}{}_{\alpha}\vec{e}_{M(\alpha')}. \tag{5.4}$$

From (5.3) we know that the vectors $\vec{e}_{M(\alpha')}$ are constanta. Consequently

$$\frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} = \left(\frac{\partial}{\partial x^\beta}\Lambda^{\alpha'}{}_{\alpha}\right)\vec{e}_{M(\alpha')}. \tag{5.5}$$

The R.H.S. of (5.5) is a linear combination of the basis vectors $\{\vec{e}_{M(\alpha')}\}$, therefore it is a vector.

Since $\frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta}$ is a vector, we must be able to express it as a linear combination of the basis vectors $\{\vec{e}_{(\mu)}\}$ we are working with, i.e.:

$$\frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta}\vec{e}_{(\mu)}, \tag{5.6}$$

where the constants $\Gamma^\mu_{\alpha\beta}$ have three indices because α indicates which basis vector $\vec{e}_{(\alpha)}$ we are differentiating, and β indicates the coordinate with respect to which the differentiation is performed. The $\Gamma^\mu_{\beta\alpha}$ are called **affine connection or Christoffel symbols**. Note that in the case of Minkowski space, the basis vectors in the Minkowskian frame are constant, thus $\Gamma^\mu_{\alpha\beta} = 0$.

Thus, coming back to eq. (5.1), the derivative of $\vec{V}$ becomes

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_{(\alpha)} + V^\alpha\Gamma^\mu_{\beta\alpha}\vec{e}_{(\mu)},$$

or relabelling the dummy indices

$$\frac{\partial \vec{V}}{\partial x^\beta} = \left[\frac{\partial V^\alpha}{\partial x^\beta} + V^\sigma\Gamma^\alpha_{\beta\sigma}\right]\vec{e}_{(\alpha)}. \tag{5.7}$$

For any fixed β, $\frac{\partial \vec{V}}{\partial x^\beta}$ is a vector field because it is a linear combination of the basis vectors $\{\vec{e}_{(\alpha)}\}$ with coefficients $\left[\frac{\partial V^\alpha}{\partial x^\beta} + V^\sigma\Gamma^\alpha_{\beta\sigma}\right]$.

If we introduce the following notation

$$V^\alpha{}_{,\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \quad , \qquad \text{and} \quad V^\alpha{}_{;\beta} = \frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha_{\beta\mu}, \tag{5.8}$$

eq. (5.7) becomes

$$\frac{\partial \vec{V}}{\partial x^\beta} = V^\alpha{}_{;\beta}\ \vec{e}_{(\alpha)}. \tag{5.9}$$

5.1.1 $V^\alpha{}_{;\beta}$ are the components of a tensor

Let us define the following quantity:

$$\nabla \vec{V} = \left[V^\alpha{}_{;\beta}\ \vec{e}_{(\alpha)} \right] \otimes \tilde{\omega}^{(\beta)}. \tag{5.10}$$

As shown in section 5.1, for any fixed value of β the quantity $\left[V^\alpha{}_{;\beta}\ \vec{e}_{(\alpha)} \right]$ is a vector; thus, $\nabla \vec{V}$ defined in eq. (5.10) is the outer product between these vectors and the basis one-forms, i.e. it is a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor. This tensor field is called **Covariant derivative of a vector**, and its components are

$$(\nabla \vec{V})^\alpha{}_\beta \equiv \nabla_\beta V^\alpha \equiv V^\alpha{}_{;\beta}. \tag{5.11}$$

NOTE THAT

In a locally inertial frame the basis vectors are constant, and consequently, according to eq. (5.6) the affine connections vanish and from eq. (5.8) it follows that

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} \Longrightarrow \frac{\partial \vec{V}}{\partial x^\beta} = V^\alpha{}_{,\beta}\ \vec{e}_{(\alpha)}. \tag{5.12}$$

Thus, in a locally inertial frame covariant and ordinary derivative coincide.

5.2 The covariant derivative of one-forms and tensors

In order to find the covariant derivative of a one-form consider a scalar field Φ . At any space point it is a number, therefore it does not depend on the coordinate basis: this implies that ordinary and covariant derivative coincide

$$\nabla_\mu \Phi = \frac{\partial \Phi}{\partial x^\mu} = (\tilde{d}\Phi)_\mu. \tag{5.13}$$

Now remember the definition of one-forms: they are linear, real valued functions of vectors such that

$$\tilde{q}(\vec{V}) = q_\alpha V^\alpha, \tag{5.14}$$

where q_α and V^α are the components of the one-form and vector fields, and $q_\alpha V^\alpha$ is a scalar function. Let us assume that the scalar field in eq. (5.13) is the the function $q_\alpha V^\alpha$; consequently its covariant derivative will be

$$\nabla_\mu \Phi \equiv \frac{\partial \Phi}{\partial x^\mu} = \frac{\partial q_\alpha}{\partial x^\mu} V^\alpha + q_\alpha \frac{\partial V^\alpha}{\partial x^\mu}.$$

Substituting $\frac{\partial V^\alpha}{\partial x^\mu}$ from eq. (5.8) we find

$$\nabla_\mu \Phi = \frac{\partial q_\alpha}{\partial x^\mu} V^\alpha + q_\alpha [V^\alpha{}_{;\mu} - V^\beta \Gamma^\alpha_{\mu\beta}],$$

and relabeling the indices

$$\begin{aligned} \nabla_\mu \Phi &= \frac{\partial q_\alpha}{\partial x^\mu} V^\alpha + q_\sigma V^\sigma{}_{;\mu} - q_\sigma V^\alpha \Gamma^\sigma_{\mu\alpha} = \\ &= [\frac{\partial q_\alpha}{\partial x^\mu} - q_\sigma \Gamma^\sigma_{\mu\alpha}] V^\alpha + q_\sigma V^\sigma{}_{;\mu}. \end{aligned} \tag{5.15}$$

Since $\nabla_\mu \Phi$ are the components of a $\left(\begin{array}{c} 0 \\ 1 \end{array}\right)$ tensor, this equation is true only if all terms on the right-hand side are the components of tensors of the same rank. Let us consider the second term: it is the result of the contraction of a $\left(\begin{array}{c} 0 \\ 1 \end{array}\right)$ and a $\left(\begin{array}{c} 1 \\ 1 \end{array}\right)$ tensor, therefore it is a $\left(\begin{array}{c} 0 \\ 1 \end{array}\right)$ tensor. The first term is a $\left(\begin{array}{c} 0 \\ 1 \end{array}\right)$ tensor only if the terms in square brackets are the components of a $\left(\begin{array}{c} 0 \\ 2 \end{array}\right)$ tensor, which we call **covariant derivative of the one-form** $\tilde{q}$

$$(\nabla \tilde{q})_{\alpha\mu} \equiv \nabla_\mu q_\alpha \equiv q_{\alpha;\mu} = q_{\alpha,\mu} - q_\sigma \Gamma^\sigma_{\mu\alpha}. \tag{5.16}$$

Thus, eq. (5.15) can be written as

$$\nabla_\mu \Phi = \nabla_\mu (q_\alpha V^\alpha) = q_{\alpha;\mu} V^\alpha + q_\alpha V^\alpha{}_{;\mu}, \tag{5.17}$$

which shows that the covariant derivative satisfies the standard property of the derivative of a product.

The same procedure can be used to define the covariant derivative of $\left(\begin{array}{c} N \\ N' \end{array}\right)$ tensors.

(do it as an exercise)

$$(\nabla T_{\mu\nu})_\beta = T_{\mu\nu,\beta} - T_{\alpha\nu} \Gamma^\alpha_{\beta\mu} - T_{\mu\alpha} \Gamma^\alpha_{\beta\nu} \tag{5.18}$$

$$(\nabla A^{\mu\nu})_\beta = A^{\mu\nu}{}_{,\beta} + A^{\alpha\nu} \Gamma^\mu_{\alpha\beta} + A^{\mu\alpha} \Gamma^\nu_{\alpha\beta} \tag{5.19}$$

$$(\nabla B^\mu{}_\nu)_\beta = B^\mu{}_{\nu,\beta} + B^\alpha{}_\nu \Gamma^\mu_{\beta\alpha} - B^\mu_\alpha \Gamma^\alpha_{\beta\nu} \tag{5.20}$$

what is the rule?

5.3 The covariant derivative of the metric tensor

The covariant derivative of $g_{\mu\nu}$ is zero

$$g_{\mu\nu;\alpha} = 0.$$

The reason is the following. We know from the principle of equivalence that at each point of spacetime we can choose a coordinate system such that $g_{\mu\nu}$ reduces to $\eta_{\mu\nu}$. The coordinate basis associated to these coordinates has constant basis vectors, therefore the affine connections also vanish (see eq. 5.6). In this frame

$$g_{\alpha\beta;\mu} = \eta_{\alpha\beta;\mu} = \frac{\partial \eta_{\alpha\beta}}{\partial x^\mu} - \Gamma^\nu_{\alpha\mu}\eta_{\nu\beta} - \Gamma^\nu_{\beta\mu}\eta_{\alpha\nu} = 0$$

$g_{\alpha\beta;\mu}$ is a $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ tensor, and if all components of a tensor are zero in a coordinate system, they are zero in **any** coordinate system therefore

$$g_{\alpha\beta;\mu} = 0 \tag{5.21}$$

always.

5.4 Symmetries of the affine connections

Consider an arbitrary scalar field Φ.
Its first covariant derivative is a one-form and coincides with the ordinary derivative. Its second covariant derivative $\nabla\nabla\Phi$ is a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor of components $\Phi_{,\beta;\alpha}$. In minkowskian coordinates, i.e. in a locally inertial frame, covariant derivative reduces to ordinary derivative:

$$\Phi_{,\beta;\alpha} = \Phi_{,\beta,\alpha} = \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\Phi, \tag{5.22}$$

and since partial derivatives commute

$$\Phi_{,\beta,\alpha} = \Phi_{,\alpha,\beta} \Rightarrow \Phi_{,\beta;\alpha} = \Phi_{,\alpha;\beta}. \tag{5.23}$$

Thus, the tensor $\nabla\nabla\Phi$ is symmetric. But if a tensor is symmetric in one basis, it is symmetric in any basis, therefore

$$\Phi_{,\beta,\alpha} - \Phi_{,\mu}\Gamma^\mu_{\beta\alpha} = \Phi_{,\alpha,\beta} - \Phi_{,\mu}\Gamma^\mu_{\alpha\beta}$$

in any coordinate system. It follows that for any Φ

$$\Phi_{,\mu}\Gamma^\mu_{\beta\alpha} = \Phi_{,\mu}\Gamma^\mu_{\alpha\beta},$$

and consequently

$$\Gamma^\mu_{\beta\alpha} = \Gamma^\mu_{\alpha\beta} \tag{5.24}$$

in any coordinate system.

5.5 The relation between the affine connections and the metric tensor

From eq. (5.21) it follows that

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^\nu_{\alpha\mu}g_{\nu\beta} - \Gamma^\nu_{\beta\mu}g_{\alpha\nu} = 0,$$

therefore

$$g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\alpha\mu} g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu}. \tag{5.25}$$

Let us now consider the following equations

$$g_{\alpha\mu,\beta} = \Gamma^{\nu}_{\alpha\beta} g_{\nu\mu} + \Gamma^{\nu}_{\mu\beta} g_{\alpha\nu},$$

$$-g_{\beta\mu,\alpha} = -\Gamma^{\nu}_{\beta\alpha} g_{\nu\mu} - \Gamma^{\nu}_{\mu\alpha} g_{\beta\nu},$$

It follows that

$$\begin{aligned} g_{\alpha\beta,\mu} &+ g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = (\Gamma^{\nu}_{\alpha\mu} - \Gamma^{\nu}_{\mu\alpha}) g_{\nu\beta} + \\ &+ (\Gamma^{\nu}_{\beta\mu} + \Gamma^{\nu}_{\mu\beta}) g_{\alpha\nu} + (\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\beta\alpha}) g_{\nu\mu}, \end{aligned}$$

where we have used $g_{\alpha\beta} = g_{\beta\alpha}$.
Since $\Gamma^{\alpha}_{\beta\gamma}$ are symmetric in β and γ, it follows that

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2\Gamma^{\nu}_{\beta\mu} g_{\alpha\nu}.$$

If we multiply by $g^{\alpha\gamma}$ and remember that since $g^{\alpha\gamma}$ is the inverse of $g_{\alpha\gamma}$

$$g^{\alpha\gamma} g_{\alpha\nu} = \delta^{\gamma}_{\nu},$$

we finally find

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \tag{5.26}$$

This expression is extremely useful, since it allows to compute the affine connection in terms of the components of the metric.

Are the $\Gamma^{\alpha}_{\beta\gamma}$ components of a tensor?
They are not, and it is easy to see why. In a locally inertial frame the $\Gamma^{\alpha}_{\beta\gamma}$ vanish, but in any other frame they don't. If it would be a tensor they should vanish in any frame.
In the first chapter we defined the Christoffel symbols as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \xi^{\lambda}} \frac{\partial^2 \xi^{\lambda}}{\partial x^{\mu} \partial x^{\nu}}. \tag{5.27}$$

This definition was a consequence of the equivalence principle. We did the following: We considered a free particle in a locally inertial frame $\{\xi^{\alpha}\}$:

$$\frac{d^2 \xi^{\alpha}}{d\tau^2} = 0. \tag{5.28}$$

Then we transformed this equation to an arbitrary coordinate system $\{x^{\alpha}\}$ and we showed that it becomes

$$\frac{d^2 x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu} \left[\frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \right] = 0, \tag{5.29}$$

with $\Gamma^{\alpha}_{\mu\nu}$ defined in eq. (5.27).
In this chapter we have defined the Γ's as those functions that satisfy the equation

$$\frac{\partial \vec{e}_{(\mu)}}{\partial x^{\nu}} = \Gamma^{\alpha}_{\mu\nu} \vec{e}_{(\alpha)}. \tag{5.30}$$

What is the relation between eq. (5.27) and eq. (5.30)?
In a localy inertial frame $\{\xi^\alpha\}$ be $\vec{e}_{M(\mu)}$ the constant basis vectors. If we make a coordinate transformation to a new coordinate system $\{x^{\alpha'}\}$, the new basis $\{\vec{e}_{(\mu')}\}$ will be

$$\vec{e}_{(\mu')} = \Lambda^\alpha{}_{\mu'}\vec{e}_{M(\alpha)} = \frac{\partial \xi^\alpha}{\partial x^{\mu'}}\vec{e}_{M(\alpha)}. \tag{5.31}$$

In this frame, eq. (5.30)which defines the affine connections can be rewritten as

$$\frac{\partial}{\partial x^{\nu'}}\left[\Lambda^\beta{}_{\mu'}\vec{e}_{M(\beta)}\right] = \Gamma^{\alpha'}_{\mu'\nu'}\Lambda^\gamma{}_{\alpha'}\vec{e}_{M(\gamma)} \tag{5.32}$$

or, being the $\vec{e}_{M(\beta)}$ constant

$$\frac{\partial \Lambda^\beta{}_{\mu'}}{\partial x^{\nu'}}\vec{e}_{M(\beta)} = \Gamma^{\alpha'}_{\mu'\nu'}\Lambda^\gamma{}_{\alpha'}\vec{e}_{M(\gamma)}. \tag{5.33}$$

This equation can be re-written as

$$\left(\frac{\partial \Lambda^\beta{}_{\mu'}}{\partial x^{\nu'}} - \Gamma^{\alpha'}_{\mu'\nu'}\Lambda^\beta{}_{\alpha'}\right)\vec{e}_{M(\beta)} = 0. \tag{5.34}$$

We now multiply eq. (5.34) by $\Lambda^{\sigma'}{}_\beta$ and find

$$\Lambda^{\sigma'}{}_\beta\frac{\partial \Lambda^\beta{}_{\mu'}}{\partial x^{\nu'}} - \Gamma^{\alpha'}_{\mu'\nu'}\Lambda^{\sigma'}{}_\beta\Lambda^\beta{}_{\alpha'} = 0. \tag{5.35}$$

Since $\Lambda^{\sigma'}{}_\beta\Lambda^\beta{}_{\alpha'} = \delta^{\sigma'}{}_{\alpha'}$, it follows that

$$\Gamma^{\sigma'}_{\mu'\nu'} = \Lambda^{\sigma'}{}_\beta\frac{\partial \Lambda^\beta{}_{\mu'}}{\partial x^{\nu'}} = \frac{\partial x^{\sigma'}}{\partial \xi^\beta}\frac{\partial^2 \xi^\beta}{\partial x^{\nu'}\partial x^{\mu'}},$$

which coincides with eq. (5.27). Thus, as expected, the two definitions are equivalent. How do the $\Gamma^\alpha_{\beta\gamma}$ transform?
The easiest way to see it is from the definition (5.27). In an arbitrary coordinate system $\{x^{\mu'}\}$ they are

$$\begin{aligned}
\Gamma^{\lambda'}_{\mu'\nu'} &= \frac{\partial x^{\lambda'}}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^{\nu'}\partial x^{\mu'}} = \\
&= \frac{\partial x^{\lambda'}}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial \xi^\alpha}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x^{\nu'}}\right) = \\
&= \frac{\partial x^{\lambda'}}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\left[\frac{\partial x^\sigma}{\partial x^{\nu'}}\frac{\partial^2 \xi^\alpha}{\partial x^\tau \partial x^\sigma}\frac{\partial x^\tau}{\partial x^{\mu'}} + \frac{\partial \xi^\alpha}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\nu'}\partial x^{\mu'}}\right] = \\
&= \frac{\partial x^{\lambda'}}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x^{\nu'}}\frac{\partial x^\tau}{\partial x^{\mu'}}\Gamma^\rho_{\tau\sigma} + \frac{\partial x^{\lambda'}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\nu'}\partial x^{\mu'}}
\end{aligned} \tag{5.36}$$

The first term is what we should get if $\Gamma^\alpha_{\beta\gamma}$ were a tensor. But we know it is not, and in fact there is an additional term.

5.6 Non coordinate basis

In Sec. 3.4 we have seen that if we pass from Minkowskian coordinates $\{x^\alpha\} \equiv (ct, x, y)$ to polar coordinates $\{x^{\alpha'}\} \equiv (ct, r, \theta)$ the coordinate basis

$$\{\vec{e}_{(\alpha)}\} \rightarrow \begin{cases} \vec{e}_{(0)} & \rightarrow & (1,0,0) \\ \vec{e}_{(1)} & \rightarrow & (0,1,0) \\ \vec{e}_{(2)} & \rightarrow & (0,0,1) \end{cases} \tag{5.37}$$

transforms to $\{\vec{e}_{(\alpha')}\}$

$$\begin{cases} \vec{e}_{(0')} = \vec{e}_{(0)} \\ \vec{e}_{(1')} = \vec{e}_{(r)} = \cos\theta \vec{e}_{(1)} + \sin\theta \vec{e}_{(2)} \\ \vec{e}_{(2')} = \vec{e}_{(\theta)} = -r\sin\theta \vec{e}_{(1)} + r\cos\theta \vec{e}_{(2)} \end{cases} \tag{5.38}$$

according to the law

$$\vec{e}_{(\alpha')} = \Lambda^\mu{}_{\alpha'} \vec{e}_{(\mu)}.$$

The new basis is a coordinate basis and the matrix $\Lambda^\mu{}_{\alpha'} = \frac{\partial x^\mu}{\partial x^{\alpha'}}$ is the matrix associated to the coordinate transformation. However we may choose a different basis for vectors. For example the vectors $\{\vec{e}_{(\alpha')}\}$ in the previous example are not normalized. In fact

$$\vec{e}_{(\alpha')} \cdot \vec{e}_{(\beta')} = g_{\alpha'\beta'} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix} \neq \eta_{\alpha'\beta'}.$$

We may decide that we want a basis composed by unit vectors, and choose

$$\begin{cases} \vec{e}_{\hat{r}} = \vec{e}_r \\ \vec{e}_{\hat{t}} = \vec{e}_t \\ \vec{e}_{\hat{\theta}} = \frac{1}{r}\vec{e}_\theta. \end{cases} \tag{5.39}$$

In this case we would find

$$\vec{e}_{(\hat{\alpha})} \cdot \vec{e}_{(\hat{\beta})} = \eta_{\hat{\alpha}\hat{\beta}}.$$

But now the question is: do there exist coordinates $\{x^{\hat{\alpha}}\}$ such that

$$e_{(\hat{\alpha})} = \Lambda^\mu{}_{\hat{\alpha}} \vec{e}_{(\mu)} = \frac{\partial x^\mu}{\partial x^{\hat{\alpha}}} \vec{e}_{(\mu)}$$

so that the basis $\{\vec{e}_{(\hat{\alpha})}\}$ is a coordinate basis? Alternatively, we can formulate the same question for the basis one-forms: if $\{\tilde{\omega}^{(\alpha')}\}$ is the coordinate basis for one-forms and $\{\omega^{(\hat{\alpha})}\}$ is the normalized basis, is $\{\tilde{\omega}^{(\hat{\alpha})}\}$ a new coordinate basis associated to some coordinates $\{x^{\hat{\alpha}}\}$? i.e.

$$\tilde{\omega}^{(\hat{\alpha})} = \Lambda^{\hat{\alpha}}{}_\beta \tilde{\omega}^{(\beta)} = \frac{\partial x^{\hat{\alpha}}}{\partial x^\beta} \tilde{\omega}^{(\beta)}?$$

For instance, in the previous example,

$$\begin{aligned} \tilde{\omega}^{\hat{1}} &= \tilde{\omega}^{\hat{r}} = \tilde{\omega}^r = \cos\theta \tilde{dx} + \sin\theta \tilde{dy} \\ \tilde{\omega}^{\hat{2}} &= \tilde{\omega}^{\hat{\theta}} = r\tilde{\omega}^\theta = -\sin\theta \tilde{dx} + \cos\theta \tilde{dy} \end{aligned} \tag{5.40}$$

The point is that if this is true, $\Lambda^{\hat{\alpha}}{}_{\beta}$ **must coincide** with the partial derivative $\frac{\partial x^{\hat{\alpha}}}{\partial x^{\beta}}$, and consequently the following condition must be satisfied for any $\Lambda^{\hat{\alpha}}{}_{\gamma}$:

$$\frac{\partial}{\partial x^{\gamma}}\Lambda^{\hat{\alpha}}{}_{\beta} = \frac{\partial^2 x^{\hat{\alpha}}}{\partial x^{\gamma}\partial x^{\beta}} = \frac{\partial^2 x^{\hat{\alpha}}}{\partial x^{\beta}\partial x^{\gamma}} = \frac{\partial}{\partial x^{\beta}}\Lambda^{\hat{\alpha}}{}_{\gamma}. \tag{5.41}$$

This is an "integrability condition" that all the components of $\Lambda^{\hat{\alpha}}{}_{\gamma}$ must satisfy in order the coordinates $\{x^{\hat{\alpha}}\}$ do exist.
For example, let us check whether the basis (5.40) is a coordinate basis. From the expression of $\tilde{\omega}^{\theta}$ we find that

$$\Lambda^{\hat{2}}{}_{1} = \frac{\partial x^{\hat{2}}}{\partial x} = -\sin\theta \qquad \Lambda^{\hat{2}}{}_{2} = \frac{\partial x^{\hat{2}}}{\partial y} = \cos\theta,$$

eq. (5.41) gives

$$\frac{\partial}{\partial y}\Lambda^{\hat{2}}{}_{1} = \frac{\partial}{\partial x}\Lambda^{\hat{2}}{}_{2} \;\Rightarrow\; \frac{\partial}{\partial y}(-\sin\theta) = \frac{\partial}{\partial x}(\cos\theta),$$

But

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2+y^2},$$

so that it should be

$$\frac{\partial}{\partial y}\left[-\frac{y}{\sqrt{x^2+y^2}}\right] = \frac{\partial}{\partial x}\left[\frac{y}{\sqrt{x^2+y^2}}\right],$$

which is certainly not true.
We conclude that the basis $\{\tilde{\omega}^{(\hat{\alpha})}\}$ **is not** a coordinate basis, since we cannot associate to it a coordinate transformation.

What are the consequences of choosing a noncoordinate basis?
As we have seen at the end of section 3.5, the gradient of a scalar field Φ is a one-form:

$$\tilde{d}\Phi \;\to\; \left\{\frac{\partial\Phi}{\partial x^{\alpha}}\right\} \equiv \{\Phi_{,\alpha}\} \;. \tag{5.42}$$

For example let us start in a 2-dimensional plane with coordinates $(x, y) = (x^1, x^2)$. Then change to polar coordinates $(r, \theta) = (x^{1'}, x^{2'})$. The gradient will transform as one-forms do:

$$\tilde{d}\Phi_{\alpha'} = \Lambda^{\beta}{}_{\alpha'}\tilde{d}\Phi_{\beta}$$

where $\tilde{d}\Phi_x = \Phi_{,x} = \frac{\partial\Phi}{\partial x}$ and $\tilde{d}\Phi_y = \Phi_{,y} = \frac{\partial\Phi}{\partial y}$.
The components of the gradient in the new coordinate basis are

$$\begin{cases} \tilde{d}\Phi_r = \Lambda^{x}{}_{r}\tilde{d}\Phi_x + \Lambda^{y}{}_{r}\tilde{d}\Phi_y = \dfrac{\partial x}{\partial r}\tilde{d}\Phi_x + \dfrac{\partial y}{\partial r}\tilde{d}\Phi_y \\ \tilde{d}\Phi_\theta = \Lambda^{x}{}_{\theta}\tilde{d}\Phi_x + \Lambda^{y}{}_{\theta}\tilde{d}\Phi_y = \dfrac{\partial x}{\partial \theta}\tilde{d}\Phi_x + \dfrac{\partial y}{\partial \theta}\tilde{d}\Phi_y. \end{cases} \tag{5.43}$$

Being

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta \;, \end{aligned}$$

$$\begin{cases} \tilde{d}\Phi_r = \cos\theta \tilde{d}\Phi_x + \sin\theta \tilde{d}\Phi_y = \dfrac{\partial \Phi}{\partial r} = \Phi_{,r} \\ \tilde{d}\Phi_\theta = -r\sin\theta \tilde{d}\Phi_x + r\cos\theta \tilde{d}\Phi_y = \dfrac{\partial \Phi}{\partial \theta} = \Phi_{,\theta}. \end{cases} \tag{5.44}$$

Thus the components of the gradient in the new coordinate basis $(\vec{e}_{(r)}, \vec{e}_{(\theta)})$ will still be

$$\tilde{d}\Phi_{j'} \rightarrow \frac{\partial \Phi}{\partial x^{j'}} .$$

But this is certainly non true if we use the non coordinate basis $\{\vec{e}_{(\hat{\alpha})}\}$: there are no coordinates associated to this basis, thus we cannot define $\tilde{d}\Phi_{\hat{j}} = \frac{\partial \Phi}{\partial x^{\hat{j}}}$!
Let us see what happens to the affine connections if we use a non-coordinate basis. We have defined $\Gamma^\alpha_{\beta\gamma}$ as

$$\nabla_\alpha \vec{e}_{(\beta)} = \frac{\partial \vec{e}_{(\beta)}}{\partial x^\alpha} = \Gamma^\nu_{\beta\alpha}\vec{e}_{(\nu)} . \tag{5.45}$$

This is a definition valid in any basis, therefore in terms of a noncoordinate basis $\{\vec{e}_{(\hat{\alpha})}\}$ eq. (5.45) becomes

$$\nabla_{\hat{\alpha}} \vec{e}_{(\hat{\beta})} = \Gamma^{\hat{\nu}}_{\hat{\beta}\hat{\alpha}}\vec{e}_{(\hat{\nu})} . \tag{5.46}$$

But now, since the $\{x^{\hat{\alpha}}\}$ do not exist, is not longer true that

$$\Phi_{,\hat{\beta};\hat{\alpha}} = \Phi_{,\hat{\alpha};\hat{\beta}} .$$

If we go back to eq.(5.23) we see that we used this condition to show the simmetry of the affine conection in the two lower indices. Thus if the basis is a non coordinate basis

$$\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} \neq \Gamma^{\hat{\alpha}}_{\hat{\gamma}\hat{\beta}}$$

and moreover eq (5.26) which gives the connections in terms of $g_{\alpha\beta}$ is no longer true as well.
In the following of this course we shall use mainly coordinate basis, and we shall explicitely specify when we will use a non coordinate basis.

EXERCISE

In this chapter we have introduced the connections as those quantities that allow to find the covariant derivative of a vector in an arbitrary frame. Given the metric components, the simplest way to compute the connection is to use eq. (5.26). As an exercise, let us compute the connection $\Gamma^\mu_{\alpha\beta}$ in a different way, using directly the definition

$$\frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta}\vec{e}_{(\mu)}. \tag{5.47}$$

Let us consider for example a 2-dimensional flat space in polar coordinates, i.e. $(x^{1'}, x^{2'}) \equiv (r, \theta)$ and remember that the basis vectors are related to the coordinate basis associated to cartesian coordinates by the equations (3.50)

$$\begin{aligned} \vec{e}_{(1')} &= \cos\theta \vec{e}_{(1)} + \sin\theta \vec{e}_{(2)} \\ \vec{e}_{(2')} &= -r\sin\theta \vec{e}_{(1)} + r\cos\theta \vec{e}_{(2)}. \end{aligned}$$

Let us indicate, for simplicity $(\vec{e}_{(1)}, \vec{e}_{(2)})$ with $(\vec{e}_{(x)}, \vec{e}_{(y)})$, and $(\vec{e}_{(1')}, \vec{e}_{(2')})$ with $(\vec{e}_{(r)}, \vec{e}_{(\theta)})$. From these expressions we find

$$\frac{\partial \vec{e}_{(r)}}{\partial r} = \frac{\partial}{\partial r}(\cos\theta \vec{e}_{(x)} + \sin\theta \vec{e}_{(y)}) = 0,$$

and consequently

$$\Gamma^{\mu}_{rr}\vec{e}_{(\mu)} = \Gamma^{r}_{rr}\vec{e}_{(r)} + \Gamma^{\theta}_{rr}\vec{e}_{(\theta)} = 0 \Longrightarrow \Gamma^{r}_{rr} = \Gamma^{\theta}_{rr} = 0.$$

Moreover

$$\begin{aligned}\frac{\partial \vec{e}_{(r)}}{\partial \theta} &= \frac{\partial}{\partial \theta}(\cos\theta \vec{e}_{(x)} + \sin\theta \vec{e}_{(y)}) = \\ &= -\sin\theta \vec{e}_{(x)} + \cos\theta \vec{e}_{(y)} = \frac{1}{r}\vec{e}_{(\theta)},\end{aligned}$$

therefore

$$\frac{1}{r}\vec{e}_{(\theta)} = \Gamma^{\mu}_{r\theta}\vec{e}_{(\mu)} = \Gamma^{r}_{r\theta}\vec{e}_{(r)} + \Gamma^{\theta}_{r\theta}\vec{e}_{(\theta)} \Longrightarrow \Gamma^{r}_{r\theta} = 0 \quad , \quad \Gamma^{\theta}_{r\theta} = \frac{1}{r}.$$

Proceeding along these lines one can show that

$$\Gamma^{r}_{\theta r} = 0 \ , \ \Gamma^{\theta}_{\theta r} = \frac{1}{r} \ , \ \Gamma^{r}_{\theta\theta} = -r \ , \ \Gamma^{\theta}_{\theta\theta} = 0.$$

It should be noted that altough we have used the cartesian basis to express $\vec{e}_{(r)}$ and $\vec{e}_{(\theta)}$ and compute their derivatives, at the end the Γ's depend only on the coordinates r and θ. Note also that the same result can be obtained by using eq. (5.26) and the metric

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} .$$

5.7 Summary of the preceeding Sections

In chapter 1 we have seen that the equation of motion of a particle which moves under the exclusive action of a gravitational field is

$$\frac{d^2x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\left[\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\right] = 0. \tag{5.48}$$

In the frame associated to the coordinates $\{x^{\mu}\}$ the line element is

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{5.49}$$

Then we have seen that the Equivalence Principle allows to find a locally inertial frame $\{\xi^{\alpha}\}$ where eq. (5.48) becomes

$$\frac{d^2\xi^{\alpha}}{d\tau^2} = 0, \tag{5.50}$$

and the line element reduces to

$$ds^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{5.51}$$

However we do not know if this transformation holds everywhere, i.e. if the spacetime is really flat, or if it holds only locally, which would mean that there is a non constant and non uniform gravitational field. It follows that the study of the motion of a single particle and the knowledge of the $\Gamma^{\alpha}_{\mu\nu}$'s do not allow to decide whether there is a non constant and non uniform gravitational field.

Then we have introduced vectors and tensors on a manifold, we have defined the metric tensor as a geometric object and we have shown that its role is not only that of defining the distance between points, but also that of mapping vectors into one-forms, and of computing the scalar product between vectors. We have shown that if we introduce at each point of the manifold a basis for vectors $\{\vec{e}_{(\alpha)}\}$ (and a dual basis for one forms $\{\tilde{\omega}^{(\beta)}\}$) any vector (or one-form) can be assigned "components" with respect to the basis

$$\vec{A} = A^{\alpha}\vec{e}_{(\alpha)}. \tag{5.52}$$

Then we have introduced an operator of **covariant derivative**, which generates a tensor according to the following rule

$$\nabla_{\beta}V^{\alpha} = V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}V^{\mu}. \tag{5.53}$$

(and similar rules for tensors). The covariant derivative coincides with ordinary derivative in two particular cases:

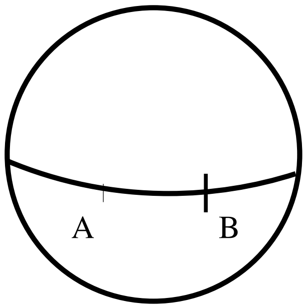
1) the spacetime is flat **and** we are using a basis where the vectors $\vec{e}_{(\alpha)}$ are constant. Consequently from the definition (5.6) it follows that $\Gamma^{\alpha}{}_{\mu\beta} = 0$.

2) the spacetime is curved, but we are in a locally inertial frame. Indeed, in this frame eq. (5.48) reduces to eq. (5.50), which means again that $\Gamma^{\alpha}{}_{\mu\beta} = 0$.

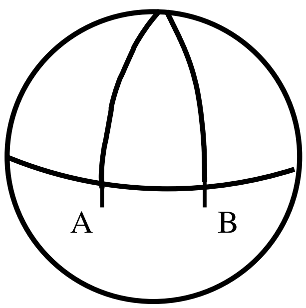
The fact that we can always find a frame where $g_{\mu\nu}$ reduces to $\eta_{\mu\nu}$ and the $\Gamma^{\alpha}{}_{\mu\beta} = 0$ (and consequently the first derivatives of $g_{\mu\nu}$ vanish) implies that in order to know if we are in the presence of a gravitational field, (i.e. if the spacetime is curved), **we need to know the second derivatives of the metric tensor** $g_{\mu\nu,\alpha,\beta}$. This result should not be surprising: in chapter 1 we introduced the 2-dimensional Gaussian geometry and we said that one can always choose a frame where the metric looks flat, but there exists a quantity, the Gaussian curvature, which tells us that the space is curved. The gaussian curvature depends on the first derivatives (non linearly) and on the second derivatives (linearly) of the metric; thus, we shall now look for a generalization of the Gaussian curvature. We already mentioned that in four dimensions we need more than one invariant to describe the intrinsic properties of a curved surface: we need six functions, and it is clear that a vector would not be enough. Thus, we need a tensor, but which tensor? The only thing we know is that it should contain the second derivatives of $g_{\mu\nu}$. In order to introduce the curvature tensor we first need to introduce the notion of parallel transport of a vector along a curve.

5.8 Parallel Transport

In chapter 1 we discussed and compared the intrinsic geometry of cones, cylinders and spheres, and we noticed that while it is flat for cones and cylinders, it is curved for spheres. That means, for example, that two lines which start parallel do not remain parallel when prolonged:



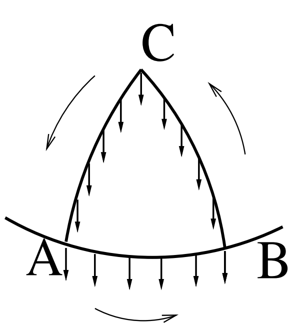
consider two segments in A and B, perpendicular to the equator, i.e. parallel.



The same lines when prolonged: they do not remain parallel.

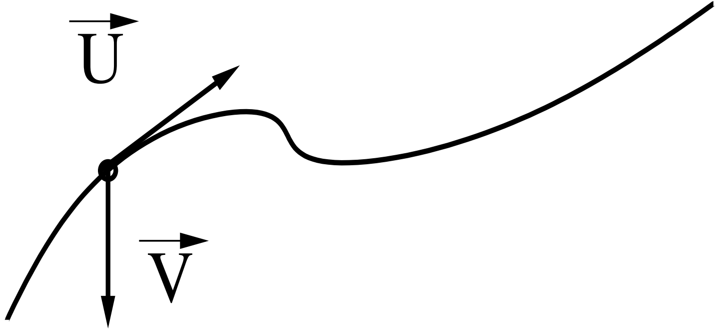
It is also interesting to see what happens when we parallely transport a vector along a path. **Parallel Transport means that for each infinitesimal displacement, the displaced vector must be parallel to the original one, and must have the same lenght**. Let us consider first the case when the path belongs to a flat space.

a) *FLAT SPACE*

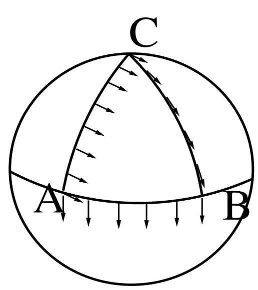


When we return to A the displaced vector coincides with the original vector in A.

b) *ON A SPHERE*



(remember that the vector must always be tangent to the sphere)



When the vector goes back to A it is rotated of 90 degrees This is a consequence of the curvature of the sphere.

On a curved manifold it is impossible to define a globally parallel vector field. The parallel transport of a vector depends on the path along which it is transported.

Let us now compute how does a vector change when it is parallely transported. Consider a curve of parameter λ and a vector field $\vec{V}$ defined at every point of the curve. Be $\vec{U} \to \{\frac{dx^\alpha}{d\lambda}\}$ the vector tangent to the curve

At every point of the curve we can choose a locally inertial frame $\{\xi^\alpha\}$. In this frame, if we move $\vec{V}$ along the curve of an infinitesimal $d\lambda$, parallel to itself and keeping its lenght unchanged, its components do not change

$$\frac{dV^\alpha}{d\lambda} = 0. \tag{5.54}$$

But

$$\frac{dV^\alpha}{d\lambda} = \frac{\partial V^\alpha}{\partial \xi^\beta}\frac{d\xi^\beta}{d\lambda} = U^\beta {V^\alpha}_{,\beta} = 0. \tag{5.55}$$

Since we are in a locally inertial frame, ordinary and covariant derivative coincide and therefore we can write

$$U^\beta V^\alpha{}_{;\beta} = 0. \tag{5.56}$$

If this equation is true in a locally inertial frame, since it is a tensor equation it must be true in any other frame. Therefore eq. (5.56) is the frame-invariant definition of the **parallel transport** of $\vec{V}$ along the curve identified by the tangent vector $\vec{U}$.

Eq. (5.56) is written in terms of the components of $\vec{V}$ and $\vec{U}$; if we want to write it in a frame-independent form we shall write

$$\nabla_{\vec{U}}\vec{V} = 0, \tag{5.57}$$

which means that the covariant derivative along the direction of the vector $\vec{U}$ is zero. Written explicitely for a generic reference frame with coordinates $\{x^\alpha\}$ eq. (5.57) gives

$$\begin{aligned} \left(\nabla_{\vec{U}}\vec{V}\right)^\alpha &\equiv U^\beta V^\alpha{}_{;\beta} \\ = \frac{dx^\beta}{d\lambda}\left[\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\nu}V^\nu\right] &= \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha{}_{\beta\nu}V^\nu U^\beta = 0. \end{aligned} \tag{5.58}$$

Thus, contrary to what happens in flat space the components of a vector parallely transported along a curve in curved space do change, and the change is given by

$$\frac{dV^\alpha}{d\lambda} = -\Gamma^\alpha{}_{\beta\nu}V^\nu U^\beta.$$

5.9 The geodesic equation

In Chapter 1 we introduced the geodesics, as the curves which describe the motion of free particles; "free" here means that no other force than gravity is acting on them. We showed that they are the solution of the *geodesic equation* (1.37)

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\beta}\left[\frac{dx^\mu}{d\tau}\frac{dx^\beta}{d\tau}\right] = 0\,. \tag{5.59}$$

A different derivation of this equation, simpler than that given in Chapter 1, makes use of the notion of covariant derivative. Let us consider a "free" particle, with worldline $x^\mu(\tau)$ and four-velocity (i.e. tangent vector to the worldline) $U^\mu = dx^\mu/d\tau$. By the equivalence principle, at any point of the worldline we can define a locally inertial frame $\{x^{\alpha'}\}$, in which the laws of special relativity hold; then, in this frame the particle four-acceleration is zero, i.e.

$$\frac{dU^{\mu'}}{d\tau} = \frac{dx^{\alpha'}}{d\tau}\frac{\partial U^{\mu'}}{\partial x^{\alpha'}} = U^{\alpha'}U^{\mu'}{}_{,\alpha'} = 0\,. \tag{5.60}$$

In a locally inertial frame ordinary and covariant derivative coincide, thus

$$U^{\alpha'}U^{\mu'}{}_{;\alpha'} = 0\,. \tag{5.61}$$

This is a tensorial equation, and the covariance principle establishes that it holds in any coordinate frame; therefore, in a generic frame we can write

$$U^\alpha U^\mu{}_{;\alpha} = 0\,. \tag{5.62}$$

Equations (5.62) and (5.59) coincide; indeed

$$U^\alpha U^\mu{}_{;\alpha} = U^\alpha U^\mu{}_{,\alpha} + U^\alpha \Gamma^\mu{}_{\alpha\beta} U^\beta\ , \tag{5.63}$$

and by substituting $U^\mu = dx^\mu/d\tau$ this equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\ , \tag{5.64}$$

which is eq. (5.59).

The parameter along a geodesic need not to be the proper time. Be s the new parameter chosen to parametrize the geodesic. Since

$$\frac{d}{d\tau} = \frac{d}{ds}\frac{ds}{d\tau}, \tag{5.65}$$

equation (5.64) becomes

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\left[\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right] = -\left[\frac{d^2s}{d\tau^2}\Big/\left(\frac{ds}{d\tau}\right)^2\right]\frac{dx^\alpha}{ds}; \tag{5.66}$$

From this equation we see that the new curve is a geodesic, i.e. has the form of equation (5.64), only if the new parameter is related to the proper time τ by a linear transformations

$$s = a\lambda + b, \qquad\qquad a, b = \text{const}; \tag{5.67}$$

in which case the right hand side of equation (5.66) vanishes. τ and s are called **affine parameters**.

Equation (5.62) was derived assuming that the geodesic was the worldline of a massive particle, i.e a timelike curve. However, this equation has a more general validity, since a geodesic can be either timelike, spacelike or null. If a geodesic is timelike, i.e. $\vec{U}\cdot\vec{U} < 0$, it can represent the wordline of a massive particle; in this case, by performing the linear transformation (5.67) it is possible to change the affine parameter in such a way that $\vec{U}\cdot\vec{U} = -1$, so that the new parameter is the particle proper time.

If, instead, a geodesic is a null curve, i.e. $\vec{U}\cdot\vec{U} = 0$, it can represent the wordline of a massless particle; in this case the affine parameter is a generic parameter, since proper time is not defined for massless particles.

If the geodesic is spacelike, i.e. $U\cdot\vec{U} > 0$, it does not represent the worldline of a particle of any kind.

According to the equation of parallel transport (5.56), the geodesic equation written in the form (5.62) is the equation of the parallel transport of the tangent vector $\vec{U}$ along the geodesic. This means that if we take the tangent vector at a point $\mathbf{p}$, and parallely transport it to a point $\mathbf{p}'$ along the geodesic line, the transported vector is tangent to the curve at $\mathbf{p}'$. Thus, a curve $\mathcal{C}$ with tangent vector $\vec{U}$ is a geodesic if

$$\nabla_{\vec{U}}\vec{U} = 0. \tag{5.68}$$

For this eason we say that: **geodesics are those curves which parallel-transport their own tangent vectors**.

Chapter 6

The Curvature Tensor

We are now in a position to introduce the curvature tensor. We will do it in two different ways.

6.1 a) A Formal Approach

Let us start writing the transformation rule for affine connections

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x^{\alpha'}}\frac{\partial x^{\rho'}}{\partial x^{\mu}}\frac{\partial x^{\sigma'}}{\partial x^{\nu}}\Gamma^{\alpha'}{}_{\rho'\sigma'} + \frac{\partial x^{\lambda}}{\partial x^{\alpha'}}\frac{\partial^2 x^{\alpha'}}{\partial x^{\mu}\partial x^{\nu}}. \tag{6.1}$$

As we already noticed (Chapter V sec. 5) if the last term on the right-hand side would be zero $\Gamma^{\lambda}{}_{\mu\nu}$ would transform as a tensor. Let us isolate the 'bad term', by multiplying eq. (6.1) by $\frac{\partial x^{\tau'}}{\partial x^{\lambda}}$:

$$\frac{\partial^2 x^{\tau'}}{\partial x^{\mu}\partial x^{\nu}} = \frac{\partial x^{\tau'}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\mu\nu} - \frac{\partial x^{\rho'}}{\partial x^{\mu}}\frac{\partial x^{\sigma'}}{\partial x^{\nu}}\Gamma^{\tau'}{}_{\rho'\sigma'}. \tag{6.2}$$

We now differentiate this equation with respect to x^{κ}

$$\begin{aligned}\frac{\partial^3 x^{\tau'}}{\partial x^{\kappa}\partial x^{\mu}\partial x^{\nu}} &= \frac{\partial^2 x^{\tau'}}{\partial x^{\kappa}\partial x^{\lambda}}\Gamma^{\lambda}{}_{\mu\nu} + \frac{\partial x^{\tau'}}{\partial x^{\lambda}}\left(\frac{\partial}{\partial x^{\kappa}}\Gamma^{\lambda}{}_{\mu\nu}\right) \\ &-\frac{\partial^2 x^{\rho'}}{\partial x^{\kappa}\partial x^{\mu}}\frac{\partial x^{\sigma'}}{\partial x^{\nu}}\Gamma^{\tau'}{}_{\rho'\sigma'} - \frac{\partial x^{\rho'}}{\partial x^{\mu}}\frac{\partial^2 x^{\sigma'}}{\partial x^{\kappa}\partial x^{\nu}}\Gamma^{\tau'}{}_{\rho'\sigma'} - \frac{\partial x^{\rho'}}{\partial x^{\mu}}\frac{\partial x^{\sigma'}}{\partial x^{\nu}}\left(\frac{\partial}{\partial x^{\kappa}}\Gamma^{\tau'}{}_{\rho'\sigma'}\right).\end{aligned} \tag{6.3}$$

We now use eq. (6.2):

$$\begin{aligned}&\frac{\partial^3 x^{\tau'}}{\partial x^{\kappa}\partial x^{\mu}\partial x^{\nu}} = \\ &+\Gamma^{\lambda}{}_{\mu\nu}\left[\frac{\partial x^{\tau'}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\lambda} - \frac{\partial x^{\beta'}}{\partial x^{\kappa}}\frac{\partial x^{\gamma'}}{\partial x^{\lambda}}\Gamma^{\tau'}{}_{\beta'\gamma'}\right] + \frac{\partial x^{\tau'}}{\partial x^{\lambda}}\left[\frac{\partial}{\partial x^{\kappa}}\Gamma^{\lambda}{}_{\mu\mu}\right] \\ &-\frac{\partial x^{\sigma'}}{\partial x^{\nu}}\Gamma^{\tau'}{}_{\rho'\sigma'}\left[\frac{\partial x^{\rho'}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\mu} - \frac{\partial x^{\beta'}}{\partial x^{\kappa}}\frac{\partial x^{\gamma'}}{\partial x^{\mu}}\Gamma^{\rho'}{}_{\beta'\gamma'}\right]\end{aligned} \tag{6.4}$$

$$-\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\left[\frac{\partial x^{\sigma\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\nu}-\frac{\partial x^{\beta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\gamma\prime}}{\partial x^{\nu}}\Gamma^{\sigma\prime}{}_{\beta\prime\gamma\prime}\right]$$
$$-\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\left(\frac{\partial}{\partial x^{\kappa}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right).$$

Let us rewrite the last term as

$$\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\eta\prime}}{\partial x^{\kappa}}\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right). \tag{6.5}$$

(The reason is that the indices of Γ have a prime, thus the derivatives must be computed with respect to the $\{x^{\alpha\prime}\}$). We now rewrite eq. (6.5) in the following way

$$\frac{\partial^3 x^{\tau\prime}}{\partial x^{\kappa}\partial x^{\mu}\partial x^{\nu}}= \tag{6.6}$$
$$\left[\frac{\partial x^{\tau\prime}}{\partial x^{\lambda}}\left(\frac{\partial}{\partial x^{\kappa}}\Gamma^{\lambda}{}_{\mu\nu}\right)+\left(\frac{\partial x^{\tau\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\lambda}\Gamma^{\lambda}{}_{\mu\nu}\right)\right]$$
$$-\left[\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\eta\prime}}{\partial x^{\kappa}}\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right)-\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\beta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\gamma\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\Gamma^{\rho\prime}{}_{\beta\prime\gamma\prime}\right]$$
$$-\left[\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\beta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\gamma\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\Gamma^{\sigma\prime}{}_{\beta\prime\gamma\prime}\right]$$
$$-\left[\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\rho\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\mu}+\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\sigma\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\nu}+\frac{\partial x^{\beta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\gamma\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\mu\mu}\Gamma^{\tau\prime}{}_{\beta\prime\gamma\prime}\right].$$

We now relabel the indices in the following way

$$\frac{\partial x^{\tau\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\lambda}\Gamma^{\lambda}{}_{\mu\nu}\to\frac{\partial x^{\tau\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\kappa\eta}\Gamma^{\eta}{}_{\mu\nu} \tag{6.7}$$

$$\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\beta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\gamma\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\Gamma^{\rho\prime}{}_{\beta\prime\gamma\prime}\to\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\eta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime}$$
$$\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\beta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\gamma\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\Gamma^{\sigma\prime}{}_{\beta\prime\gamma\prime}\to\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\eta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\lambda\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\sigma\prime}$$

$$\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\rho\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\mu}\to\frac{\partial x^{\rho\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\sigma\prime\rho\prime}\frac{\partial x^{\sigma\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\kappa\mu}$$
$$\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\sigma\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{\kappa\nu}\to\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\sigma\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\kappa\nu}$$
$$\frac{\partial x^{\beta\prime}}{\partial x^{\kappa}}\frac{\partial x^{\gamma\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\mu\mu}\Gamma^{\tau\prime}{}_{\beta\prime\gamma\prime}\to\frac{\partial x^{\rho\prime}}{\partial x^{\kappa}}\frac{\partial x^{\sigma\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\mu\mu}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}$$

With these changes the terms can be collected in the following way

$$\frac{\partial^3 x^{\tau\prime}}{\partial x^{\kappa}\partial x^{\mu}\partial x^{\nu}}=\frac{\partial x^{\tau\prime}}{\partial x^{\lambda}}\left[\left(\frac{\partial}{\partial x^{\kappa}}\Gamma^{\lambda}{}_{\mu\nu}\right)+\Gamma^{\lambda}{}_{\kappa\eta}\Gamma^{\eta}{}_{\mu\nu}\right] \tag{6.8}$$
$$-\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\eta\prime}}{\partial x^{\kappa}}\left[\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right)-\Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime}-\Gamma^{\tau\prime}{}_{\rho\prime\lambda\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\sigma\prime}\right]$$
$$-\frac{\partial x^{\sigma\prime}}{\partial x^{\lambda}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\left[\Gamma^{\lambda}{}_{\kappa\mu}\frac{\partial x^{\rho\prime}}{\partial x^{\nu}}+\Gamma^{\lambda}{}_{\kappa\nu}\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}+\Gamma^{\lambda}{}_{\mu\nu}\frac{\partial x^{\rho\prime}}{\partial x^{\kappa}}\right].$$

We now subtract from this expression the same expression with κ and ν interchanged

$$
\begin{aligned}
&\frac{\partial^3 x^{\tau\prime}}{\partial x^\kappa \partial x^\mu \partial x^\nu} - \frac{\partial^3 x^{\tau\prime}}{\partial x^\nu \partial x^\mu \partial x^\kappa} = 0 = \qquad (6.9)\\
&\frac{\partial x^{\tau\prime}}{\partial x^\lambda}\left[\left(\frac{\partial}{\partial x^\kappa}\Gamma^\lambda{}_{\mu\nu}\right) + \Gamma^\lambda{}_{\kappa\eta}\Gamma^\eta{}_{\mu\nu}\right]\\
&-\frac{\partial x^{\rho\prime}}{\partial x^\mu}\frac{\partial x^{\sigma\prime}}{\partial x^\nu}\frac{\partial x^{\eta\prime}}{\partial x^\kappa}\left[\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right) - \Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime} - \Gamma^{\tau\prime}{}_{\rho\prime\lambda\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\sigma\prime}\right]\\
&-\frac{\partial x^{\sigma\prime}}{\partial x^\lambda}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\left[\Gamma^\lambda{}_{\kappa\mu}\frac{\partial x^{\rho\prime}}{\partial x^\nu} + \Gamma^\lambda{}_{\kappa\nu}\frac{\partial x^{\rho\prime}}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\nu}\frac{\partial x^{\rho\prime}}{\partial x^\kappa}\right] -\\
&\frac{\partial x^{\tau\prime}}{\partial x^\lambda}\left[\left(\frac{\partial}{\partial x^\nu}\Gamma^\lambda{}_{\mu\kappa}\right) + \Gamma^\lambda{}_{\nu\eta}\Gamma^\eta{}_{\mu\kappa}\right]\\
&+\frac{\partial x^{\rho\prime}}{\partial x^\mu}\frac{\partial x^{\sigma\prime}}{\partial x^\kappa}\frac{\partial x^{\eta\prime}}{\partial x^\nu}\left[\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right) - \Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime} - \Gamma^{\tau\prime}{}_{\rho\prime\lambda\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\sigma\prime}\right]\\
&+\frac{\partial x^{\sigma\prime}}{\partial x^\lambda}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\left[\Gamma^\lambda{}_{\nu\mu}\frac{\partial x^{\rho\prime}}{\partial x^\kappa} + \Gamma^\lambda{}_{\nu\kappa}\frac{\partial x^{\rho\prime}}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\kappa}\frac{\partial x^{\rho\prime}}{\partial x^\nu}\right]
\end{aligned}
$$

collecting all terms we find

$$
\begin{aligned}
&\frac{\partial x^{\tau\prime}}{\partial x^\lambda}\left[\frac{\partial}{\partial x^\kappa}\Gamma^\lambda{}_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\lambda{}_{\mu\kappa} + \Gamma^\lambda{}_{\kappa\eta}\Gamma^\eta{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\eta}\Gamma^\eta{}_{\mu\kappa}\right] \qquad (6.10)\\
&-\frac{\partial x^{\rho\prime}}{\partial x^\mu}\frac{\partial x^{\sigma\prime}}{\partial x^\nu}\frac{\partial x^{\eta\prime}}{\partial x^\kappa}\left[\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime} - \frac{\partial}{\partial x^{\sigma\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\eta\prime} + \Gamma^{\tau\prime}{}_{\lambda\prime\eta\prime}\Gamma^{\lambda\prime}{}_{\sigma\prime\rho\prime} - \Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime}\right] = 0.
\end{aligned}
$$

If we now define the following [1]

$$
R^\lambda{}_{\mu\nu\kappa} = -\left[\frac{\partial}{\partial x^\kappa}\Gamma^\lambda{}_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\lambda{}_{\mu\kappa} + \Gamma^\lambda{}_{\kappa\eta}\Gamma^\eta{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\eta}\Gamma^\eta{}_{\mu\kappa}\right], \qquad (6.11)
$$

we can write eq. (6.10) as the transformation law for the tensor

$$
R^{\sigma\prime}{}_{\alpha\prime\beta\prime\gamma\prime} = \frac{\partial x^{\sigma\prime}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha\prime}}\frac{\partial x^\nu}{\partial x^{\beta\prime}}\frac{\partial x^\kappa}{\partial x^{\gamma\prime}}R^\lambda{}_{\mu\nu\kappa}. \qquad (6.12)
$$

The tensor (6.11) is **The Curvature Tensor**, also called **The Riemann Tensor**, and it can be shown that it is the only tensor that can be constructed by using the metric, its first and second derivatives, and which is linear in the second derivatives.

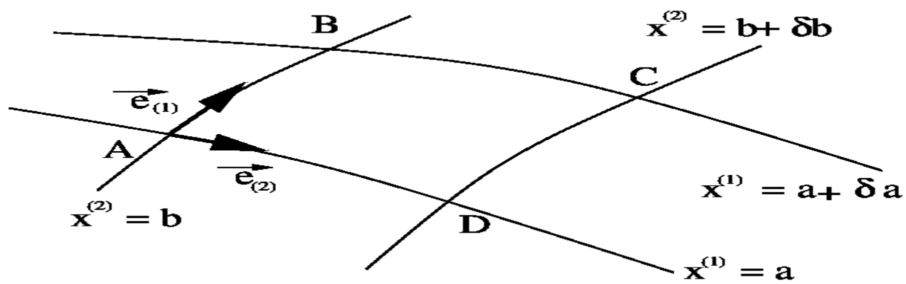
This way of defining the Riemann tensor is the "old-fashioned way": it is based on the transformation properties of the affine connections. The idea underlying this derivation is that the information about the curvature of the space must be contained in the second derivative of the metric, and therefore in the first derivative of the $\Gamma^\alpha{}_{\mu\nu}$. But since we want to find a tensor out of them, we must eliminate in eq. (6.1) the part which does not transform as a tensor, and we do this in eq. (6.9).

[1] The - sign does not agree with the definition given in Weinberg, but it does agree with the definition given in many other textbooks. As we shall see in the next section it is irrelevant. What is important is to write the Einstein equations with the right signs!

6.2 b) The curvature tensor and the curvature of the spacetime

We shall now rederive the curvature tensor in a different way that explicitely shows why it espresses the curvature of a spacetime. This derivation, due to Levi Civita, will use the notion of parallel transport of a vector along a closed loop.

Consider a closed loop whose four sides are the coordinates lines $x^1 = a$, $x^1 = a + \delta a$, $x^2 = b$, $x^2 = b + \delta b$



Take a generic vector $\vec{V}$ and parallely transport $\vec{V}$ along AB, i.e. consider $\nabla_{\vec{e}_{(1)}} \vec{V} = 0$. From eq. (5.57) it follows that

$$e^{\mu}_{(1)} V^{\alpha}{}_{;\mu} = 0. \tag{6.13}$$

Since $\vec{e}_{(1)}$ has only $e^1_{(1)} \neq 0$ then

$$\frac{\partial V^{\alpha}}{\partial x^1} + \Gamma^{\alpha}{}_{\beta 1} V^{\beta} = 0. \tag{6.14}$$

This equation can be integrated along the line AB:

$$\delta V^{\alpha}_{AB} = -\int_{A(x^2=b)}^{B} \Gamma^{\alpha}{}_{\beta 1} V^{\beta} dx^1. \tag{6.15}$$

In a similar way, if we go from B to C along the line $x^1 = a + \delta a$

$$\frac{\partial V^{\alpha}}{\partial x^2} = -\Gamma^{\alpha}{}_{\beta 2} V^{\beta} \qquad \rightarrow \qquad \delta V^{\alpha}_{BC} = -\int_{B(x^1=a+\delta a)}^{C} \Gamma^{\alpha}{}_{\beta 2} V^{\beta} dx^2. \tag{6.16}$$

From C to D

$$\frac{\partial V^{\alpha}}{\partial x^1} = -\Gamma^{\alpha}{}_{\beta 1} V^{\beta} \qquad \rightarrow \qquad \delta V^{\alpha}_{CD} = -\int_{C(x^2=b+\delta b)}^{D} \Gamma^{\alpha}{}_{\beta 1} V^{\beta} dx^1, \tag{6.17}$$

and from D back to A

$$\frac{\partial V^{\alpha}}{\partial x^2} = -\Gamma^{\alpha}{}_{\beta 2} V^{\beta} \qquad \rightarrow \qquad \delta V^{\alpha}_{DA} = -\int_{D(x^1=a)}^{A} \Gamma^{\alpha}{}_{\beta 2} V^{\beta} dx^2. \tag{6.18}$$

The change in $\vec{V}$ due to this parallel transport will be a vector $\delta\vec{V}$ whose components can be found by adding eqs. (6.15)-(6.18):

$$\delta V^\alpha = -\int_{D(x^1=a)}^{A} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 \tag{6.19}$$
$$- \int_{B(x^1=a+\delta a)}^{C} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 - \int_{C(x^2=b+\delta b)}^{D} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1$$
$$- \int_{A(x^2=b)}^{B} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1.$$

If the spacetime is flat V^α do not change when the vector is paralleley transported, i.e. $\delta V^\alpha = 0$. **But in curved spacetime δV^α will in general be different from zero**.

If we consider an infinitesimal loop, i.e. δa and δb tend to zero, we can take an expansion of eq. (6.19) to first order in δa and δb:

$$\delta V^\alpha \simeq -\int_{A(x^2=b)}^{B} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 - \tag{6.20}$$
$$\left[\int_{B(x^1=a)}^{C} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 + \frac{\partial}{\partial x^1}\left(\int_B^C \Gamma^\alpha{}_{\beta 2} V^\beta dx^2\right)\delta a\right]$$
$$-\left[\int_{C(x^2=b)}^{D} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 + \frac{\partial}{\partial x^2}\left(\int_C^D \Gamma^\alpha{}_{\beta 1} V^\beta dx^1\right)\delta b\right]$$
$$-\int_{D(x^1=a)}^{A} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2\,,$$

Since

$$A = (a, b), \quad C = (a + \delta a, b + \delta b), \quad B = (a + \delta a, b), \quad \text{and} \quad D = (a, b + \delta b), \tag{6.21}$$

the previous equation becomes

$$\delta V^\alpha \simeq -\int_a^{a+\delta a} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 \tag{6.22}$$
$$-\int_b^{b+\delta b} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 - \left[\int_b^{b+\delta b} \frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\beta 2} V^\beta\right) dx^2\right]\delta a$$
$$+\int_a^{a+\delta a} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 + \left[\int_a^{a+\delta a} \frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\beta 1} V^\beta\right) dx^1\right]\delta b$$
$$+\int_b^{b+\delta b} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2\,,$$

i.e.

$$\delta V^\alpha \simeq -\delta a \int_b^{b+\delta b} \frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\beta 2} V^\beta\right) dx^2 \tag{6.23}$$
$$+\delta b \int_a^{a+\delta a} \frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\beta 1} V^\beta\right) dx^1 \simeq \delta a \delta b \left[-\frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\beta 2} V^\beta\right) + \frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\beta 1} V^\beta\right)\right].$$

Eq. (6.23) can be further developed by using eq. (6.14)

$$\frac{\partial V^\kappa}{\partial x^1} = -\Gamma^\kappa{}_{\beta 1} V^\beta, \qquad \frac{\partial V^\kappa}{\partial x^2} = -\Gamma^\kappa{}_{\beta 2} V^\beta ; \tag{6.24}$$

it becomes

$$\begin{aligned} \delta V^\alpha &= \delta a \delta b \left[\frac{\partial \Gamma^\alpha{}_{\beta 1}}{\partial x^2} V^\beta + \Gamma^\alpha{}_{\kappa 1} \frac{\partial V^\kappa}{\partial x^2} - \frac{\partial \Gamma^\alpha{}_{\beta 2}}{\partial x^1} V^\beta - \Gamma^\alpha{}_{\kappa 2} \frac{\partial V^\kappa}{\partial x^1} \right] \\ &= \delta a \delta b \left[\frac{\partial \Gamma^\alpha{}_{\beta 1}}{\partial x^2} - \frac{\partial \Gamma^\alpha{}_{\beta 2}}{\partial x^1} - \Gamma^\alpha{}_{\kappa 1} \Gamma^\kappa{}_{\beta 2} + \Gamma^\alpha{}_{\kappa 2} \Gamma^\kappa{}_{\beta 1} \right] V^\beta. \end{aligned} \tag{6.25}$$

Note that:

- δa and δb are the non vanishing components of the displacement vectors $\vec{\delta x}_{(1)}$ and $\vec{\delta x}_{(2)}$ along the direction of the basis vectors $\vec{e}_{(1)}$ and $\vec{e}_{(2)}$, i.e.

$$\vec{\delta x}_{(1)} = \delta a \vec{e}_{(1)} , \qquad \vec{\delta x}_{(2)} = \delta a \vec{e}_{(2)} \tag{6.26}$$

whose components in the basis $\{\vec{e}_{(\alpha)}\}$ are

$$\begin{aligned} \delta x^\mu_{(1)} &= (0, \delta a, 0, 0) = \delta a \ \delta^\mu_1 , \\ \delta x^\mu_{(2)} &= (0, 0, \delta b, 0) = \delta b \ \delta^\mu_2 . \end{aligned} \tag{6.27}$$

Thus, we can write eq. (6.25) as follows

$$\delta V^\alpha = \delta x^\nu_{(1)} \ \delta x^\mu_{(2)} \ \left[\frac{\partial \Gamma^\alpha{}_{\beta\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha{}_{\beta\mu}}{\partial x^\nu} - \Gamma^\alpha{}_{\kappa\nu} \Gamma^\kappa{}_{\beta\mu} + \Gamma^\alpha{}_{\kappa\mu} \Gamma^\kappa{}_{\beta\nu} \right] V^\beta . \tag{6.28}$$

- The term in square

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{\kappa\nu} \Gamma^\kappa{}_{\beta\mu} + \Gamma^\alpha{}_{\kappa\mu} \Gamma^\kappa{}_{\beta\nu} . \tag{6.29}$$

Note that it is antisymmetric in ν and μ; indeed, it must be because, if we interchange $\vec{\delta x}_{(1)}$ and $\vec{\delta x}_{(2)}$, δV^α changes sign, because we would go around the loop in the opposite direction. This shows that the sign of (6.29) can be chosen arbitrarily, and this is the reason why the definitions of the Riemann tensor given in textbooks may differ for a sign.

We have already shown that the object given in eq. (6.29) is a tensor, by looking at the way it transforms under a coordinate transformation (eq. 6.12). However, we want to see if it also agrees with the definition of tensors given in chapter 4. Let us contract eq. (6.28) with V_α.

$$\delta V^\alpha V_\alpha = \delta x^\nu_{(1)} \ \delta x^\mu_{(2)} \ \left[\frac{\partial \Gamma^\alpha{}_{\beta\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha{}_{\beta\mu}}{\partial x^\nu} - \Gamma^\alpha{}_{\kappa\nu} \Gamma^\kappa{}_{\beta\mu} + \Gamma^\alpha{}_{\kappa\mu} \Gamma^\kappa{}_{\beta\nu} \right] V^\beta V_\alpha. \tag{6.30}$$

The result of this contraction is, of course, a number. On the right-hand side there are the components of 3 vectors i.e.: $\delta x^\nu_{(1)}$, $\delta x^\mu_{(2)}$ and V^β; moreover there are the components of the one-form V_α. The four geometrical objects (three vectors and one one-form) are contracted with the quantity within brackets, and the result is a number. In addition, we note that (6.30) is linear in $V^\beta, V_\alpha, \delta x^\nu_{(1)}\ \delta x^\mu_{(2)}$. For instance, if we consider a displacement $\delta x^\nu_{(1a)} + \delta x^\nu_{(1b)}$ along $\vec{e}_{(1)}$ it is immediate to see that

$$\delta V^\alpha V_\alpha = \delta x^\nu_{(1a)} \delta x^\mu_{(2)} \, [...] \, V^\beta V_\alpha + \delta x^\nu_{(1b)} \delta x^\mu_{(2)} \, [...] \, V^\beta V_\alpha \,, \tag{6.31}$$

and similarly for the other quantities. If we consider a generic $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ tensor, $T^\alpha{}_{\beta\gamma\delta}$, since by definition it is a linear function of one one-form and three vectors, when supplied with these arguments (for example the one-form $\tilde{V}$, and the three vectors $\vec{V}$, $\vec{\delta x}_{(1)}$ and $\vec{\delta x}_{(2)}$ it will produce the following number

$$T(\tilde{V}, \vec{V}, \vec{\delta x}_{(2)}, \vec{\delta x}_{(1)}) = T^\alpha{}_{\beta\rho\delta} V_\alpha V^\beta \delta x^\rho_{(2)} \ \delta x^\delta_{(1)} \ . \tag{6.32}$$

Eq. (6.32) has the same structure of eq. (6.30). Therefore we are entitled to define the components of the Riemann tensor as in eq. (6.29).

It should now be clear why the Riemann tensor deserves its name of **Curvature Tensor**: it tells us how does a vector change when it is parallely transported along a loop, due to the curvature of the spacetime. If the spacetime is flat

$$\delta V^\alpha = 0 \qquad \text{along any closed loop} \qquad \to \qquad R^\alpha{}_{\beta\gamma\delta} = 0, \tag{6.33}$$

in any reference frame. Indeed, if a tensor vanishes in a given frame, then it vanishes in any other frame.

The components of the Riemann tensor assume a very nice form when computed in a locally inertial frame:

$$R^\alpha{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left[g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu} \right], \tag{6.34}$$

or lowering the index α

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^\lambda{}_{\beta\mu\nu} = \frac{1}{2} \left[g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu} \right]. \tag{6.35}$$

It should be stressed that

1) The Riemann tensor is linear in the second derivatives of $g_{\mu\nu}$, and non linear in the first derivatives.

2) In a locally inertial frame the $\Gamma^\alpha{}_{\nu\sigma}$ vanish and therefore the non-linear part of the Riemann tensor vanishes as well.

6.3 Symmetries

From eq. (6.35) it is easy to verify that

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}, \tag{6.36}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0. \tag{6.37}$$

Since $R_{\alpha\beta\mu\nu}$ is a tensor, these symmetry properties are valid in any reference frame. The symmetries of the Riemann tensor reduce the number of independent components to 20.

6.4 The Riemann tensor gives the commutator of covariant derivatives

Let us consider the second covariant derivatives of a vector field $\vec{V}$

$$\nabla_\alpha \nabla_\beta V^\mu = \nabla_\alpha (V^\mu{}_{;\beta}) = (V^\mu{}_{;\beta})_{,\alpha} + \Gamma^\mu{}_{\sigma\alpha} V^\sigma{}_{;\beta} - \Gamma^\sigma{}_{\beta\alpha} V^\mu{}_{;\sigma}. \tag{6.38}$$

In a locally inertial frame $\Gamma^\mu{}_{\sigma\alpha} = 0$, and eq. (6.38) becomes

$$\nabla_\alpha \nabla_\beta V^\mu = (V^\mu{}_{;\beta})_{,\alpha} = V^\mu{}_{,\beta,\alpha} + \Gamma^\mu{}_{\nu\beta,\alpha} V^\nu. \tag{6.39}$$

By interchanging α and β

$$\nabla_\beta \nabla_\alpha V^\mu = (V^\mu{}_{;\alpha})_{,\beta} = V^\mu{}_{,\alpha,\beta} + \Gamma^\mu{}_{\nu\alpha,\beta} V^\nu. \tag{6.40}$$

The commutator of the covariant derivatives then is

$$[\nabla_\alpha, \nabla_\beta]\, V^\mu = \nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = (\Gamma^\mu{}_{\nu\beta,\alpha} - \Gamma^\mu{}_{\nu\alpha,\beta})\, V^\nu. \tag{6.41}$$

Since in a locally inertial frame

$$R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu{}_{\nu\beta,\alpha} - \Gamma^\mu{}_{\nu\alpha,\beta} \tag{6.42}$$

(equivalent to eq. 6.34), eq. (6.41) becomes

$$[\nabla_\alpha, \nabla_\beta]\, V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu. \tag{6.43}$$

This is a tensor equation and since it is valid in a given reference frame, it will be valid in **any** frame. Eq. (6.43) implies that in curved spacetime covariant derivatives **do not commute** and therefore the order in which they appear is important.

6.5 The Bianchi identities

Let us differentiate eq. (6.35) with respect to x^λ (and rememeber that it is valid in a locally inertial frame)

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}\left[g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}\right]. \tag{6.44}$$

By using the fact that $g_{\alpha\beta}$ is symmetric and eq. (6.44) one can show that

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0. \tag{6.45}$$

Since it is valid in a locally inertial frame and it is a tensor equation, it will be valid in any frame:

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0, \tag{6.46}$$

where we have replaced the ordinary derivative with the covariant derivative. **These are the Bianchi identities that, as we shall see, play an important role in the development of the theory.**

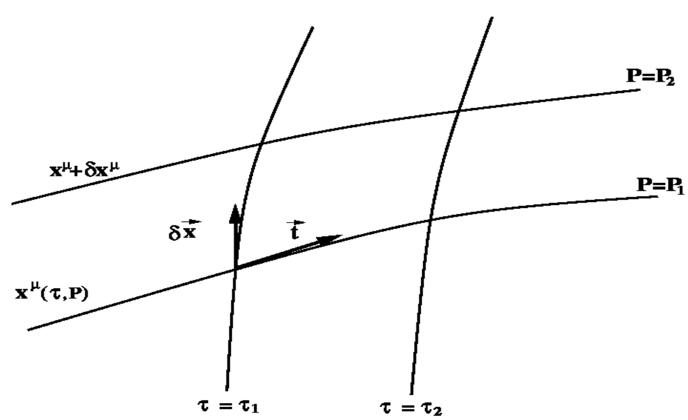
Chapter 7

The Geodesic deviation

The Principle of equivalence establishes that we can always choose a locally inertial frame where the affine connections vanish and the metric becomes that of a flat spacetime. Conversely, if the spacetime is flat we can always define a coordinate system which "simulates", locally, the existence of any arbitrary gravitational field. In this frame we could measure the "simulated" gravitational force by studying the motion of a single particle, but these measurements would never allow us to know whether that force is simulated or real: this can be understood only by comparing the motion of close particles, i.e. by comparing the behaviour of close geodesics.

7.1 The equation of geodesic deviation

Consider two particles moving along the trajectories $x^\mu(\tau)$ and $x^\mu(\tau)+\delta x^\mu(\tau)$, where δx^μ is the vector of separation between the two close geodesics, and τ is an **affine parameter**. This is equivalent to say: consider a two-parameter family of geodesics $x^\mu(\tau, p)$, where the parameter p labels different geodesics



Be

$$t^\alpha = \frac{\partial x^\alpha}{\partial \tau} \tag{7.1}$$

the tangent vector to the geodesic line, and be

$$\delta x^\alpha = \frac{\partial x^\alpha}{\partial p}. \tag{7.2}$$

Note that

$$\frac{\partial t^\alpha}{\partial p} = \frac{\partial \delta x^\alpha}{\partial \tau}. \tag{7.3}$$

We now compute the covariant derivative of the vector $\vec{t}$ along the curve $\tau = \text{const}$ whose tangent vector is δx^μ, i.e. $\nabla_{\vec{\delta x}}\,\vec{t}$. The components of this vector are

$$\left(\nabla_{\vec{\delta x}}\,\vec{t}\right)^\alpha = \frac{\partial x^\mu}{\partial p}\left[\frac{\partial t^\alpha}{\partial x^\mu} + \Gamma^\alpha{}_{\mu\nu}t^\nu\right] = \frac{\partial t^\alpha}{\partial p} + \Gamma^\alpha{}_{\mu\nu}t^\nu\delta x^\mu. \tag{7.4}$$

Similarly, the covariant derivative of the vector $\vec{\delta x}$ along the curve $p = \text{const}$, i.e. along the geodesic, has components

$$\left(\nabla_{\vec{t}}\,\vec{\delta x}\right)^\alpha = t^\mu \delta x^\alpha{}_{;\mu} = \frac{\partial \delta x^\alpha}{\partial \tau} + \Gamma^\alpha{}_{\mu\nu}\delta x^\nu t^\mu. \tag{7.5}$$

From eq. (7.3) and from the symmetry of $\Gamma^\alpha{}_{\mu\nu}$ in the lower indices it follows that

$$\nabla_{\vec{t}}\,\vec{\delta x} = \nabla_{\vec{\delta x}}\,\vec{t}. \tag{7.6}$$

The quantities $\left(\nabla_{\vec{t}}\,\vec{\delta x}\right)^\alpha$ or $\left(\nabla_{\vec{\delta x}}\,\vec{t}\right)^\alpha$ involve only the affine connections, and therefore they do not give significant information on the gravitational field. We then compute the second covariant derivative of the vector $\vec{\delta x}$ along the curve $p = const$, i.e $\nabla_{\vec{t}}\left(\nabla_{\vec{t}}\,\vec{\delta x}\right)$. We define the following operator:

$$\frac{D}{d\tau}\delta x^\alpha \equiv \left(\nabla_{\vec{t}}\,\vec{\delta x}\right)^\alpha = t^\mu \delta x^\alpha{}_{;\mu}\ . \tag{7.7}$$

With this definition,

$$\frac{D^2\delta x^\alpha}{d\tau^2} = \left(\nabla_{\vec{t}}\left(\nabla_{\vec{t}}\,\vec{\delta x}\right)\right)^\alpha. \tag{7.8}$$

This quantity, called **geodesic deviation**, is a vector describing the relative acceleration of two nearby geodesics.

In order to compute the geodesic deviation, let us consider the commutator

$$\left[\nabla_{\vec{t}}, \nabla_{\vec{\delta x}}\right]\vec{t} = \nabla_{\vec{t}}\left(\nabla_{\vec{\delta x}}\,\vec{t}\right) - \nabla_{\vec{\delta x}}\left(\nabla_{\vec{t}}\,\vec{t}\right). \tag{7.9}$$

whose components are

$$\begin{aligned}\left[\nabla_{\vec{t}}\left(\nabla_{\vec{\delta x}}\,\vec{t}\right)\right]^\alpha &= t^\mu\left(\delta x^\nu\ t^\alpha{}_{;\nu}\right)_{;\mu} - \delta x^\mu\left(t^\nu\ t^\alpha{}_{;\nu}\right)_{;\mu} \\ &= t^\mu\ \delta x^\nu{}_{;\mu}t^\alpha{}_{;\nu} + t^\mu\ \delta x^\nu\ t^\alpha{}_{;\nu;\mu} - \delta x^\mu\ t^\nu{}_{;\mu}\ t^\alpha{}_{;\nu} - \delta x^\mu\ t^\nu t^\alpha{}_{;\nu;\mu} \\ &= \left(t^\mu\ \delta x^\nu{}_{;\mu} - \delta x^\mu\ t^\nu{}_{;\mu}\right)t^\alpha{}_{;\nu} + \left(t^\alpha{}_{;\nu;\mu} - t^\alpha{}_{;\mu;\nu}\right)t^\mu\ \delta x^\nu.\end{aligned} \tag{7.10}$$

From eq. (7.6) we find that

$$t^\mu\ \delta x^\nu{}_{;\mu} = \delta x^\mu\ t^\nu{}_{;\mu},$$

and eq. (7.11) becomes

$$\left[\nabla_{\vec{t}}\left(\nabla_{\vec{\delta x}}\ \vec{t}\right)\right]^{\alpha} = \left(t^{\alpha}{}_{;\nu:\mu} - t^{\alpha}{}_{;\mu;\nu}\right) t^{\mu}\delta x^{\nu}. \tag{7.11}$$

We now remind that, according to eq. (6.43), the commutator of covariant derivatives is

$$\left(t^{\alpha}{}_{;\nu:\mu} - t^{\alpha}{}_{;\mu;\nu}\right) = R^{\alpha}{}_{\beta\mu\nu}t^{\beta}, \tag{7.12}$$

therefore eq. (7.11) becomes

$$\left[\nabla_{\vec{t}}\left(\nabla_{\vec{\delta x}}\ \vec{t}\right)\right]^{\alpha} = R^{\alpha}{}_{\beta\mu\nu}t^{\beta}t^{\mu}\delta x^{\nu}. \tag{7.13}$$

Moreover, since t^{μ} is the geodesic tangent vector, when it is parallel-transported along the geodesic it gives (see Section 5.9)

$$\nabla_{\vec{t}}\,\vec{t} = 0; \tag{7.14}$$

as a consequence $\nabla_{\vec{\delta x}}\left(\nabla_{\vec{t}}\,\vec{t}\right) = 0$ and the commutator (7.9) can be rewritten as

$$\left(\left[\nabla_{\vec{t}}, \nabla_{\vec{\delta x}}\right]\ \vec{t}\right)^{\alpha} = \left(\nabla_{\vec{t}}\left(\nabla_{\vec{t}}\ \vec{\delta}x\right)\right)^{\alpha} = R^{\alpha}{}_{\beta\mu\nu}t^{\beta}t^{\mu}\delta x^{\nu}. \tag{7.15}$$

By direct substitution of this expression in eq. (7.8) we finally find

$$\frac{D^2\delta x^{\alpha}}{d\tau^2} = R^{\alpha}{}_{\beta\mu\nu}\ t^{\beta}\ t^{\mu}\ \delta x^{\nu}. \tag{7.16}$$

This is the equation of geodesic deviation, which shows that the relative acceleration of nearby particles moving along geodesics depends on the curvature tensor. Since the Riemann tensor is zero if and only if the gravitational field is either zero or constant and uniform, the equation of the geodesic deviation really contains the information on the gravitational field in a given spacetime.

Chapter 8

The stress-energy tensor

Now we know that there exists a tensor which allows to understand if the spacetime is curved or flat, i.e. if we are in the presence of a non-constant, non-uniform gravitational field. But in order to derive Einstein's equations, we still need to answer the following question: **how do we describe matter and fields in general relativity?** This question is relevant because we want to find what to put on the right-hand-side of the equations as a source of the gravitational field.

We shall first define the stress-energy tensor in flat spacetime, and then generalize this notion to a generic spacetime.

In Special Relativity, we define the **energy-momentum** four-vector of a particle of mass m and velocity $\mathbf{v} = \frac{d\boldsymbol{\xi}}{dt}$ in the following way

$$p^\alpha = mcu^\alpha, \qquad \alpha = 0, 3, \tag{8.1}$$

where $u^\alpha = \frac{d\xi^\alpha}{d\tau}$ is the four-velocity ($u^\alpha u_\alpha = -1$); τ , which has the dimensions of a length, is related to the particle proper time by the equation: $\left[\textbf{proper time} = \frac{\mathbf{1}}{\mathbf{c}}\boldsymbol{\tau}\right]$. In what follows, we shall indicate in boldface tri-vectors, for instance $\mathbf{v}$, whereas four-vectors will be indicated with an arrow, i.e. $\vec{A}$. Also remember that $\{\xi^\alpha\}$ are Minkowskian coordinates of flat spacetime, or of a locally inertial frame.

Note that $\xi^0 = ct$ and, defining

$$\gamma = \frac{d\xi^0}{d\tau}\,, \tag{8.2}$$

we have:

$$\begin{aligned} u^0 &= \gamma \\ u^i &= \frac{d\xi^i}{d\tau} = \frac{d\xi^i}{dt}\frac{dt}{d\tau} = v^i\frac{\gamma}{c} \\ u^\alpha u^\beta \eta_{\alpha\beta} &= -\gamma^2\left(1 - \frac{v^2}{c^2}\right) = -1 \qquad \Rightarrow \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \end{aligned} \tag{8.3}$$

We have then

$$p^\mu = m(c\gamma, \gamma\mathbf{v})\,. \tag{8.4}$$

The time-component of the energy-momentum vector does represent the energy of the particle

$$p^0 = \frac{E}{c}, \quad \text{and} \quad E = mc^2\gamma. \tag{8.5}$$

The space-components are the components of the three-dimensional momentum

$$\mathbf{p} = m\gamma\mathbf{v}. \tag{8.6}$$

What does it change if we are dealing with a continuous or discrete distribution of matter and energy? In that case we should be able to measure some other quantities, as the mass and the energy which are contained in a unitary volume, or the flux of energy and momentum that flows across the different faces of this volume. This information is contained in the **stress-energy tensor** we are now going to define.

Let us consider the simple case of a system of n non-interacting particles located at some points $\xi_n(t)$, each with an energy-momentum vector p_n^α.
We define the **density of energy** as

$$T^{00} \equiv \sum_n cp_n^0(t)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)) = \sum_n E_n\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)), \tag{8.7}$$

the **density of momentum** $\frac{1}{c}T^{0i}$, where T^{0i} is defined as

$$T^{0i} \equiv \sum_n cp_n^i(t)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)), \qquad i = 1,3 \tag{8.8}$$

and the **current** of momentum as

$$T^{ki} \equiv \sum_n p_n^k(t)\frac{d\xi_n^i(t)}{dt}\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)), \qquad k = 1,3 \quad i = 1,3. \tag{8.9}$$

$\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n)$ is the Dirac delta-function defined by the statement that for any smooth function $f(\boldsymbol{\xi})$

$$\int d^3\xi \; f(\boldsymbol{\xi})\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n) = f(\boldsymbol{\xi}_n), \tag{8.10}$$

and if $\boldsymbol{\xi}_n = (x0, y0, z0)$

$$\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n) = \delta(x - x0)\delta(y - y0)\delta(z - z0), \tag{8.11}$$

or, in polar coordinates

$$\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n) = \frac{1}{r^2\sin^2\theta}\delta(r - r0)\delta(\theta - \theta_0)\delta(\varphi - \varphi_0). \tag{8.12}$$

Thus, according to the definition (8.10) the three-dimensional δ-function has the dimensions of the inverse of a cubic lenght l^{-3}. For this reason, for example, T^{00} is, dimensionally, an energy ($[cp^0]$) divided by a volume ($[\delta^3]$) and therefore it is the energy density of the system[1] The definitions (8.7),(8.8) and (8.9) can be unified into a single formula

$$T^{\alpha\beta} = \sum_n p_n^\alpha \frac{d\xi_n^\beta(t)}{dt} \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)), \qquad \alpha, \beta = 0, 3, \tag{8.13}$$

or, since

$$p_n^\alpha = \frac{E_n}{c^2} \frac{d\xi_n^\alpha(t)}{dt}, \tag{8.14}$$

eq. (8.13) can also be written as

$$T^{\alpha\beta} = c^2 \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)), \tag{8.15}$$

which clearly shows that $T^{\alpha\beta}$ is **symmetric**

$$T^{\alpha\beta} = T^{\beta\alpha}. \tag{8.16}$$

Finally, an alternative way of writing eq. (8.13) is

$$T^{\alpha\beta} = c \sum_n \int p_n^\alpha \frac{d\xi_n^\beta}{d\tau_n} \delta^4(\vec{\xi} - \vec{\xi}_n(\tau_n)) d\tau_n, \tag{8.17}$$

where

$$\delta^4(\vec{\xi} - \vec{\xi}_{\mathbf{n}}) = \delta(\xi^{\mathbf{0}} - \xi_{\mathbf{n}}^{\mathbf{0}})\delta(\xi^{\mathbf{1}} - \xi_{\mathbf{n}}^{\mathbf{1}})\delta(\xi^{\mathbf{2}} - \xi_{\mathbf{n}}^{\mathbf{2}})\delta(\xi^{\mathbf{3}} - \xi_{\mathbf{n}}^{\mathbf{3}}); \tag{8.18}$$

indeed, using the property (8.10) of the δ-function it is easy to see that

$$\begin{aligned}
T^{\alpha\beta} &= c \sum_n \int p_n^\alpha \frac{d\xi_n^\beta}{d\tau_n} \delta^4(\vec{\xi} - \vec{\xi}_n(\tau_n)) \, d\tau_n \\
&= c \sum_n \int \left[p_n^\alpha \frac{d\xi_n^\beta}{d\tau_n} \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(\tau_n)) \right] \delta(\xi^0 - \xi_n^0(\tau_n)) \frac{d\tau_n}{d\xi_n^0} \, d\xi_n^0 \\
&= c \sum_n \left[p_n^\alpha \frac{d\xi_n^\beta}{d\xi_n^0} \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(\tau_n)) \right]_{\xi^0(\tau_n) = \xi^0} \\
&= c \sum_n p_n^\alpha \frac{d\xi_n^\beta}{d\xi^0} \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(\xi^0)) = \sum_n p_n^\alpha \frac{d\xi_n^\beta}{dt} \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(\xi^0))
\end{aligned} \tag{8.19}$$

which coincides with (8.15)

[1] Properties of the δ-function

$$\delta(x) = \delta(-x), \qquad \delta(cx) = \frac{1}{|c|}\delta(x)$$

$$\delta[g(x)] = \sum_j \frac{1}{|g'(x^j)|}\delta(x - x^j) \qquad x\delta(x) = 0$$

$$\int dx f(x)\delta'(x - x_0) = -f'(x_0).$$

Summarizing, the meaning of the different components is the following

T^{00} = energy-density. In the non-relativistic case $v << c$, $p^0_n \sim m_n c$ and $T^{00} \sim \sum_n m_n c^2 \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t))$ reduces to the density of matter ρc^2 where

$$\rho = \sum_n m_n \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)) \tag{8.20}$$

(remember the dimensions of the δ-function) .

$\frac{1}{c} T^{0i}$ = density of momentum. Since the dimensions of the momentum $\vec{p}$ are those of an energy divided by a velocity, $[p^0] = [E/c]$, it follows that cT^{0i} has the dimensions of $[\frac{E}{tS}]$, i.e. it is the energy which flows across the unit surface orthogonal to the axis ξ^i per unit time (i=1,3) (see eq. (8.8)).

Similar dimensional considerations allow us the interpret T^{ik} as the flux of the i-th component of the three-momentum $\mathbf{p}$ across the unit surface orthogonal to the axis ξ^k (i,k=1,3) (see eq. (8.9)).

Now we must check several things:

1) is $T^{\alpha\beta}$ a tensor?

2) does it satisfy any conservation law? (remember that the energy-momentum four vector does satisfy a conservation law).

3) if it does, how to write this law in a curved spacetime, i.e. in the presence of a gravitational field?

1) **is $T^{\alpha\beta}$ a tensor?**

Let us consider a **generic** coordinate transformation

$$\{\xi^\alpha\} \longrightarrow \{x^{\alpha'}\} \quad \longrightarrow \quad \xi^\alpha = \Lambda^\alpha{}_{\gamma'}\, x^{\gamma'}, \tag{8.21}$$

The four-momentum and the four-velocity transform as

$$p^\alpha = \Lambda^\alpha{}_{\gamma'}\, p^{\gamma'}, \qquad\qquad u^\alpha \equiv \frac{d\xi^\alpha}{d\tau} = \Lambda^\alpha{}_{\gamma'}\, u^{\gamma'}. \tag{8.22}$$

In order to see how $T^{\alpha\beta}$ transforms we need a brief digression to show how to transform $\delta^4(x)$.

In a four dimensional spacetime the volume element which is invariant under a generic coordinate transformation is $\sqrt{-g}\; d^4x$, i.e.

$$\sqrt{-g}\; d^4x = \sqrt{-g'}\; d^4x'. \tag{8.23}$$

Indeed,

$$d^4x = |J|\; d^4x', \tag{8.24}$$

where $J = \det\left(\frac{\partial x^\alpha}{\partial x^{\beta'}}\right)$ is the Jacobian associated to the coordinate transformation. Since

$$g_{\alpha'\beta'} = \frac{\partial x^\mu}{\partial x^{\alpha'}}\frac{\partial x^\nu}{\partial x^{\beta'}} g_{\mu\nu}, \tag{8.25}$$

taking the determinant of both member we get

$$g' = J^2 g \qquad \text{and} \quad \text{therefore} \qquad \sqrt{-g} = \frac{1}{|J|}\sqrt{-g'}. \tag{8.26}$$

Thus, if $\{\xi^\alpha\}$ is a Minkowskian frame, and $\{x^\alpha\}$ is a generic frame,

$$d^4\xi = \sqrt{-g}\; d^4x. \tag{8.27}$$

Let us now consider a delta-function in Minkowski's spacetime; by definition, for every function $f(\vec{\xi})$,

$$\int d^4\xi \; f(\vec{\xi})\delta^4(\vec{\xi} - \vec{\xi}_n) = f(\vec{\xi}_n)\,, \tag{8.28}$$

and, in a generic frame,

$$\int d^4x \; f(\vec{x})\delta^4(\vec{x} - \vec{x}_n) = f(\vec{x}_n)\,. \tag{8.29}$$

Let us now perform a coordinate transformation $\xi^\alpha \to x^\alpha$, with $x^\alpha = x^\alpha(\xi^\mu)$, and define $\tilde{f}(\vec{\xi}) = f(\vec{x}(\vec{\xi}))$; multiplying and dividing Eq. (8.29) by $\sqrt{-g}$, it gives

$$\int \sqrt{-g}\; d^4x \; f(\vec{x})\frac{\delta^4(\vec{x} - \vec{x}_n)}{\sqrt{-g}} = \int \tilde{f}(\vec{\xi})\frac{\delta^4(\vec{x} - \vec{x}_n)}{\sqrt{-g}}\; d^4\xi = \tilde{f}(\vec{\xi}_n) \tag{8.30}$$

which is valid for every function f. Comparing eqs. (8.28) and (8.30) it follows

$$\delta^4(\vec{\xi} - \vec{\xi}_n) = \frac{\delta^4(\vec{x} - \vec{x}_n)}{\sqrt{-g}}. \tag{8.31}$$

Using eqs. (8.17), (8.22) and (8.31) it is now easy to find the transformation rule for $T^{\alpha\beta}$:

$$T^{\alpha\beta} = c\sum_n \int \Lambda^\alpha{}_{\gamma'}\,\Lambda^\beta{}_{\delta'}\,p_n^{\gamma'}\,\frac{dx_n^{\delta'}}{d\tau_n}\;\frac{\delta^4(\vec{x} - \vec{x}_n)}{\sqrt{-g}}\; d\tau_n. \tag{8.32}$$

Therefore if we define

$$T^{\alpha\beta} = c\sum_n \int \frac{1}{\sqrt{-g}}\; p_n^\alpha\; \frac{dx_n^\beta}{d\tau_n}\; \delta^4(\vec{x} - \vec{x}_n)\; d\tau_n \;, \tag{8.33}$$

under a generic coordinate transformation it will transform like

$$T^{\alpha\beta} = \Lambda^\alpha{}_{\gamma'}\,\Lambda^\beta{}_{\delta'}\,T^{\gamma'\,\delta'}. \tag{8.34}$$

and therefore it is a tensor. In flat spacetime, and in a locally inertial frame $\sqrt{-g} = 1$ and we recover the definition (8.17). In conclusion, eq. (8.33) is the stress-energy tensor appropriate to describe a cloud of non interacting particles both in flat and in curved spacetime. Of course we may have different kind of matter and/or energy: a fluid, an electromagnetic field, etc. In that case it is possible to show that the corresponding stress-energy tensor can be derived by writing the action of the considered field, and by varying this action with respect to $g_{\mu\nu}$. However, the physical meaning of the different components of $T^{\alpha\beta}$ will be the same.

We shall now use the tensor we have derived to answer the second important question we raised. The answer will be valid for the stress-energy tensor of any sort of matter-energy.

2) **Does $T^{\alpha\beta}$ satisfy a conservation law?**

Let us assume that we are in flat spacetime. Let us differentiate the (αi)-components of the stress-energy tensor given in eq. (8.13):

$$\frac{\partial T^{\alpha i}}{\partial \xi^i} = \sum_n p_n^\alpha(t)\, \frac{d\xi_n^i(t)}{dt}\, \frac{\partial}{\partial \xi^i}\, \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)), \tag{8.35}$$

where $\alpha = 0,3$ and $i = 1,3$[2]. Since

$$\frac{\partial}{\partial \xi^i}\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)) = -\frac{\partial}{\partial \xi_n^i}\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)), \tag{8.36}$$

eq. (8.35) becomes

$$\begin{aligned} \frac{\partial T^{\alpha i}}{\partial \xi^i} &= -\sum_n p_n^\alpha(t)\, \frac{d\xi_n^i(t)}{dt}\, \frac{\partial}{\partial \xi_n^i}\, \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)) \\ &= -\sum_n p_n^\alpha(t)\, \frac{\partial}{\partial t}\, \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)). \end{aligned} \tag{8.37}$$

Let us now differentiate the $T^{\alpha 0}$-component with respect to $\xi^0 = ct$:

$$\frac{\partial T^{\alpha 0}}{\partial \xi^0} = \sum_n \frac{dp_n^\alpha(t)}{dt}\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)) + \sum_n p_n^\alpha(t)\, \frac{\partial}{\partial t}\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)) \tag{8.38}$$

Since

$$\frac{dp_n^\alpha(t)}{dt} = \frac{dp_n^\alpha(\tau)}{d\tau}\frac{d\tau}{dt} = \frac{d\tau}{dt}f_n^\alpha, \tag{8.39}$$

where f_n^α is the relativistic force, the first term in eq. (8.38) can be considered as a *density of force* G^α defined as

$$G^\alpha(\boldsymbol{\xi}, t) = \sum_n \frac{dp_n^\alpha(t)}{dt}\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t)) = \sum_n \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_n(t))\frac{d\tau}{dt}f_n^\alpha. \tag{8.40}$$

It is a density because the δ-function is $[l^{-3}]$. If the particles are free, $f_n^\alpha = 0$ and adding eq. (8.37) and eq. (8.38) we find

$$\frac{\partial}{\partial \xi^0}T^{\alpha 0} + \frac{\partial}{\partial \xi^i}T^{\alpha i} = 0 \quad \to \quad \frac{\partial T^{\alpha\beta}}{\partial \xi^\beta} = 0, \tag{8.41}$$

or

$$T^{\alpha\beta}{}_{,\beta} = 0, \tag{8.42}$$

[2] Remind: $\frac{\partial T^{\alpha i}}{\partial \xi^i} = \frac{\partial T^{\alpha 1}}{\partial \xi^1} + \frac{\partial T^{\alpha 2}}{\partial \xi^2} + \frac{\partial T^{\alpha 3}}{\partial \xi^3}$.

which is the conservation law we were looking for.

Why is $T^{\alpha\beta}{}_{,\beta} = 0$ a conservation law? To answer this question, let us derive a conservation law of classical electrodynamics. Consider, as an example, a collection of charged particles of density $\rho = dq/dV$, enclosed in a volume V.

$$\frac{\partial}{\partial t}\int_V \rho dV \tag{8.43}$$

will be the variation of charge inside the volume V. Be S the surface enclosing the volume, and $\mathbf{n}$ the normal vector, which is assumed to be positive if pointing outward.

$$\rho\mathbf{v}\cdot\mathbf{n}dS \tag{8.44}$$

will be the charge which flows across dS per unit time. It is positive if the charge goes out, negative if it flows in. Thus

$$\int_S \rho\mathbf{v}\cdot\mathbf{n}dS \tag{8.45}$$

is the total charge per unit time which flows across the surface S enclosing the volume V. The continuity equation then says that

$$\frac{\partial}{\partial t}\int_V \rho\ dV = -\int_S \rho\mathbf{v}\cdot\mathbf{n}\ dS. \tag{8.46}$$

The minus sign is because the right-hand side is positive if the charge contained in V increases. If we now introduce the three-dimensional current

$$\mathbf{J} = \rho\mathbf{v}, \tag{8.47}$$

eq. (8.46) becomes

$$\frac{\partial}{\partial t}\int_V \rho\ dV = -\int_S \mathbf{J}\cdot\mathbf{n}\ dS. \tag{8.48}$$

Since

$$\mathbf{J}\cdot\mathbf{n}\ dS = \eta_{ik}n^iJ^k\ dS = J^kn_kdS,$$

by putting $dS_k = n_kdS$, and using Gauss' theorem

$$\int_S J^kdS_k = \int_V div\ \mathbf{J}\ dV, \tag{8.49}$$

eq. (8.48) becomes

$$\frac{\partial}{\partial t}\int_V \rho\ dV = -\int_V div\ \mathbf{J}\ dV. \tag{8.50}$$

Since the volume V is arbitrary, we can write

$$div\ \mathbf{J} = -\frac{\partial\rho}{\partial t}, \tag{8.51}$$

or

$$\frac{\partial J^1}{\partial\xi^1} + \frac{\partial J^2}{\partial\xi^2} + \frac{\partial J^3}{\partial\xi^3} = -\frac{\partial\rho}{\partial t}, \tag{8.52}$$

which is the **continuity equation** in a differential form. Let us now transform eq. (8.51) in a four-dimensional form. We define a four-current

$$J^\alpha = \rho \frac{d\xi^\alpha}{dt} = (\rho c, \mathbf{J}), \tag{8.53}$$

and eq. (8.51) becomes

$$\frac{\partial J^\alpha}{\partial \xi^\alpha} = 0, \qquad \alpha = 0, 3. \tag{8.54}$$

We are now going to show that any current $J^\alpha(x)$ which satisfies the conservation law (8.54) is associated to a total charge Q defined as

$$Q = \int_V J^0 dV, \tag{8.55}$$

which is conserved. **The integral in eq. (8.55) is evaluated at some fixed time, thus we say that the integration is performed on a hypersurface $\xi^0 = const$ over the whole three-dimensional space.** The total charge Q is a conserved quantity for the following reason. By virtue of eq. (8.54)

$$\frac{1}{c}\frac{dQ}{dt} = \int_{allspace} \frac{1}{c}\frac{\partial}{\partial t} J^0\ dV = -\int_{allspace} div\ \mathbf{J}\ dV = -\int_{surface} J^k dS_k. \tag{8.56}$$

The last equality follows from Gauss' theorem, and the subscript 'surface' means that we are considering the flux of $\mathbf{J}$ across the surface which encloses the whole space. If $\mathbf{J}$ goes to zero at infinity, the last term in eq. (8.56) vanishes, and therefore the total charge Q is a conserved quantity.

Let us now go back to equation (8.42), and set, as an example, $\alpha = 0$:

$$\frac{\partial T^{00}}{\partial \xi^0} = -\left[\frac{\partial T^{01}}{\partial \xi^1} + \frac{\partial T^{02}}{\partial \xi^2} + \frac{\partial T^{03}}{\partial \xi^3}\right]. \tag{8.57}$$

By integrating over a volume V which, as before, extends to all space, we get

$$\frac{\partial}{\partial \xi^0}\int_V T^{00} dV = -\int_V \frac{\partial T^{0k}}{\partial \xi^k}\ dV = -\int_S T^{0k} dS_k. \tag{8.58}$$

Since T^{0k} is the energy which flows across the unit surface orthogonal to ξ^k, if we assume that this energy flow is zero at infinity we finally find

$$\frac{\partial}{\partial \xi^0}\int_V T^{00} dV = 0 \quad \to \quad \int_V T^{00} dV = constant. \tag{8.59}$$

which expresses the conservation of energy. A similar procedure can be used to find the conservation of momentum by putting $\alpha = i = 1, 2, 3$. In this case we find

$$\frac{\partial}{\partial \xi^0}\int_V T^{i0} dV = -\int_V \frac{\partial T^{ik}}{\partial \xi^k}\ dV = -\int_S T^{ik} dS_k. \tag{8.60}$$

Assuming that the momentum currents vanish at infinity, we find

$$\frac{\partial}{\partial \xi^0}\int_V T^{i0}dV = 0 \quad \rightarrow \quad \int_V T^{i0}dV = constant. \tag{8.61}$$

In conclusion, we can define a vector

$$P^\alpha = \int_V T^{\alpha 0}dV, \qquad \alpha = 0, 3, \tag{8.62}$$

which can be identified as the conserved energy-momentum vector of the system. It should be reminded that this derivation has been carried out in the framework of Special Relativity.

3) **How do we write this conservation law in curved spacetime?**

In order to answer this question we need to state The Principle of General Covariance which will be the foundation of the theory of General Relativity:

8.1 The Principle of General Covariance

A physical law is true if:

1) it is true in the absence of gravity, i.e. it reduces to the laws of special relativity when $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ *and* $\Gamma^\alpha{}_{\mu\nu}$ *vanish.* It is clear that this first proposition includes the Equivalence Principle.

2) In order to preserve their form under an arbitrary coordinate transformation, all equations must be generally covariant. This means that all equations must be expressed in a tensor form.

The physical content of the Principle of General Covariance is that if a tensor equation is true in absence of gravity, then it is true in the presence of an *arbitrary* gravitational field. It should also be stressed that the Principle of General Covariance can be applied only on scales that are small compared with the typical distances associated to the gravitational field, (for example to the curvature) , because only on these scales one can construct locally inertial frames.

And now we can give an answer to the question 3). First we note that eq. (8.42) is valid in special relativity, i.e. in the absence of gravity, therefore, according to the Principle of Equivalence, it will hold in a locally inertial frame of a curved spacetime. In this frame, the covariant and ordinary derivative coincide, therefore we can write eq. (8.42) in the alternative form

$$T^{\alpha\beta}{}_{;\beta} = 0. \tag{8.63}$$

Then we observe that in the light of the Principle of General Covariance, since the conservation law (8.42) is a tensor equation, it will hold in any arbitrary frame. Thus in order to transform a generic tensor equation valid in Special Relativity to a *generally covariant* form it will suffice to replace the comma with a semi-colon. The general conservation law satisfied by the stress-energy tensor therefore is eq. (8.63).

Is this a conservation law?

To answer this question we need to compute the covariant divergence of a tensor. From the expression of the affine connections in terms of the metric we find that

$$\Gamma^\mu{}_{\lambda\mu} = \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{\rho\lambda}}{\partial x^\mu} + \frac{\partial g_{\rho\mu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\rho}\right). \tag{8.64}$$

The first and the third term give

$$g^{\mu\rho}\frac{\partial g_{\rho\lambda}}{\partial x^\mu} - g^{\mu\rho}\frac{\partial g_{\lambda\mu}}{\partial x^\rho} = g^{\mu\rho}\frac{\partial g_{\rho\lambda}}{\partial x^\mu} - g^{\rho\mu}\frac{\partial g_{\mu\lambda}}{\partial x^\rho} = 0, \tag{8.65}$$

due to the symmetry of $g_{\alpha\beta}$, therefore

$$\Gamma^\mu{}_{\lambda\mu} = \frac{1}{2}g^{\mu\rho}\frac{\partial g_{\rho\mu}}{\partial x^\lambda}. \tag{8.66}$$

For any arbitrary matrix M

$$\text{Tr}\left[M^{-1}(x)\frac{\partial}{\partial x^\lambda}M(x)\right] = \frac{\partial}{\partial x^\lambda}ln[|DetM(x)|]. \tag{8.67}$$

But this is what we have on the right-hand side of eq. (8.66), therefore, if we call $Det(g) = g$, eq. (8.66) becomes (since $g < 0$)

$$\Gamma^\mu{}_{\lambda\mu} = \frac{1}{2}\frac{\partial}{\partial x^\lambda}ln[-g] = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\sqrt{-g}. \tag{8.68}$$

Thus for example, if V^μ is a vector

$$V^\lambda{}_{;\lambda} = V^\lambda{}_{,\lambda} + \Gamma^\lambda{}_{\alpha\lambda}V^\alpha = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}V^\lambda\right), \tag{8.69}$$

and for $T^{\mu\nu}$

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) + \Gamma^\nu{}_{\lambda\mu}T^{\mu\lambda}. \tag{8.70}$$

In particular, if $F^{\mu\nu}$ is antisymmetric, the last term in eq. (8.70) is zero and

$$F^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g}F^{\mu\nu}). \tag{8.71}$$

Now we go back to eq. (8.63). By using eq. (8.70) it becomes

$$\frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) = -\sqrt{-g}\Gamma^\nu{}_{\lambda\mu}T^{\mu\lambda}, \tag{8.72}$$

and this is not a conservation law. Thus we cannot define a conserved four-momentum as we did in Special Relativity. We may be tempted to define

$$P^\alpha = \int_V \sqrt{-g}T^{\alpha 0}dV, \qquad \alpha = 0, 3, \tag{8.73}$$

but this would not be a vector. The physical reason for this failure is that now we are in General Relativity, and we must take into account not only the energy and momentum associated to matter, but also the energy which is carried by the gravitational field itself, and the momentum which may be carried by gravitational waves. However we shall see that if the spacetime admits some symmetry (for example if it is spherically or plane-symmetric, or it is invariant under time-translations etc.) conserved quantities can be defined.

Chapter 9

The Einstein equations

We now have all the elements needed to derive the equations of the gravitational field. We expect they will be more complicated than the linear equations of the electromagnetic field. For example electromagnetic waves are produced as a consequence of the motion of charged particles, but the energy and the momentum they carry **are not** a source for the electromagnetic field, and they do not appear on the right-hand side of the equations. In gravity the situation is different. The equation

$$E = mc^2, \tag{9.1}$$

establishes that mass and energy can transform one into another: they are different manifestation of the same physical quantity. It follows that if the mass is the source of the gravitational field, so must be the energy, and consequently both mass and energy should appear on the right-hand side of the field equations. This implies that the equations we are looking for will be non linear. For example a system of arbitrarily moving masses will radiate gravitational waves, which carry energy, which is in turn source of the gravitational field and must appear on the right-hand-side of the equations. However, since newtonian gravity works remarkably well when we are dealing with non relativistic particles, or in general when the gravitational field is weak, in formulating the new theory we shall require that in the weak field limit the new equations reduce to the Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho, \tag{9.2}$$

where ρ is the matter density, Φ is the newtonian potential and ∇^2 is the Laplace operator in cartesian coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{9.3}$$

Let us start by asking how the equations should look in the weak field limit.

9.1 The geodesic equations in the weak field limit

Consider a non-relativistic particle which moves in a **weak and stationary** gravitational field. Be τ/c the proper time. Since $v << c$, it follows that

$$\frac{dx^i}{dt} << c \qquad \rightarrow \qquad \frac{dx^i}{d\tau} << \frac{cdt}{d\tau} = \frac{dx^0}{d\tau}. \tag{9.4}$$

In an arbitrary coordinate system the geodesic equations are

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \quad \rightarrow \quad \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{cdt}{d\tau}\right)^2 = 0. \tag{9.5}$$

From the expressions of the affine connections in terms of $g_{\mu\nu}$ we easily find that

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\sigma}\left(2g_{0\sigma,0} - g_{00,\sigma}\right). \tag{9.6}$$

In addition, if the field is stationary $g_{0\sigma,0} = 0$, and

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\sigma}\frac{\partial g_{00}}{\partial x^\sigma}. \tag{9.7}$$

Since we have assumed that the gravitational field is weak, we can choose a coordinate system such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1, \tag{9.8}$$

where $h_{\mu\nu}$ is a small perturbation of the flat metric. In other words, we are assuming that the field is so weak that the metric is nearly flat. Since we are interested only in first order terms, we shall raise and lower indices with the flat metric $\eta^{\mu\nu}$. For example

$$h^\lambda{}_\nu = g^{\lambda\rho}h_{\rho\nu} \sim \eta^{\lambda\rho}h_{\rho\nu} + O(h^2_{\mu\nu}).$$

If we substitute eq. (9.8) into eq. (9.7), and retain only the terms up to first order in $h_{\mu\nu}$ we find

$$\Gamma^\mu_{00} \sim -\frac{1}{2}\eta^{\mu\sigma}\frac{\partial h_{00}}{\partial x^\sigma}, \tag{9.9}$$

and the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\alpha}\frac{\partial h_{00}}{\partial x^\alpha}\left(\frac{cdt}{d\tau}\right)^2, \tag{9.10}$$

or, splitting the time- and the space-components

$$\frac{d^2\mathbf{x}}{d\tau^2} = \frac{1}{2}\boldsymbol{\nabla} h_{00}\left(\frac{cdt}{d\tau}\right)^2, \qquad \text{and} \qquad \frac{d^2ct}{d\tau^2} = -\frac{1}{2}\frac{\partial h_{00}}{\partial ct}\left(\frac{cdt}{d\tau}\right)^2 = 0, \tag{9.11}$$

where

$$\boldsymbol{\nabla} \rightarrow \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{9.12}$$

is the gradient in cartesian coordinates. The second equation vanishes because we have assumed that the field is stationary ($\frac{\partial h_{00}}{\partial t} = 0$). We can rescale the time coordinate in such a way that $\frac{cdt}{d\tau} = 1$ and the first of eqs. (9.11) becomes

$$\frac{d^2\mathbf{x}}{d\tau^2} = \frac{1}{2}\boldsymbol{\nabla} h_{00}. \tag{9.13}$$

We should remember that the corresponding newtonian equation is

$$\frac{d^2\mathbf{x}}{dt^2} = -\boldsymbol{\nabla}\Phi, \tag{9.14}$$

where Φ is the gravitational potential given by the Poisson equation (9.2). By comparing eqs. (9.14) and (9.13), and since $\tau = ct$ we see that it must be

$$h_{00} = -2\frac{\Phi}{c^2} + const. \tag{9.15}$$

For example if the field is stationary and spherically symmetric, the newtonian potential is

$$\Phi = -\frac{GM}{r}, \tag{9.16}$$

and if we require that h_{00} vanishes at infinity, the constant must be zero and eq. (9.15) gives

$$h_{00} = -2\frac{\Phi}{c^2}, \qquad \text{and} \qquad g_{00} = -(1 + 2\frac{\Phi}{c^2}). \tag{9.17}$$

Thus we have shown that in the weak field limit the geodesic equations reduce to the newtonian law of gravitation. This suggests the form that the field equations should have. In fact if the field is weak, matter will behave non-relativistically, i.e. $T^{00} = T_{00} \sim \rho c^2$ and therefore the generalization of Laplace's equation (9.2) could be

$$\nabla^2 g_{00} = -\frac{8\pi G}{c^4} T_{00}. \tag{9.18}$$

But this equation is not even Lorentz-invariant! It doesn't work. However it suggests that if in place of a stationary field, we would have an arbitrary distribution of energy and matter, we should construct a tensor starting from $g_{\mu\nu}$ and its derivatives such that the field equations are

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{9.19}$$

where $G_{\mu\nu}$ is an operator acting on $g_{\mu\nu}$ which we shall now define. It should be stressed that, by the Principle of General Covariance, if equation (9.19) holds in a given reference frame, it will hold in any other frame.

9.2 Einstein's field equations

Let us first see which derivatives and of which order do we expect in $G_{\mu\nu}$. A comparison with the Laplace equation shows that $G_{\mu\nu}$ must have the dimensions of a second derivative. In fact, suppose that it contains terms of this type

$$\frac{\partial^3 g_{\mu\nu}}{\partial x_\mu^3}, \qquad \frac{\partial^2 g_{\mu\nu}}{\partial x_\mu^2}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\nu}, \qquad \frac{\partial g_{\mu\nu}}{\partial x_\nu}, \tag{9.20}$$

then, in order to be dimensionally homogeneous each term should be multiplied by a constant having the dimensions of a suitable power of a lenght

$$\frac{\partial^3 g_{\mu\nu}}{\partial x_\mu^3}\cdot l, \qquad \frac{\partial^2 g_{\mu\nu}}{\partial x_\mu^2}\frac{\partial g_{\mu\nu}}{\partial x_\nu}\cdot l, \qquad \frac{\partial g_{\mu\nu}}{\partial x_\nu}\cdot\frac{1}{l}. \tag{9.21}$$

In this case, a gravitational field acting on small or on very large scale would be described by equations where some of the terms would be negligible with respect to some others. This is unacceptable, because we want a set of equations that are valid at any scale, and consequently the only terms we can accept in $G_{\mu\nu}$ are those containing the second derivatives of $g_{\mu\nu}$ in a linear form and products of first derivatives. Let us summarize the assumptions that we need to make on $G_{\mu\nu}$:

1) it must be a tensor

2) it must be linear in the second derivatives, and it must contain products of first derivatives of $g_{\mu\nu}$.

3) Since $T_{\mu\nu}$ is symmetric, $G_{\mu\nu}$ also must be symmetric.

4) Since $T_{\mu\nu}$ satisfies the "conservation law" $T^{\mu\nu}{}_{;\mu} = 0$, $G_{\mu\nu}$ must satisfy the same conservation law.

$$G^{\mu\nu}{}_{;\nu} = 0. \tag{9.22}$$

5) In the weak field limit it must reduce to (compare with eq. (9.18)

$$G_{00} \sim -\nabla^2 g_{00}. \tag{9.23}$$

In this last assumption the Principle of Equivalence and the weak field limit explicitely appear.

In the preceeding section we have shown that there exists a tensor which is linear in the second derivatives of $g_{\mu\nu}$ and non linear in the first derivatives. It is the Riemann tensor, given in eq. (6.34), and it contains the information on the gravitational field. However we cannot use it directely in the field equations we are looking for, since it has four indices (it is a $\begin{pmatrix}1\\3\end{pmatrix}$ tensor) while we need a $\begin{pmatrix}2\\0\end{pmatrix}$ (or $\begin{pmatrix}0\\2\end{pmatrix}$) tensor. In addition, the covariant divergence of the stress-energy tensor vanishes, and so must be also for the tensor we shall put on the left-hand side of eq. (9.19).

By contracting the Riemann tensor with the metric we can construct a $\begin{pmatrix}0\\2\end{pmatrix}$ tensor, i.e. **the Ricci tensor**:

$$R_{\mu\nu} = g^{\kappa\alpha} R_{\kappa\mu\alpha\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \tag{9.24}$$

which is a symmetric tensor because of the symmetry property of the Riemann tensor

$$R_{\kappa\mu\alpha\nu} = R_{\alpha\nu\kappa\mu}, \tag{9.25}$$

and a scalar, called the **scalar curvature**

$$R = R^{\alpha}{}_{\alpha}. \tag{9.26}$$

The contraction in eq. (9.26) has the following meaning

$$R^{\alpha}{}_{\alpha} = R^{0}{}_{0} + R^{1}{}_{1} + R^{2}{}_{2} + R^{3}{}_{3}. \tag{9.27}$$

It can be shown, by using the symmetries of the Riemann tensor, that $R_{\mu\nu}$ and R are the only second rank tensor and scalar that can be constructed by contraction of $R_{\kappa\mu\alpha\nu}$ with the metric. Both in $R_{\mu\nu}$ and R the second derivatives of $g_{\mu\nu}$ appear linearly. Therefore the tensor we are looking for should have the following form

$$G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R, \tag{9.28}$$

where C_1 and C_2 are constants to be determined. The tensor $G_{\mu\nu}$ satisfies the points 1,2 and 3. Condition 4 requires that

$$G^{\mu\nu}{}_{;\mu} = C_1 R^{\mu\nu}{}_{;\mu} + C_2 g^{\mu\nu} R_{;\mu} = 0. \tag{9.29}$$

(remember that the covariant derivative of $g_{\mu\nu}$ vanishes). Now a very remarkable thing happens: eq. (9.29) is satisfied because of the Bianchi identities

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0. \tag{9.30}$$

In fact by contracting these equations we find

$$\begin{aligned} g^{\lambda\nu}\left(R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu}\right) &= g^{\lambda\nu}\left(R_{\lambda\mu\nu\kappa;\eta} - R_{\lambda\mu\nu\eta;\kappa}\right) + g^{\lambda\nu} R_{\lambda\mu\kappa\eta;\nu} \\ &= \left(R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}{}_{\mu\kappa\eta;\nu}\right) = 0. \end{aligned} \tag{9.31}$$

Contracting again

$$g^{\mu\kappa}\left(R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}{}_{\mu\kappa\eta;\nu}\right) = R_{;\eta} - R^{\kappa}{}_{\eta;\kappa} - R^{\nu}{}_{\eta;\nu} = 0. \tag{9.32}$$

The last expression can be rewritten in the following form

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\nu} = 0. \tag{9.33}$$

Therefore, the Bianchi identities say that if

$$\frac{C_2}{C_1} = -\frac{1}{2}, \tag{9.34}$$

eq. (9.33) will be satisfied. We still need C_1.

In the weak field limit [1]

$$|T_{ij}| << |T_{00}|, \qquad i,j = 1,3, \tag{9.37}$$

and therefore

$$|G_{ij}| << |G_{00}|, \qquad i,j = 1,3. \tag{9.38}$$

From eqs. (9.28) and (9.34) it follows

$$|C_1 \left(R_{ij} - \frac{1}{2} g_{ij} R \right)| << |G_{00}|, \tag{9.39}$$

hence

$$R_{ij} \simeq \frac{1}{2} g_{ij} R. \tag{9.40}$$

Since $g_{ij} \simeq \eta_{ij}$

$$R_{kk} \simeq \frac{1}{2} R, \qquad\qquad k = 1,3 \tag{9.41}$$

consequently

$$R = g^{\mu\nu} R_{\mu\nu} \simeq \eta^{\mu\nu} R_{\mu\nu} = -R_{00} + \sum_k R_{kk} = -R_{00} + \frac{3}{2} R, \tag{9.42}$$

and

$$R \simeq 2 R_{00}. \tag{9.43}$$

Since

$$G_{\mu\nu} = C_1 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right), \tag{9.44}$$

we find

$$G_{00} \simeq C_1 2 R_{00}. \tag{9.45}$$

If we now compute R_{00} in the weak field limit (assuming the field is stationary), we find that the non linear part is second order. Retaining only the first order terms and imposing stationarity we get

$$R_{00} \simeq -\frac{1}{2} \eta^{ij} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j} = -\frac{1}{2} \nabla^2 g_{00}, \qquad i,k = 1,3 \tag{9.46}$$

namely

$$G_{00} \simeq -C_1 \nabla^2 g_{00}, \tag{9.47}$$

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[1]The fact that in the weak field limit $|T_{ik}| << T_{00}$ can be easily understood if we consider, as an example, a system on non-interacting particles. If ρ is the mass density

$$\rho = \sum_n m_n \delta^3(\mathbf{r} - \mathbf{r_n}), \tag{9.35}$$

where $\mathbf{r_n}$ denotes the positions of the particles, the stress-energy tensor (8.15) can be also written as

$$T^{\mu\nu} = \rho c^2 \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \tag{9.36}$$

It is clear that, if $\frac{dx^i}{d\tau} << \frac{dx^0}{d\tau}$ $i = 1,3$ the dominant term will be T^{00} .

A comparison of this equation with eq. (9.23) shows that if we require that the relativistic field equations reduce to the newtonian equations in the weak field limit it must be

$$C_1 = 1. \tag{9.48}$$

In conclusion, the Einstein's field equations are [2]

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{9.49}$$

where

$$G_{\mu\nu} = \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right), \tag{9.50}$$

and it is called **The Einstein tensor**. An alternative form is

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \tag{9.51}$$

In vacuum $T_{\mu\nu} = 0$ and the Einstein equations reduce to

$$R_{\mu\nu} = 0. \tag{9.52}$$

Therefore, in vacuum the Ricci tensor vanishes, but the Riemann tensor does not, unless the gravitational field vanishes or is constant and uniform. We may still add to eqs. (9.49) the following term

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} \right) = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{9.53}$$

where λ is a constant. This term satisfies the conditions 1,2,3 and 4, but not the condition 5. This means that it must be very small in such a way that in the weak field limit the equations reduce to the newtonian equations.

9.3 Gauge invariance of the Einstein equations

Since there are 10 independent components of $G_{\mu\nu}$, Einstein's equations provide 10 equations for the 10 independent components of $g_{\mu\nu}$. However these equations are not independent, because, as we have seen, the Bianchi identities imply the "conservation law" ${G^{\mu\nu}}_{;\nu} = 0$, which provides 4 relations that the Einstein tensor must satisfy. Thus the number of independent equations reduces to six.

Do we have six equations and 10 unknown functions? Why do we have these four degrees of freedom? The reason is the following. Be $g_{\mu\nu}$ a solution of the equations. If we make a coordinate transformation $x^{\mu'} = x^{\mu'}(x^\alpha)$ the 'transformed' tensor $g'_{\mu\nu} = g_{\mu'\nu'}$ is again

[2] Although we call these equations the Einstein equations, they were derived independently (and in a more elegant form) by D. Hilbert in the same year. However Einstein showed the implications of these equations in the theory of the solar system, and in particular that the precession of the perihelion of Mercury has a relativistic origin. This led to the theory's acceptance and since then the equations have been called the Einstein equations.

a solution, as established by the Principle of General Covariance. This also means that $g_{\mu\nu}$ and $g'_{\mu\nu}$ do represent the same physical solution (the same geometry) seen in different reference frames.

The coordinate transformation involves 4 arbitrary functions $x^{\mu\prime}(x^\alpha)$, therefore the four degrees of freedom derive from the freedom of choosing the coordinate system, and disappear when we choose it. For example, we may choose a frame where four of the ten $g_{\mu\nu}$ are zero.

Thus Einstein's equations do not determine the solution $g_{\mu\nu}$ in a unique way, but only up to an arbitrary coordinate transformation. A similar situation arises in the case of Maxwell's equations in Special Relativity. In that case the equations for the vector potential[3] A^μ are

$$\Box A_\alpha - \frac{\partial^2 A^\beta}{\partial x^\alpha \partial x^\beta} = -\frac{4\pi}{c} J_\alpha. \tag{9.54}$$

(where $\Box = -\frac{\partial^2}{c^2\partial t^2} + \nabla^2 = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}$). These are four equations for the four components of the vector potential. However they do not determine A^μ uniquely, because of the conservation law

$$J^\mu{}_{,\mu} = 0, \qquad \text{i.e.} \qquad \frac{\partial}{\partial x^\mu}\left(\Box A^\mu - \eta^{\mu\alpha}\frac{\partial^2 A^\beta}{\partial x^\alpha \partial x^\beta}\right) = 0. \tag{9.55}$$

Equation (9.55) plays the same role as the Bianchi identities do in our context. It provides **one** condition which must be satisfied by the components of A^μ, therefore the number of independent Maxwell equations is three. The extra degree of freedom corresponds to a gauge invariance, which means the following.
If A_α is a solution,

$$A'_\alpha = A_\alpha + \frac{\partial \Phi}{\partial x^\alpha}, \tag{9.56}$$

will also be a solution. In fact, by direct substitution we find

$$\Box A'_\alpha - \frac{\partial}{\partial x^\alpha}\Box\Phi - \frac{\partial^2 A'^\beta}{\partial x^\alpha \partial x^\beta} + \eta^{\beta\delta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\frac{\partial \Phi}{\partial x^\delta} = -\frac{4\pi}{c} J_\alpha, \tag{9.57}$$

and since the second and the last term on the left hand-side cancel, it becomes

$$\Box A'_\alpha - \frac{\partial^2 A'^\beta}{\partial x^\alpha \partial x^\beta} = -\frac{4\pi}{c} J_\alpha, \tag{9.58}$$

q.e.d.

Since Φ is arbitrary, we can chose it in such a way that

$$\frac{\partial}{\partial x^\beta} A'^\beta = 0 \tag{9.59}$$

[3]Eq. (9.54) is the four-dimensional version of the wave equation for the vector potential

$$\Box \mathbf{A} = grad(div\mathbf{A}) = -\frac{4\pi}{c}\mathbf{J}.$$

and eq. (9.58) becomes

$$\Box A'_\alpha = -\frac{4\pi}{c} J_\alpha, \tag{9.60}$$

This is the Lorenz gauge.

Summaryzing: in the electromagnetic case the extra degree of freedom on A_μ is due to the fact that the vector potential is defined up to a function Φ defined in eq. (9.56). In our case the **four** extra degrees of freedom are due to the fact that $g_{\mu\nu}$ is defined up to a coordinate transformation. This gauge freedom is particularly useful when one is looking for exact solutions of Einstein's equations.

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9.4 Example: The armonic gauge.

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The armonic gauge is defined by the condition

$$\Gamma^\lambda = g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0. \tag{9.61}$$

As we shall see in a next lecture, this gauge is of particular interest when we study the propagation of gravitational waves, because it simplifies the equations in a way similar to that of Maxwell's equations when written in the Lorenz gauge. It is always possible to choose this gauge indeed, given a generic coordinate transformation, the affine connections $\Gamma^\alpha_{\beta\gamma}$ transform as (see eq. (5.36))

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\rho}\frac{\partial x^\tau}{\partial x^{\mu'}}\frac{\partial x^\sigma}{\partial x^{\nu'}}\Gamma^\rho_{\tau\sigma} + \frac{\partial x^{\lambda'}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\mu'}\partial x^{\nu'}}. \tag{9.62}$$

When contracted with $g^{\mu'\nu'}$ this equation gives

$$\Gamma^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^\rho}\Gamma^\rho + g^{\mu'\nu'}\frac{\partial x^{\lambda'}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\mu'}\partial x^{\nu'}}.$$

The last term can be written in the following form

$$g^{\mu'\nu'}\frac{\partial x^{\lambda'}}{\partial x^\sigma}\frac{\partial}{\partial x^{\mu'}}\left\{\frac{\partial x^\sigma}{\partial x^{\nu'}}\right\} = g^{\mu'\nu'}\left\{\frac{\partial}{\partial x^{\mu'}}\left[\frac{\partial x^{\lambda'}}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x^{\nu'}}\right] - \frac{\partial x^\sigma}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\mu'}\partial x^\sigma}\right\}$$
$$= g^{\mu'\nu'}\left\{\frac{\partial}{\partial x^{\mu'}}\delta^{\lambda'}{}_{\nu'} - \frac{\partial x^\sigma}{\partial x^{\nu'}}\frac{\partial x^\rho}{\partial x^{\mu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^\rho\partial x^\sigma}\right\},$$

from which we find

$$\Gamma^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^\rho}\Gamma^\rho - g^{\rho\sigma}\frac{\partial^2 x^{\lambda'}}{\partial x^\rho\partial x^\sigma}. \tag{9.63}$$

Therefore, if Γ^λ is non zero, we can always find a frame where $\Gamma^{\rho'} = 0$ and reduce to the armonic gauge. The condition $\Gamma^\lambda = 0$ can be rewritten in a more elegant form remembering the expression of the affine connections in terms of the metric tensor

$$\Gamma^\lambda = \frac{1}{2}g^{\mu\nu}g^{\lambda\kappa}\left\{\frac{\partial g_{\kappa\mu}}{\partial x^\nu} + \frac{\partial g_{\kappa\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right\} = 0. \tag{9.64}$$

Since

$$g^{\lambda\kappa}\frac{\partial g_{\kappa\mu}}{\partial x^\nu} = -g_{\kappa\mu}\frac{\partial g^{\lambda\kappa}}{\partial x^\nu}, \tag{9.65}$$
$$\frac{1}{2}g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^\kappa} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\kappa}\sqrt{-g}\,,$$

it follows that

$$\Gamma^\lambda = \frac{1}{2} g^{\mu\nu} \left\{ -g_{\kappa\mu} \left[\frac{\partial g^{\lambda\kappa}}{\partial x^\nu} \right] - g_{\kappa\nu} \left[\frac{\partial g^{\lambda\kappa}}{\partial x^\mu} \right] \right\} - \frac{g^{\lambda\kappa}}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \sqrt{-g} = 0. \tag{9.66}$$

The term in brackets is symmetric in μ and ν, therefore

$$\Gamma^\lambda = -\frac{1}{2} \left\{ 2 g^{\mu\sigma} g_{\kappa\mu} \frac{\partial g^{\lambda\kappa}}{\partial x^\sigma} \right\} - \frac{g^{\lambda\kappa}}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \sqrt{-g} = 0, \tag{9.67}$$

and, since $g^{\mu\sigma} g_{\kappa\mu} = \delta^\sigma{}_\kappa$

$$\Gamma^\lambda = -\frac{\partial g^{\lambda\kappa}}{\partial x^\kappa} - \frac{g^{\lambda\kappa}}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \sqrt{-g} = 0, \tag{9.68}$$

from which we find

$$-\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \left(\sqrt{-g} g^{\lambda\kappa} \right) = 0. \tag{9.69}$$

This means that

$$\Gamma^\lambda = 0 \qquad \text{implies} \qquad \frac{\partial}{\partial x^\kappa} \left(\sqrt{-g} g^{\lambda\kappa} \right) = 0. \tag{9.70}$$

The reason why this gauge is called 'armonic' is the following. A function Φ is armonic if

$$\Box \Phi = 0, \tag{9.71}$$

where the operator $\Box$ is the **covariant d'Alambertian operator** defined as

$$\Box \Phi = g^{\lambda\kappa} \nabla_\lambda \nabla_\kappa \Phi, \tag{9.72}$$

and ∇_λ is the covariant derivative. Since

$$g^{\lambda\kappa} \nabla_\lambda \nabla_\kappa \Phi = g^{\lambda\kappa} \left(\frac{\partial \Phi_{;\lambda}}{\partial x^\kappa} - \Gamma^\alpha_{\lambda\kappa} \Phi_{;\alpha} \right) = \tag{9.73}$$

$$g^{\lambda\kappa} \left[\frac{\partial^2 \Phi}{\partial x^\kappa \partial x^\lambda} - \Gamma^\alpha_{\lambda\kappa} \frac{\partial \Phi}{\partial x^\alpha} \right] = g^{\lambda\kappa} \frac{\partial^2 \Phi}{\partial x^\kappa \partial x^\lambda} - \Gamma^\alpha \frac{\partial \Phi}{\partial x^\alpha}.$$

If $\Gamma^\lambda = 0$ the armonic gauge condition becomes

$$\Box \Phi = g^{\lambda\kappa} \frac{\partial^2 \Phi}{\partial x^\kappa \partial x^\lambda} = 0. \tag{9.74}$$

If $\Gamma^\lambda = 0$ then the **coordinates itself are armonic functions**, in fact putting $\Phi = x^\mu$ in eq. (9.74) one finds

$$\Box x^\mu = g^{\lambda\kappa} \frac{\partial^2 x^\mu}{\partial x^\kappa \partial x^\lambda} = g^{\lambda\kappa} \frac{\partial}{\partial x^\kappa} \delta^\mu_\lambda = 0, \tag{9.75}$$

q.e.d. If the spacetime is flat, armonic coordinates coincide with minkowskian coordinates.

Chapter 10

Symmetries

H. Weyl: "*Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.*"

The solution of a physical problem can be considerably simplified if it allows some symmetries. Let us consider for example the equations of Newtonian gravity. It is easy to find a solution which is spherically symmetric, but it may be difficult to find the analytic solution for an arbitrary mass distribution.

In euclidean space a symmetry is related to an invariance with respect to some operation. For example plane symmetry implies invariance of the physical variables with respect to translations on a plane, spherically symmetric solutions are invariant with respect to translation on a sphere, and the equations of Newtonian gravity are symmetric with respect to time translations

$$t' \to t + \tau.$$

Thus, a symmetry corresponds to invariance under translations along certain lines or over certain surfaces. This definition can be applied and extended to Riemannian geometry. A solution of Einstein's equations has a symmetry if there exists an n-dimensional manifold, with $1 \leq n \leq 4$, such that the solution is invariant under translations which bring a point of this manifold into another point of the same manifold. For example, for spherically symmetric solutions the manifold is the 2-sphere, and n=2. This is a simple example, but there exhist more complicated four-dimensional symmetries. These definitions can be made more precise by introducing the notion of Killing vectors.

10.1 The Killing vectors

Consider a vector field $\vec{\xi}(x^\mu)$ defined at every point x^α of a spacetime region. $\vec{\xi}$ identifies a symmetry if an infinitesimal translation along $\vec{\xi}$ leaves the line-element unchanged, i.e.

$$\delta(ds^2) = \delta(g_{\alpha\beta}dx^a dx^b) = 0. \tag{10.1}$$

This implies that

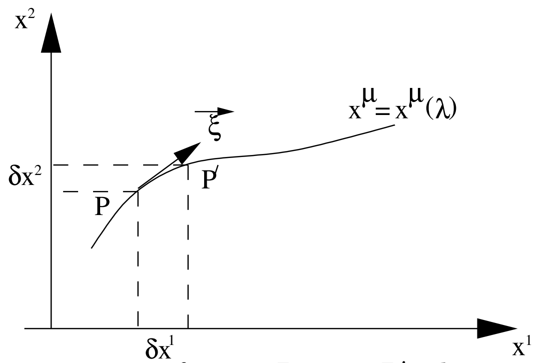
$$\delta g_{\alpha\beta}dx^a dx^b + g_{\alpha\beta}\left[\delta(dx^a)dx^b + dx^a\delta(dx^b)\right] = 0. \tag{10.2}$$

$\vec{\xi}$ is the tangent vector to some curve $x^\alpha(\lambda)$, i.e. $\xi^\alpha = \frac{\delta x^\alpha}{d\lambda}$, therefore an infinitesimal translation in the direction of $\vec{\xi}$ is an infinitesimal translation along the curve from a point $P(\lambda)$ to the point $P'(\lambda + d\lambda)$. Putting

$$\delta x^\alpha = x^\alpha(\lambda + d\lambda) - x^\alpha(\lambda) = \frac{dx^\alpha}{d\lambda} d\lambda = \xi^\alpha d\lambda \,,$$

the coordinates of $P(\lambda)$ and $P'(\lambda + d\lambda)$ are, respectively,

$$P = (x^\alpha) \qquad \text{and} \qquad P' = (x^\alpha + \delta x^\alpha).$$



$$P = (x^1, x^2)$$
$$P' = (x^1 + \delta x^1, x^2 + \delta x^2)$$

When we move from P to P' the metric components change as follows

$$\begin{aligned} g_{\alpha\beta}(P') &\simeq g_{\alpha\beta}(P) + \frac{\partial g_{\alpha\beta}}{\partial \lambda} d\lambda + ... \\ &= g_{\alpha\beta}(P) + \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \frac{dx^\mu}{d\lambda} d\lambda + ... \\ &= g_{\alpha\beta}(P) + g_{\alpha\beta,\mu} \xi^\mu d\lambda, \end{aligned} \tag{10.3}$$

hence

$$\delta g_{\alpha\beta} = g_{\alpha\beta,\mu} \xi^\mu d\lambda. \tag{10.4}$$

Moreover, since the operators δ and d commute, we find

$$\begin{aligned} \delta(dx^a) &= d(\delta x^\alpha) = d(\xi^\alpha d\lambda) = d\xi^\alpha d\lambda \\ &= \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu d\lambda = \xi^\alpha_{,\mu} dx^\mu d\lambda \,. \end{aligned} \tag{10.5}$$

Thus, using eqs. (10.5) and (10.4), eq. (10.2) becomes

$$g_{\alpha\beta,\mu}\xi^\mu d\lambda dx^\alpha dx^\beta + g_{\alpha\beta}\left[\xi^\alpha_{,\mu} dx^\mu d\lambda dx^\beta + \xi^\beta_{,\gamma} dx^\gamma d\lambda dx^\alpha\right] = 0, \tag{10.6}$$

and, after relabelling the indices,

$$\left[g_{\alpha\beta,\mu}\xi^\mu + g_{\delta\beta}\xi^\delta_{,\alpha} + g_{\alpha\delta}\xi^\delta_{,\beta}\right] dx^\alpha dx^\beta d\lambda = 0. \tag{10.7}$$

In conclusion, a solution of Einstein's equations is invariant under translations along $\vec{\xi}$, if and only if

$$g_{\alpha\beta,\mu}\xi^\mu + g_{\delta\beta}\xi^\delta_{,\alpha} + g_{\alpha\delta}\xi^\delta_{,\beta} = 0. \tag{10.8}$$

In order to find the Killing vectors of a given a metric $g_{\alpha\beta}$ we need to solve eq. (10.8), which is a system of differential equations for the components of $\vec{\xi}$. If eq. (10.8) does not admit a solution, the spacetime has no symmetries. It may look like eq. (10.8) is not covariant, since it contains partial derivatives, but it is easy to show that it is equivalent to the following covariant equation (see appendix A)

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0. \tag{10.9}$$

This is the **Killing equation**.

10.1.1 Lie-derivative

The variation of a tensor under an infinitesimal translation along the direction of a vector field $\vec{\xi}$ is the **Lie-derivative** ($\vec{\xi}$ must not necessarily be a Killing vector), and it is indicated as $L_{\vec{\xi}}$. For a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor

$$L_{\vec{\xi}} T_{\alpha\beta} = T_{\alpha\beta,\mu}\xi^\mu + T_{\delta\beta}\xi^\delta_{,\alpha} + T_{\alpha\delta}\xi^\delta_{,\beta} \ . \tag{10.10}$$

For the metric tensor

$$L_{\vec{\xi}} g_{\alpha\beta} = g_{\alpha\beta,\mu}\xi^\mu + g_{\delta\beta}\xi^\delta_{,\alpha} + g_{\alpha\delta}\xi^\delta_{,\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} \ ; \tag{10.11}$$

if $\vec{\xi}$ is a Killing vector the Lie-derivative of $g_{\alpha\beta}$ vanishes.

10.1.2 Killing vectors and the choice of coordinate systems

The existence of Killing vectors remarkably simplifies the problem of choosing a coordinate system appropriate to solve Einstein's equations. For instance, if we are looking for a solution which admits a timelike Killing vector $\vec{\xi}$, it is convenient to choose, at each point of the manifold, the timelike basis vector $\vec{e}_{(0)}$ aligned with $\vec{\xi}$; with this choice, the time coordinate lines coincide with the worldlines to which $\vec{\xi}$ is tangent, i.e. with the **congruence of worldlines** of $\vec{\xi}$, and the components of $\vec{\xi}$ are

$$\xi^\alpha = (\xi^0, 0, 0, 0) \ . \tag{10.12}$$

If we parametrize the coordinate curves associated to $\vec{\xi}$ in such a way that ξ^0 is constant or equal unity, then

$$\xi^\alpha = (1, 0, 0, 0) \ , \tag{10.13}$$

and from eq. (10.8) it follows that

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0 \ . \tag{10.14}$$

This means that **if the metric admits a timelike Killing vector, with an appropriate choice of the coordinate system it can be made independent of time.**

A similar procedure can be used if the metric admits a spacelike Killing vector. In this case, by choosing one of the spacelike basis vectors, say the vector $\vec{e}_{(1)}$, parallel to $\vec{\xi}$, and by a suitable reparametrization of the corresponding conguence of coordinate lines, one can write

$$\xi^\alpha = (0,1,0,0) \ , \tag{10.15}$$

and with this choice the metric is independent of x^1, i.e. $\partial g_{\alpha\beta}/\partial x^1 = 0$.

If the Killing vector is null, starting from the coordinate basis vectors $\vec{e}_{(0)}, \vec{e}_{(1)}, \vec{e}_{(2)}, \vec{e}_{(3)}$, it is convenient to construct a set of new basis vectors

$$\vec{e}_{(\alpha')} = \Lambda^\beta_{\alpha'} \vec{e}_{(\beta)} \ , \tag{10.16}$$

such that the vector $\vec{e}_{(0')}$ is a null vector. Then, the vector $\vec{e}_{(0')}$ can be chosen to be parallel to $\vec{\xi}$ at each point of the manifold, and by a suitable reparametrization of the corresponding coordinate lines

$$\xi^\alpha = (1,0,0,0) \ , \tag{10.17}$$

and the metric is independent of $x^{0'}$, i.e. $\partial g_{\alpha\beta}/\partial x^{0'} = 0$.

The map

$$f_t : \ \mathcal{M} \to \mathcal{M}$$

under which the metric is unchanged is called an *isometry*, and the Killing vector field is the generator of the isometry.

The **congruence of worldlines** of the vector $\vec{\xi}$ can be found by integrating the equations

$$\frac{\delta x^\mu}{d\lambda} = \xi^\mu(x^\alpha). \tag{10.18}$$

10.2 Examples

1) Killing vectors of flat spacetime

The Killing vectors of Minkowski's spacetime can be obtained very easily using cartesian coordinates. Since all Christoffel symbols vanish, the Killing equation becomes

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0 \ . \tag{10.19}$$

By combining the following equations

$$\xi_{\alpha,\beta\gamma} + \xi_{\beta,\alpha\gamma} = 0 \ , \qquad \xi_{\beta,\gamma\alpha} + \xi_{\gamma,\beta\alpha} = 0 \ , \qquad \xi_{\gamma,\alpha\beta} + \xi_{\alpha,\gamma\beta} = 0 \ , \tag{10.20}$$

and by using eq. (10.19) we find

$$\xi_{\alpha,\beta\gamma} = 0 \ , \tag{10.21}$$

whose general solution is

$$\xi_\alpha = c_\alpha + \epsilon_{\alpha\gamma} x^\gamma \ , \tag{10.22}$$

where c_α, $\epsilon_{\alpha\beta}$ are constants. By substituting this expression into eq. (10.19) we find

$$\epsilon_{\alpha\gamma} x^\gamma_{,\beta} + \epsilon_{\beta\gamma} x^\gamma_{,\alpha} = \epsilon_{\alpha\gamma}\delta^\gamma_\beta + \epsilon_{\beta\gamma}\delta^\gamma_\alpha = \epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = 0$$

Therefore eq. (10.22) is the solution of eq. (10.19) only if

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \ . \tag{10.23}$$

The general Killing vector field of the form (10.22) can be written as the linear combination of ten Killing vector fields $\xi^{(A)}_\alpha = \{\xi^{(1)}_\alpha, \xi^{(2)}_\alpha, \ldots, \xi^{(10)}_\alpha\}$ corresponding to ten independent choices of the constants c_α, $\epsilon_{\alpha\beta}$:

$$\xi^{(A)}_\alpha = c^{(A)}_\alpha + \epsilon^{(A)}_{\alpha\gamma} x^\gamma \qquad A = 1, \ldots, 10\,. \tag{10.24}$$

For instance, we can choose

$$\begin{aligned}
c^{(1)}_\alpha &= (1,0,0,0) \qquad \epsilon^{(1)}_{\alpha\beta} = 0\\
c^{(2)}_\alpha &= (0,1,0,0) \qquad \epsilon^{(2)}_{\alpha\beta} = 0\\
c^{(3)}_\alpha &= (0,0,1,0) \qquad \epsilon^{(3)}_{\alpha\beta} = 0\\
c^{(4)}_\alpha &= (0,0,0,1) \qquad \epsilon^{(4)}_{\alpha\beta} = 0\\
c^{(5)}_\alpha &= 0 \qquad \epsilon^{(5)}_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}\\
c^{(6)}_\alpha &= 0 \qquad \epsilon^{(6)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}\\
c^{(7)}_\alpha &= 0 \qquad \epsilon^{(7)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}\\
c^{(8)}_\alpha &= 0 \qquad \epsilon^{(8)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}\\
c^{(9)}_\alpha &= 0 \qquad \epsilon^{(9)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}\\
c^{(10)}_\alpha &= 0 \qquad \epsilon^{(10)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}
\end{aligned} \tag{10.25}$$

Therefore, flat spacetime admits ten linearly independent Killing vectors.

The symmetries generated by the Killing vectors with $A = 1, \ldots, 4$ are spacetime translations; the symmetries generated by the Killing vectors with $A = 5, 6, 7$ are Lorentz's boosts; the symmetries generated by the Killing vectors with $A = 8, 9, 10$ are space rotations.

2) Killing vectors of a spherical surface

Let us consider a sphere of unit radius

$$ds^2 = d\theta^2 + sin^2\theta d\varphi^2 = (dx^1)^2 + sin^2 x^1 (dx^2)^2\ . \tag{10.26}$$

Eq. (10.8)

$$g_{\alpha\beta,\mu}\xi^{\mu} + g_{\delta\beta}\xi^{\delta}_{,\alpha} + g_{\alpha\delta}\xi^{\delta}_{,\beta} = 0$$

gives

$$\begin{array}{lll} 1) & \alpha=\beta=1 & 2g_{\delta 1}\xi^{\delta}_{,1} = 0 \to \xi^{1}_{,1} = 0 \\ 2) & \alpha=1,\beta=2 & g_{\delta 2}\xi^{\delta}_{,1} + g_{1\delta}\xi^{\delta}_{,2} = 0 \to \xi^{1}_{,2} + \sin^2\theta\xi^{2}_{,1} = 0 \\ 3) & \alpha=\beta=2 & g_{22,\mu}\xi^{\mu} + 2g_{\delta 2}\xi^{\delta}_{,2} = 0 \to \cos\theta\xi^{1} + \sin\theta\xi^{2}_{,2} = 0. \end{array} \quad (10.27)$$

The general solution is

$$\xi^1 = Asin(\varphi + a), \qquad \xi^2 = Acos(\varphi + a)cot\theta + b. \quad (10.28)$$

Therefore a spherical surface admits three linearly independent Killing vectors, associated to the choice of the integration constants (A, a, b).

10.3 Conserved quantities in geodesic motion

Killing vectors are important because they are associated to conserved quantities, which may be hidden by an unsuitable coordinate choice.

Let us consider a massive particle moving along a geodesic of a spacetime which admits a Killing vector $\vec{\xi}$. The geodesic equations written in terms of the particle four-velocity $\vec{U} = \frac{\delta x^\alpha}{d\tau}$ read

$$\frac{dU^\alpha}{d\tau} + \Gamma^{\alpha}{}_{\beta\nu}U^\beta U^\nu = 0. \quad (10.29)$$

By contracting eq. (10.29) with $\vec{\xi}$ we find

$$\xi_\alpha\left[\frac{dU^\alpha}{d\tau} + \Gamma^{\alpha}{}_{\beta\nu}U^\beta U^\nu\right] = \frac{d(\xi_\alpha U^\alpha)}{d\tau} - U^\alpha\frac{d\xi_\alpha}{d\tau} + \Gamma^{\alpha}{}_{\beta\nu}U^\beta U^\nu\xi_\alpha\ . \quad (10.30)$$

Since

$$U^\alpha\frac{d\xi_\alpha}{d\tau} = U^\beta\frac{d\xi_\beta}{d\tau} = U^\beta\frac{\partial\xi_\beta}{\partial x^\nu}\frac{\delta x^\nu}{d\tau} = U^\beta U^\nu\frac{\partial\xi_\beta}{\partial x^\nu}\ , \quad (10.31)$$

eq. (10.30) becomes

$$\frac{d(\xi_\alpha U^\alpha)}{d\tau} - U^\beta U^\nu\left[\frac{\partial\xi_\beta}{\partial x^\nu} - \Gamma^{\alpha}{}_{\beta\nu}\xi_\alpha\right] = 0\ , \quad (10.32)$$

i.e.

$$\frac{d(\xi_\alpha U^\alpha)}{d\tau} - U^\beta U^\nu\xi_{\beta;\nu} = 0\ . \quad (10.33)$$

Since $\xi_{\beta;\nu}$ is antisymmetric in β and ν, while $U^\beta U^\nu$ is symmetric, the term $U^\beta U^\nu\xi_{\beta;\nu}$ vanishes, and eq. (10.33) finally becomes

$$\frac{d(\xi_\alpha U^\alpha)}{d\tau} = 0 \qquad \to \qquad \xi_\alpha U^\alpha = const\ , \quad (10.34)$$

i.e. the quantity $(\xi_\alpha U^\alpha)$ is a constant of the particle motion. Thus, for every Killing vector there exists an associated conserved quantity.

Eq. (10.34) can be written as follows:

$$g_{\alpha\mu}\xi^\mu U^\alpha = const \ . \tag{10.35}$$

Let us now assume that $\vec{\xi}$ is a timelike Killing vector. In section 10.1.2 we have shown that the coordinate system can be chosen in such a way that $\xi^\mu = \{1, 0, 0, 0\}$, in which case eq. (10.35) becomes

$$g_{\alpha 0}\xi^0 U^\alpha = const \qquad \rightarrow \qquad g_{\alpha 0} U^\alpha = const \ . \tag{10.36}$$

If the metric is asymptotically flat, as it is for instance when the gravitational field is generated by a distribution of matter confined in a finite region of space, at infinity $g_{\alpha\beta}$ reduces to the Minkowski metric $\eta_{\alpha\beta}$, and eq. (10.36) becomes

$$\eta_{00} U^0 = const \qquad \rightarrow \qquad U^0 = const \ . \tag{10.37}$$

Since in flat spacetime the energy-momentum vector of a massive particle is $p^\alpha = mcU^\alpha = \{E/c, mv^i\gamma\}$, the previous equation becomes

$$\frac{E}{c} = const \ , \tag{10.38}$$

i.e. at infinity the conservation law associated to a timelike Killing vector reduces to the energy conservation for the particle motion. For this reason we say that, when the metric admits a timelike Killing vector, eq. (10.34) expresses the energy conservation for the particle motion along the geodesic.

If the Killing vector is spacelike, by choosing the coordinate system such that, say, $\xi^\mu = \{0, 1, 0, 0\}$, eq. (10.34) reduces to

$$g_{\alpha 1}\xi^1 U^\alpha = const \qquad \rightarrow \qquad g_{\alpha 1} U^\alpha = const \ .$$

At infinity this equation becomes

$$\eta_{11} U^1 = const \qquad \rightarrow \qquad \frac{p^1}{mc} = const \ ,$$

showing that the component of the energy-momentum vector along the x^1 direction is constant; thus, when the metric admit a spacelike Killing vector eq. (10.34) expresses momentum conservation along the geodesic motion.

If the particle is massless, the geodesic equation cannot be parametrized with the proper time. In this case the particle worldline has to be parametrized using an affine parameter λ such that the geodesic equation takes the form (10.29), and the particle four-velocity is $U^\alpha = \frac{dx^\alpha}{d\lambda}$. The derivation of the constants of motion associated to a spacetime symmetry, i.e. to a Killing vector, is similar as for massive particles, reminding that by a suitable choice of the parameter along the geodesic $p^\alpha = \{E, p^i\}$.

It should be mentioned that in Riemannian spaces there may exist conservation laws which cannot be traced back to the presence of a symmetry, and therefore to the existence of a Killing vector field.

10.4 Killing vectors and conservation laws

In Chapter 7 we have shown that the stress-energy tensor satisfies the "conservation law"

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{10.39}$$

and we have shown that in general this is not a genuine conservation law. If the spacetime admits a Killing vector, then

$$\left(\xi_\mu T^{\mu\nu}\right)_{;\nu} = \xi_{\mu;\nu} T^{\mu\nu} + \xi_\mu T^{\mu\nu}{}_{;\nu} = 0. \tag{10.40}$$

Indeed, the second term vanishes because of eq. (10.39) and the first vanishes because $\xi_{\mu;\nu}$ is antisymmetric in μ an ν, whereas $T^{\mu\nu}$ is symmetric.

Since there is a contraction on the index μ, the quantity $(\xi_\mu T^{\mu\nu})$ is a vector, and according to eq. (8.69)

$$V^\nu{}_{;\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu}\left(\sqrt{-g} V^\nu\right) , \tag{10.41}$$

therefore eq. (10.40) is equivalent to

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu}\left[\sqrt{-g}\left(\xi_\mu T^{\mu\nu}\right)\right] = 0 , \tag{10.42}$$

which expresses the conservation of the following quantity and accordingly, a conserved quantity can be defined as

$$T = \int_{(x^0=const)} \sqrt{-g}\left(\xi_\mu T^{\mu 0}\right) dx^1 dx^2 dx^3 , \tag{10.43}$$

as shown in Chapter 7.

In classical mechanics energy is conserved when the hamiltonian is independent of time; thus, conservation of energy is associated to a symmetry with respect to time translations. In section 10.1.2 we have shown that if a metric admits a timelike Killing vector, with a suitable choice of coordinates it can me made time independent (where now "time" indicates more generally the x^0-coordinate). Thus, in this case it is natural to interpret the quantity defined in eq. (10.43) as a conserved energy.

In a similar way, when the metric addmits a spacelike Killing vector, the associated conserved quantities are indicated as "momentum" or "angular momentum", although this is more a matter of definition.

It should be stressed that the energy of a gravitational system can be defined in a non ambiguous way only if there exists a timelike Killing vector field.

10.5 Hypersurface orthogonal vector fields

Given a vector field $\vec{V}$ it identifies a **congruence of worldlines**, i.e. the set of curves to which the vector is tangent at any point of the considered region. If there exists a family of surfaces $f(x^\mu) = const$ such that, at each point, the worldlines of the congruence are perpendicular to that surface, $\vec{V}$ is said to be **hypersurface orthogonal**. This is equivalent to require that $\vec{V}$ is orthogonal to all vectors $\vec{t}$ tangent to the hypersurface, i.e.

$$\vec{t} \cdot \vec{V} = 0 \quad \rightarrow \quad t^\alpha V^\beta g_{\alpha\beta} = 0 \ . \tag{10.44}$$

We shall now show that, as consequence, $\vec{V}$ is parallel to the gradient of f. As described in Chapter 3, section 5, the gradient of a function $f(x^\mu)$ is a one-form

$$\tilde{d}f \rightarrow (\frac{\partial f}{\partial x^0}, \frac{\partial f}{\partial x^1}, ... \frac{\partial f}{\partial x^n}) = \{f_{,\alpha}\}. \tag{10.45}$$

When we say that $\vec{V}$ is parallel to $\tilde{d}f$ we mean that the one-form dual to $\vec{V}$, i.e. $\tilde{V} \rightarrow \{g_{\alpha\beta}V^\beta \equiv V_\alpha\}$ satisfies the equation

$$V_\alpha = \lambda f_{,\alpha} \ , \tag{10.46}$$

where λ is a function of the coordinates $\{x^\mu\}$. This equation is equivalent to eq. (10.44). Indeed, given any curve $x^\alpha(s)$ lying on the hypersurface, and being $t^\alpha = dx^\alpha/ds$ its tangent vector, since $f(x^\mu) = const$ the directional derivative of $f(x^\mu)$ along the curve vanishes, i.e.

$$\frac{df}{ds} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{ds} = f_{,\alpha}t^\alpha = \lambda^{-1}V_\alpha t^\alpha = 0 \ , \tag{10.47}$$

i.e. eq. (10.44).

If (10.46) is satisfied, it follows that

$$\begin{aligned} V_{\alpha;\beta} - V_{\beta;\alpha} &= (\lambda f_{,\alpha})_{;\beta} - (\lambda f_{,\beta})_{;\alpha} \\ &= \lambda\left(f_{,\alpha;\beta} - f_{,\beta;\alpha}\right) + f_{,\alpha}\lambda_{;\beta} - f_{,\beta}\lambda_{;\alpha} = \\ &= \lambda\left(f_{,\alpha,\beta} - f_{,\beta,\alpha} - \Gamma^\mu{}_{\beta\alpha}f_{,\mu} + \Gamma^\mu{}_{\alpha\beta}f_{,\mu}\right) + f_{,\alpha}\lambda_{,\beta} - f_{,\beta}\lambda_{,\alpha} \\ &= V_\alpha\frac{\lambda_{,\beta}}{\lambda} - V_\beta\frac{\lambda_{,\alpha}}{\lambda} \ , \end{aligned} \tag{10.48}$$

i.e.

$$V_{\alpha;\beta} - V_{\beta;\alpha} = V_\alpha\frac{\lambda_{,\beta}}{\lambda} - V_\beta\frac{\lambda_{,\alpha}}{\lambda} \ . \tag{10.49}$$

If we now define the following quantity, which is said *rotation*

$$\omega^\delta = \frac{1}{2}\epsilon^{\delta\alpha\beta\mu}V_{[\alpha;\beta]}V_\mu \ , \tag{10.50}$$

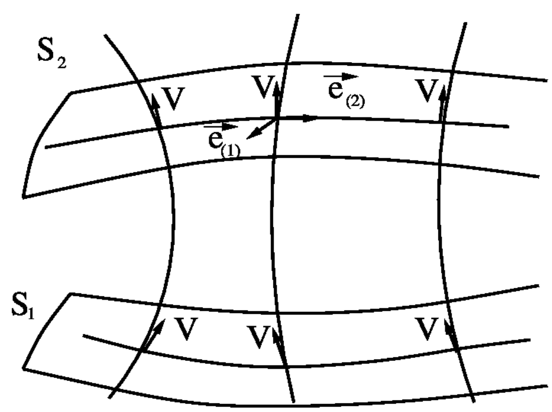
using the definition of the antisymmetric unit pseudotensor $\epsilon^{\delta\alpha\beta\mu}$ given in Appendix B, it follows that

$$\omega^\delta = 0. \tag{10.51}$$

Then, if the vector field $\vec{V}$ is hypersurface horthogonal, (10.51) is satisfied. Actually, (10.51) is a necessary and sufficient condition for $\vec{V}$ to be hypersurface horthogonal; this result is the *Frobenius theorem*.

10.5.1 Hypersurface-orthogonal vector fields and the choice of coordinate systems

The existence of a hypersurface-orthogonal vector field allows to choose a coordinate frame such that the metric has a much simpler form. Let us consider, for the sake of simplicity, a three-dimensional spacetime (x^0, x^1, x^2).



Be S_1 and S_2 two surfaces of the family $f(x^\mu) = cost$, to which the vector field $\vec{V}$ is orthogonal. As an example, we shall assume that $\vec{V}$ is timelike, but a similar procedure can be used if $\vec{V}$ is spacelike. If $\vec{V}$ is timelike, it is convenient to choose the basis vector $\vec{e}_{(0)}$ parallel to $\vec{V}$, and the remaining basis vectors as the tangent vectors to some curves lying on the surface, so that

$$g_{00} = g(\vec{e}_{(0)}, \vec{e}_{(0)}) = \vec{e}_{(0)} \cdot \vec{e}_{(0)} \neq 0 \qquad (10.52)$$
$$g_{0i} = g(\vec{e}_{(0)}, \vec{e}_{(i)}) = 0, \qquad i = 1, 2.$$

Thus, with this choice, the metric becomes

$$ds^2 = g_{00}(dx^0)^2 + g_{ik}(dx^i)(dx^k), \qquad i, k = 1, 2 \; . \qquad (10.53)$$

The generalization of this example to the four-dimensional spacetime, in which case the surface S is a hypersurface, is straightforward.

In general, given a timelike vector field $\vec{V}$, we can always choose a coordinate frame such that $\vec{e}_{(0)}$ is parallel to $\vec{V}$, so that in this frame

$$V^\alpha(x^\mu) = (V^0(x^\mu), 0, 0, 0) \, . \qquad (10.54)$$

Such coordinate system is said **comoving**. If, in addition, $\vec{V}$ is hypersurface-horthogonal, then $g_{0i} = 0$ and, as a consequence, the one-form associated to $\vec{V}$ also has the form

$$V_\alpha(x^\mu) = (V_0(x^\mu), 0, 0, 0) \, , \qquad (10.55)$$

since $V_i = g_{i\mu}V^\mu = g_{i0}V^0 + g_{ik}V^k = 0$.

10.6 Appendix A

We want to show that eq. (10.8) is equivalent to eq. (10.9).

$$\xi_{\alpha;\beta} = (g_{\alpha\mu}\xi^{\mu})_{;\beta} = g_{\alpha\mu}\xi^{\mu}_{;\beta} = g_{\alpha\mu}\left(\xi^{\mu}_{,\beta} + \Gamma^{\mu}{}_{\delta\beta}\xi^{\delta}\right), \tag{10.56}$$

hence

$$\begin{aligned}\xi_{\alpha;\beta} + \xi_{\beta;\alpha} &= g_{\alpha\mu}\left(\xi^{\mu}_{,\beta} + \Gamma^{\mu}{}_{\delta\beta}\xi^{\delta}\right) \\ &+ g_{\beta\mu}\left(\xi^{\mu}_{,\alpha} + \Gamma^{\mu}{}_{\alpha\delta}\xi^{\delta}\right) \\ &= g_{\alpha\mu}\xi^{\mu}_{,\beta} + g_{\beta\mu}\xi^{\mu}_{,\alpha} + \left(g_{\alpha\mu}\Gamma^{\mu}{}_{\delta\beta} + g_{\beta\mu}\Gamma^{\mu}{}_{\alpha\delta}\right)\xi^{\delta}.\end{aligned} \tag{10.57}$$

The term in parenthesis can be written as

$$\begin{aligned}&\frac{1}{2}\left[g_{\alpha\mu}g^{\mu\sigma}\left(g_{\delta\sigma,\beta} + g_{\sigma\beta,\delta} - g_{\delta\beta,\sigma}\right) + g_{\beta\mu}g^{\mu\sigma}\left(g_{\alpha\sigma,\delta} + g_{\sigma\delta,\alpha} - g_{\alpha\delta,\sigma}\right)\right] \\ &= \frac{1}{2}\left[\delta^{\sigma}_{\alpha}\left(g_{\delta\sigma,\beta} + g_{\sigma\beta,\delta} - g_{\delta\beta,\sigma}\right) + \delta^{\sigma}_{\beta}\left(g_{\alpha\sigma,\delta} + g_{\sigma\delta,\alpha} - g_{\alpha\delta,\sigma}\right)\right] \\ &= \frac{1}{2}\left[g_{\delta\alpha,\beta} + g_{\alpha\beta,\delta} - g_{\delta\beta,\alpha} + g_{\alpha\beta,\delta} + g_{\beta\delta,\alpha} - g_{\alpha\delta,\beta}\right] \\ &= g_{\alpha\beta,\delta},\end{aligned} \tag{10.58}$$

and eq. (10.57) becomes

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = g_{\alpha\mu}\xi^{\mu}_{,\beta} + g_{\beta\mu}\xi^{\mu}_{,\alpha} + g_{\alpha\beta,\delta}\xi^{\delta} \tag{10.59}$$

which coincides with eq. (10.8).

10.7 Appendix B: The Levi-Civita completely antisymmetric pseudotensor

We define the *Levi-Civita symbol* (also said *Levi-Civita tensor density*), $e_{\alpha\beta\gamma\delta}$, as an object whose components change sign under interchange of any pair of indices, and whose non-zero components are ± 1. Since it is completely antisymmetric, all the components with two equal indices are zero, and the only non-vanishing components are those for which all four indices are different. We set

$$e_{0123} = 1. \tag{10.60}$$

Under general coordinate transformations, $e_{\alpha\beta\gamma\delta}$ does not transform as a tensor; indeed, under the transformation $x^{\alpha} \to x^{\alpha\prime}$,

$$\frac{\partial x^{\alpha}}{\partial x^{\alpha\prime}}\frac{\partial x^{\beta}}{\partial x^{\beta\prime}}\frac{\partial x^{\gamma}}{\partial x^{\gamma\prime}}\frac{\partial x^{\delta}}{\partial x^{\delta\prime}}\, e_{\alpha\beta\gamma\delta} = J\, e_{\alpha\prime\beta\prime\gamma\prime\delta\prime} \tag{10.61}$$

where J is defined (see Chapter 7) as

$$J \equiv \det\left(\frac{\partial x^{\alpha}}{\partial x^{\alpha\prime}}\right) \tag{10.62}$$

and we have used the definiton of determinant.

We now define the *Levi-Civita pseudo-tensor* as

$$\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g}\, e_{\alpha\beta\gamma\delta}\,. \tag{10.63}$$

Since, from (8.26), for a coordinate transformation $x^\alpha \to x^{\alpha\prime}$

$$|J| = \frac{\sqrt{-g'}}{\sqrt{-g}}\,, \tag{10.64}$$

then

$$\epsilon_{\alpha\beta\gamma\delta} \to \epsilon_{\alpha\prime\beta\prime\gamma\prime\delta\prime} = \mathrm{sign}(J)\,\frac{\partial x^\alpha}{\partial x^{\alpha\prime}}\frac{\partial x^\beta}{\partial x^{\beta\prime}}\frac{\partial x^\gamma}{\partial x^{\gamma\prime}}\frac{\partial x^\delta}{\partial x^{\delta\prime}}\,\epsilon_{\alpha\beta\gamma\delta}\,. \tag{10.65}$$

Thus, $\epsilon_{\alpha\beta\gamma\delta}$ is not a tensor but a pseudo-tensor, because it transforms as a tensor times the sign of the Jacobian of the transformation. It transforms as a tensor only under a subset of the general coordinate transformations, i.e. that with $\mathrm{sign}(J) = +1$.

Warning: do not confuse the Levi-Civita symbol, $e_{\alpha\beta\gamma\delta}$, with the Levi-Civita pseudo-tensor, $\epsilon_{\alpha\beta\gamma\delta}$.

Chapter 11

The Schwarzschild solution

The Schwarzschild solution was first derived by Karl Schwarzschild in 1916, although a complete understanding of the Schwarzschild spacetime was achieved much recently. The paper was communicated to the Berlin Academy by Einstein on 13 January 1916, just about two months after he had published the seminal papers on the theory of General Relativity. In those years Schwarzschild was very ill. He had contracted a fatal desease in 1915 while serving the German army at the eastern front. He died on 11 May 1916, and during his illness he wrote two papers in General Relativity, one describing the solution for the gravitational field exterior to a spherically symmetric non rotating body, which we are going to derive, and the second describing the interior solution for a star of constant density which we shall discuss later.

We now want to find an exact solution of Einstein's equations in vacuum, which is spherically symmetric and static. This will be the relativistic generalization of the newtonian solution for a pointlike mass

$$V = -\frac{GM}{r}, \tag{11.1}$$

and it will describe the gravitational field in the exterior of a non rotating body. Let us first discuss the symmetries of the problem.

11.1 The symmetries of the problem

a) *Symmetry with respect to time.*

Time-symmetric spacetimes can be stationary or static. A spacetime is said to be **stationary** if it admits a **timelike Killing vector** $\vec{\xi}$. It follows from the Killing equations that the metric of a stationary spacetime does not depend on time

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0. \tag{11.2}$$

A spacetime is **static** if it admits a **hypersurface-orthogonal, timelike Killing vector**. In this case, as shown in Chapter 12, we can choose the coordinates in such a way that

$\vec{\xi} \to (1,0,0,0)$, and the line-element takes the simple form

$$ds^2 = g_{00}(x^i)(dx^0)^2 + g_{kn}(x^i)dx^k dx^n, \qquad i,k,n = 1,3, \qquad (11.3)$$
$$\text{where} \qquad g_{00} = g(\vec{\xi},\vec{\xi}) = \vec{\xi}\cdot\vec{\xi}.$$

From this equation we see that the metric is not only independent of time, but also invariant under time reversal $t \to -t$. (If terms like $dx^0 dx^i$ were present this would not be true).
b) *Spatial symmetry.*

We now take care of the spatial part of the metric. The basic idea is that we want to "fill" the space with concentric spherical surfaces. We start with the 2-sphere of radius a in flat space

$$ds^2_{(2)} = g_{22}(dx^2)^2 + g_{33}(dx^3)^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (11.4)$$

The surface of this sphere is

$$A = \int \sqrt{g} d\theta d\varphi = \int_0^\pi a^2 \sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi a^2, \qquad (11.5)$$

and the lenght of the circumference

$$\theta = \frac{\pi}{2}, \qquad dl = a d\varphi, \qquad C = 2\pi a. \qquad (11.6)$$

These results continue to hold if a is an arbitrary function of the remaining coordinates x^0, x^1

$$ds^2_{(2)} = a^2(x^0,x^1)(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (11.7)$$

But since we have already established that the metric does not depend on time, we put $a = a(x^1)$. We are now free to make a coordinate transformation and put

$$r = a(x^1). \qquad (11.8)$$

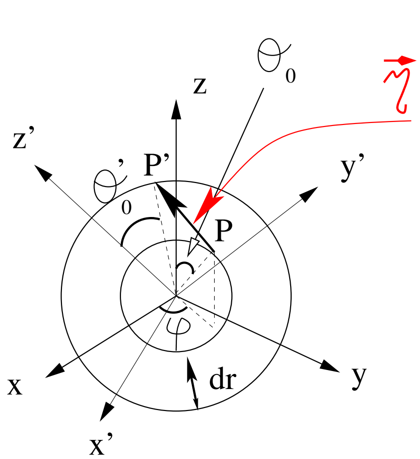
Thus we **define** the radial coordinate as being half the ratio between the surface and the circumference of the 2-sphere. However, it should be noted that in principle the coordinate r has nothing to do with the distance between the center of the sphere and the surface , as we shall later show.

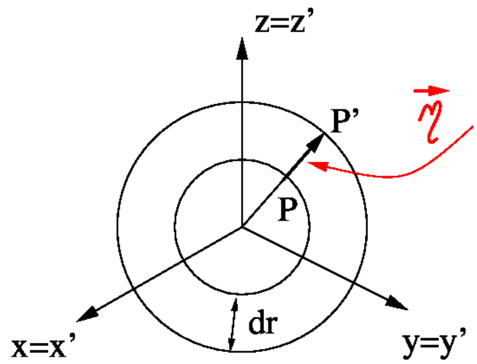
Then we go to the next sphere at $r + dr$. We may label the points of the second sphere with different (θ', φ') as indicated in the figure

If the poles of the two sferes are not aligned, the vector $\vec{\eta}$ which maps the point $P = (\theta_0, \varphi_0)$ on the internal sphere (θ_0, φ_0 constants), to the point $P' = (\theta'_0, \varphi'_0)$ on the external sphere (with $\theta'_0 = \theta_0$ and $\varphi'_0 = \varphi_0$), is directed as indicated in the figure. Conversely, if the poles are aligned $\vec{\eta}$ is orthogonal to the two spheres, and therefore it is orthogonal to $\frac{\partial}{\partial\theta} = \vec{e}_{(\theta)}$ and $\frac{\partial}{\partial\varphi} = \vec{e}_{(\varphi)}$, which are the basis vectors on the sphere. Thus in this case $\vec{\eta}$ is **hypersurface-orthogonal**.

Since we want angular coordinates (θ, φ) defined in a unique way on the whole set of spheres filling the space, we require that $\vec{\eta}$ is indeed orthogonal to the spheres. In this case $\vec{\eta}$ is the vector tangent to the coordinate line $(\theta = const, \varphi = const)$, therefore $\vec{\eta} = \frac{\partial}{\partial r} = \vec{e}_{(r)}$. The orthogonality condition then gives

$$\vec{e}_{(r)} \cdot \vec{e}_{(\theta)} = g_{r\theta} = 0, \qquad \vec{e}_{(r)} \cdot \vec{e}_{(\varphi)} = g_{r\varphi} = 0. \qquad (11.9)$$





Under these assumptions, the metric of the three-space becomes

$$ds^2_{(3)} = g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{11.10}$$

and that of the four-dimensional spacetime finally is

$$ds^2 = g_{00}(dx^0)^2 + g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{11.11}$$

At this point the two metric components g_{00} and g_{rr} should, in principle, depend on (r, θ, ϕ). However, this is not the case. Indeed, if we consider a set of new polar coordinates (θ', ϕ') to label the points on the two sferes that fill the space, neither the vector $\vec{e}_0$, nor the vector $\vec{e}_r$ will change and therefore they cannot depend on the angular coordinates we choose. As a consequence g_{00} and g_{rr} do not depend on (θ, ϕ) either, and we can write

$$g_{00} = g_{00}(r), \qquad \text{and} \qquad g_{rr} = g_{rr}(r).$$

It is convenient to rewrite the metric in the following form

$$ds^2 = -e^{2\nu}(dx^0)^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{11.12}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. Let us now compute the distance between two points $P1 = (x^0_*, r_1, \theta_*, \varphi_*)$, and $P2 = (x^0_*, r_2, \theta_*, \varphi_*)$

$$l = \int_{r_1}^{r_2} e^{\lambda} dr\,. \tag{11.13}$$

This distance **does not** coincide with $(r_2 - r_1)$.

We now write the components of the Einstein tensor in terms of the metric (11.13). They are

$$
\begin{aligned}
a) \quad & G_{00} = \frac{1}{r^2} e^{2\nu} \frac{d}{dr} \left[r(1 - e^{-2\lambda}) \right] \\
b) \quad & G_{rr} = -\frac{1}{r^2} e^{2\lambda} \left[(1 - e^{-2\lambda}) \right] + \frac{2}{r} \nu_{,r} \\
c) \quad & G_{\theta\theta} = r^2 e^{-2\lambda} \left[\nu_{,rr} + \nu_{,r}^2 + \frac{\nu_{,r}}{r} - \nu_{,r}\lambda_{,r} - \frac{\lambda_{,r}}{r} \right] \\
d) \quad & G_{\varphi\varphi} = \sin^2\theta G_{\theta\theta}
\end{aligned}
\tag{11.14}
$$

The remaining components identically vanish. Since we are looking for a vacuum solution, the equations to solve are

$$G_{\mu\nu} = 0, \tag{11.15}$$

and eq. (11.14a) gives

$$r(1 - e^{-2\lambda}) = K, \tag{11.16}$$

where K is an integration constant. Hence

$$e^{2\lambda} = \frac{1}{1 - \frac{K}{r}}. \tag{11.17}$$

From eq. (11.14b) we find

$$\nu_{,r} = \frac{1}{2} \frac{K}{r(r - K)}, \tag{11.18}$$

and therefore

$$\nu = \frac{1}{2} \log\left(1 - \frac{K}{r}\right) + \nu_0, \qquad \rightarrow \qquad e^{2\nu} = \left(1 - \frac{K}{r}\right) e^{2\nu_0}, \tag{11.19}$$

where ν_0 is a constant. We can rescale the time coordinate

$$t \rightarrow e^{\nu_0} t,$$

in such a way that $e^{2\nu}$ becomes

$$e^{2\nu} = 1 - \frac{K}{r}. \tag{11.20}$$

The final form of the solution is

$$ds^2 = -\left(1 - \frac{K}{r}\right) c^2 dt^2 + \frac{1}{1 - \frac{K}{r}} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{11.21}$$

This is the **Schwarzschild solution**. When $r \rightarrow \infty$ the metric reduces to that of a flat spacetime, therefore we say that the metric is **asymptotically flat.**

Now we want to understand what is the meaning of the integration constant K. In Chapter 8 section 1, we showed that in the weak-field limit, the geodesic equations reduce to the newtonian equations of motion, and consequently

$$g_{00} \sim -\left(1 + \frac{2\Phi}{c^2}\right) = -\left(1 - \frac{2GM}{c^2 r}\right), \qquad \text{where} \tag{11.22}$$

$\Phi = -\frac{GM}{r}$ is the newtonian potential generated by a spherical distribution of matter. From eq. (11.20) we see that when $r \to \infty$ g_{00} tends to unity as

$$-g_{00} = e^{2\nu} = 1 - \frac{K}{r}. \tag{11.23}$$

By comparing eq. (11.22) and (11.23) we find

$$K = \frac{2GM}{c^2}. \tag{11.24}$$

Therefore the constant K is the physical mass multiplied by $\frac{2G}{c^2}$. It is easy to check that the solution (11.21) satisfies eq. (11.14c).

11.2 The Birkhoff theorem

The solution (11.21) has been found by imposing that the spacetime is static and spherically symmetric, therefore it represents the gravitational field external to a non-rotating, sperically symmetric body whose structure is time-independent. However, it is more general than that. In fact Birkhoff's theorem establishes that it is the only spherically symmetric and asymptotically flat solution of the vacuum Einstein field equations. Let us assume that the functions (ν, λ) in the metric (11.12) depend both on the radial coordinate **and** on time. To prove Birkhoff's theorem we only need the components R_{0r} and $R_{\theta\theta}$ of the Ricci tensor:

$$a) \quad R_{0r} = \frac{2}{r}\frac{\partial \lambda}{\partial x^0} = 0, \tag{11.25}$$

$$b) \quad R_{\theta\theta} = 1 - e^{-2\lambda}\left[1 + r\frac{\partial(\nu - \lambda)}{\partial r}\right] = 0.$$

From eq. (11.25a) it follows that λ must depend only upon the radial coordinate r. Then from eq. (11.25b) it follows that also $\frac{\partial \nu}{\partial r}$ must be independent of x^0 and consequently

$$\nu = \nu(r) + f(x^0). \tag{11.26}$$

This means that the coefficient of $(dx^0)^2$ in the line element is $e^{2\nu(r)}e^{2f(x^0)}$. But the term $e^{2f(x^0)}$ can be 'reabsorbed' by a coordinate transformation

$$dt' = e^{f(x^0)}dt, \tag{11.27}$$

so that the new metric coefficients are

$$\nu = \nu(r), \qquad \lambda = \lambda(r), \tag{11.28}$$

and the metric is time independent. This means that even if we impose that the central object evolves in time, as it would be for example in the case of a star radially pulsating, or in a spherical collapse, we would find, in the exterior, the same Schwarzschild metric, and since the spacetime remains static even in these cases, gravitational waves could not be emitted. The conclusion is that spherically symmetric systems can never emit gravitational waves. A similar situation occurs in electrodynamics: a spherically symmetric distribution of charges and currents does not radiate.

11.3 Geometrized units

From eq. (11.23) and (11.24) is easy to see that $K = \frac{2GM}{c^2}$ must have the dimension of a lenght. Remembering that

$$G = 6.67 \times 10^{-8}\ \frac{cm^3}{gs^2}, \qquad c = 2.998 \times 10^{10}\ cm/s,$$

the ratio $\frac{G}{c^2}$ is

$$\frac{G}{c^2} = 0.7425 \times 10^{-28} cm \cdot g^{-1}. \tag{11.29}$$

It is often convenient to put

$$G = c = 1, \tag{11.30}$$

which means that we measure the mass, as the lenghts, in cm. We shall often adopt this convention, and we will indicate the geometrical mass (i.e. the mass in cm) as m.

In these unities the Schwarzschild solution becomes

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1-\frac{2m}{r}} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{11.31}$$

11.4 The singularities of Schwarzschild solution

Let us examine the metric (11.31) in some more detail. We immediately see that there is a problem when $r \to 2m$: $g_{00} \to 0$, and $g_{rr} \to \infty$. Moreover, when $r \to 0$, $g_{00} \to \infty$, and $g_{rr} \to 0$. In both cases we say that there is a singularity, but of a different nature. In order to check wheter a singularity is a genuine curvature singularity, we should compute the scalars which we can construct from the Riemann tensor and see if they diverge. To check whether the Riemann tensor is well-behaved is not enough, in fact for the Schwarzschild metric the components of $R^{\alpha}{}_{\beta\gamma\delta}$ are

$$\begin{aligned}
R^{t}{}_{rtr} &= -2\frac{m}{r^3}\left(1 - \frac{2m}{r}\right)^{-1} \\
R^{t}{}_{\theta t\theta} &= \frac{1}{\sin^2\theta} R^{t}{}_{\varphi t\varphi} = \frac{m}{r^5} \\
R^{\theta}{}_{\varphi\theta\varphi} &= 2\frac{m}{r^5}\sin^2\theta \\
R^{r}{}_{\theta r\theta} &= \frac{1}{\sin^2\theta} R^{r}{}_{\varphi r\varphi} = -\frac{m}{r^5}
\end{aligned} \tag{11.32}$$

and they diverge both at $r = 0$, and at $r = 2m$. However, if we compute the scalar invariants, like $R^{abcd}R_{abcd}$, we find that they diverge only at $r = 0$. We conclude that $r = 0$ is a true curvature singularity, while $r = 2m$ is only a *coordinate singularity*, due to an unappropriate choice of the coordinates.

We shall now analyse the properties of the surface $r = 2m$.

11.5 Spacelike, Timelike and Null Surfaces

In a curved background hypersurfaces are classified in the following way. Consider a generic hypersurface Σ

$$\Sigma(x^\mu) = 0, \tag{11.33}$$

Be $\vec{n}$ the normal vector dual to the gradient one-form

$$n_\alpha = \Sigma_{,\alpha} \tag{11.34}$$

If t^α is a tangent vector to the surface, then $t^\alpha n_\alpha = 0$. Indeed, $t^\alpha = \frac{dx^\alpha}{d\lambda}$ with $x^\alpha(\lambda)$ curve on Σ; therefore,

$$t^\alpha n_\alpha = \frac{dx^\alpha}{d\lambda}\frac{\partial\Sigma}{\partial x^\alpha} = \frac{d\Sigma}{d\lambda} = 0\,. \tag{11.35}$$

At any point of the hypersurface we can introduce a locally inertial frame, and rotate it in such a way that the components of $\vec{n}$ are

$$n^\alpha = (n^0, n^1, 0, 0) \qquad \text{and} \qquad n_\alpha n^\alpha = (n^1)^2 - (n^0)^2. \tag{11.36}$$

Consider a vector t^α tangent to Σ at the same point. t^α must be orthogonal to $\vec{n}$

$$n_\alpha t^\alpha = -n^0 t^0 + n^1 t^1 = 0 \qquad \to \qquad \frac{t^0}{t^1} = \frac{n^1}{n^0}. \tag{11.37}$$

From eq. (11.37) it follows that

$$t^\alpha = \Lambda(n^1, n^0, a, b) \qquad \text{with a, b e} \quad \Lambda \qquad \text{costant and arbitrary.} \tag{11.38}$$

Consequently the norm of $\vec{t}$ is

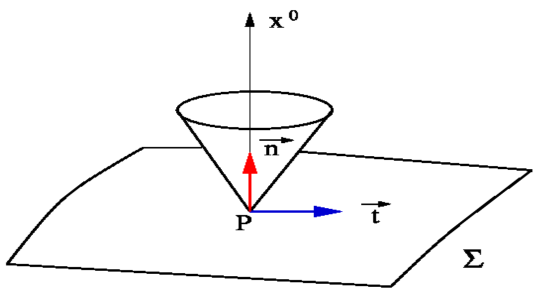
$$t_\alpha t^\alpha = \Lambda^2[-(n^1)^2 + (n^0)^2 + (a^2+b^2)] = \Lambda^2[-n_\alpha n^\alpha + (a^2+b^2)]. \tag{11.39}$$

There are three possibilities:

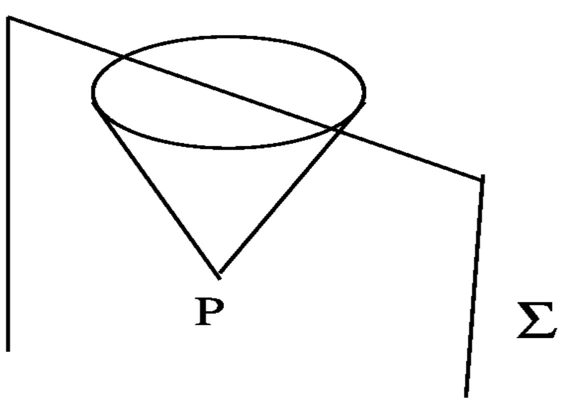
1) $n_\alpha n^\alpha < 0$, $\to$ n^α is a timelike vector $\to$ Σ **is spacelike**
2) $n_\alpha n^\alpha > 0$, $\to$ n^α is a spacelike vector $\to$ Σ **is timelike**
3) $n_\alpha n^\alpha = 0$, $\to$ n^α is a null vector $\to$ Σ **is null**

We shall now see how the normal and the tangent vectors are directed in order to understand the disposition of the light-cones with respect to the hypersurface.

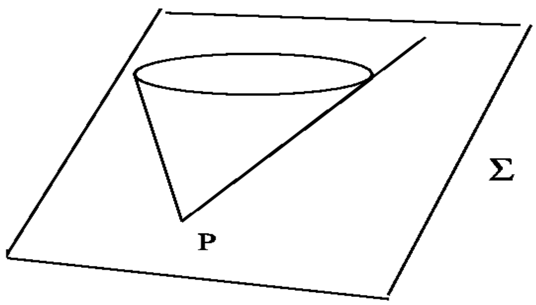
1) If $n_\alpha n^\alpha < 0$, then $t_\alpha t^\alpha > 0$ and $\vec{t}$ is spacelike. Consequently no tangent vector to Σ in P lies inside, or on the light-cone through P. Since a massive particle which starts at P must move inside the light-cone (or on the light-cone if it is massless), this means that **a spacelike hypersurface can be crossed only in one direction.**



2) If $n_\alpha n^\alpha > 0$, then $t_\alpha t^\alpha$ can be positive, negative or null depending on the value of $a^2 + b^2$. Therefore there will be some tangent vectors which lie inside the light-cone . Consequently a **timelike hypersurface can be crossed inward and outward.**



3) If $n_\alpha n^\alpha = 0$, then $t_\alpha t^\alpha$ is positive (t^α is spacelike), or null if $a = b = 0$. In this case there will be only one tangent vector (and all its multiples) at P which lies on Σ and on the light-cone



For example, in Minkowski spacetime $t = const$ is a spacelike surface, and any physical object can pass it in only one direction without violating causality. $x = const$ is a timelike surface, and physical objects can pass it in either direction, $x - ct = 0$ is a null surface. Let us now try to understand what kind of surface $r = 2m$ is.

Consider a generic hypersurface $r = const$ in the Schwarzschild geometry

$$\Sigma = r - cost = 0. \tag{11.40}$$

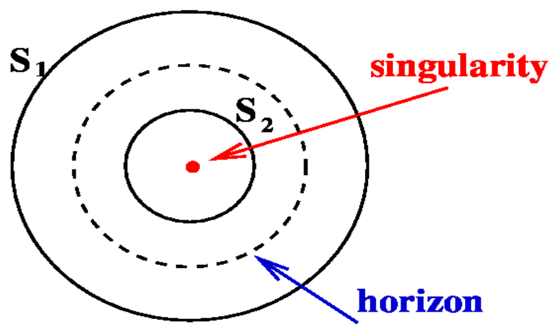
The norm of the normal vector is

$$n_\alpha n^\alpha = g^{\alpha\beta} n_\alpha n_\beta = g^{\alpha\beta} \Sigma_{,\alpha} \Sigma_{,\beta} = \\ g^{00}\Sigma_{,0}^2 + g^{rr}\Sigma_{,r}^2 + g^{\theta\theta}\Sigma_{,\theta}^2 + g^{\varphi\varphi}\Sigma_{,\varphi}^2 = g^{rr}\Sigma_{,r}^2 = \left(1 - \frac{2m}{r}\right). \tag{11.41}$$

From eq. (11.41) it follows that

$$\begin{array}{lllll} r > 2m & \rightarrow & n_\alpha n^\alpha > 0, & \Sigma & \text{is timelike} \\ r = 2m & \rightarrow & n_\alpha n^\alpha = 0, & \Sigma & \text{is null} \\ r < 2m & \rightarrow & n_\alpha n^\alpha < 0, & \Sigma & \text{is spacelike} \end{array}$$

Consider for example S_1 and S_2 as shown in the following figure



Any signal which starts at some point of S_1 can be sent both toward the origin and outward, since S_1 is timelike. Conversely, a signal which starts at a point of S_2 in the interior of $r = 2m$ must necessarily go inward, and be captured by the sigularity at $r = 0$, since S_2 is spacelike. The surface $r = 2m$ is a null surface, which is basically the transition from a spacelike to a timelike hypersurface. On the surface $r = 2m$ the timelike Killing vector $\vec{\xi}_{(t)}$ becomes null and it is spacelike for $r < 2m$.

The Schwarzschild solution is said to represent the gravitational field of a **black hole**, and the hypersurface $r = 2m$ is called the **event horizon**. The reason for these names is that if we are outside $r = 2m$ we can send a signal both inward and outward, but as soon as we cross the horizon any signal will inevitably bend toward the singularity: there is no way to know what happens inside the horizon.

As we mentioned before, $r = 0$ is a genuine curvature singularity. Thus General Relativity predicts the existence of singularities hidden by a horizon.

11.6 How to remove a coordinate singularity

In general, it is not a simple problem to understand whether a singularity is a genuine curvature singularity or it is only a coordinate singularity. The first thing to do is to compute the Riemann tensor and the scalars which can be computed from it, like $R^{abcd}R_{abcd}$ and check whether they diverge somewhere. If this is not the case, the singularity is due to a bad choice of the coordinate system, and a suitable choice of a new set of coordinates should remove it. If this can be done, we say that we are extending our original spacetime $(M, g_{\alpha\beta})$ to a larger spacetime $(\tilde{M}, \tilde{g}_{\alpha\beta})$ which includes the original one. Before analyzing the Schwarzschild case, let us consider two examples.

Consider the two-dimensional spacetime

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2, \qquad 0 < t < \infty, \qquad -\infty < x < \infty. \tag{11.42}$$

$(c = G = 1.)$ The metric is singular at $t = 0$. The coordinate transformation

$$t' = \frac{1}{t} \quad \rightarrow \quad dt' = -\frac{1}{t^2}dt, \tag{11.43}$$

gives

$$ds^2 = -(dt')^2 + dx^2, \tag{11.44}$$

Thus the metric (11.42) represents a flat spacetime. The metric (11.44) is defined for any t' , i.e. $-\infty < t' < \infty$, therefore it describes regions of the spacetime which where "unaccessible" to the coordinates (11.42). In fact in that case $0 < t < \infty$, which corresponds only to the section $0 < t' < \infty$, of our new spacetime. This is the reason why we say that the new coordinates provide an extension of the spacetime. The coordinate singularity $t = 0$ is mapped onto the line $t' \rightarrow \infty$. The new spacetime is said to be **geodesically complete** because any geodesic which starts at any given point of the spacetime, can be extended for arbitrarily large values of the affine parameter. Conversely, the original spacetime (11.42) was **geodesically incomplete** for the following reason. We have established that the spacetime is flat, and it extends from $-\infty$ to $+\infty$ in both coordinates (t', x'). In eq. (11.42) we were trying to cover our infinite spacetime with coordinates which vary in a semi-infinite range $(0 < t < \infty)$. This is the reason why the singularity $t = 0$ appears. With those coordinates we were able to cover only the region $(0 < t' < \infty)$ of the complete spacetime, but not the region $(-\infty < t' < 0)$. Consequently, geodesics which start in the region $t' < 0$, cross the axis and continue in the region $t' > 0$, cannot be completely represented in the spacetime described by (t, x) : they will terminate for a finite value of the proper time.

Another example is the Rindler spacetime, which has interesting similarities with the Schwarzschild geometry. The line-element is

$$ds^2 = -x^2dt^2 + dx^2, \qquad -\infty < t < \infty, \qquad 0 < x < \infty. \tag{11.45}$$

The metric is singular at $x = 0$. The determinant g vanishes at $x = 0$, therefore $g^{\mu\nu}$ is also singular. Let us consider goedesics in this spacetime. Since the metric is independent of time, it admits a timelike Killing vector $\vec{\xi}_{(t)} \rightarrow (1, 0)$. According to eq. (10.35)

$$\xi_{(t)\alpha}U^\alpha = g_{\alpha\beta}\xi^\alpha_{(t)}U^\beta = const = -E, \qquad \rightarrow \qquad U^0 = \frac{E}{x^2}, \tag{11.46}$$

where $U^\beta = \frac{dx^\beta}{d\lambda}$ and λ is an affine parameter (not necessarily the proper time). Therefore,

$$\frac{dt}{d\lambda} = \frac{E}{x^2}\,. \tag{11.47}$$

Since the norm of the vector tangent to the worldline of a massive particle is -1, then

$$U^\mu U^\nu g_{\mu\nu} = -x^2\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2 = -1, \tag{11.48}$$

thus

$$\left(\frac{dx}{d\lambda}\right)^2 = x^2\left(\frac{dt}{d\lambda}\right)^2 - 1 = \frac{E^2}{x^2} - 1\,. \tag{11.49}$$

Hence

$$\frac{dx}{d\lambda} = \pm\sqrt{\frac{E^2}{x^2} - 1}, \quad \to \quad \lambda = \int^x \frac{xdx}{\sqrt{E^2 - x^2}} = -\sqrt{E^2 - x^2} + const. \tag{11.50}$$

Thus a particle starting at some point x reaches $x = 0$ in a finite interval of the affine parameter: Rindler spacetime is **geodesically incomplete**. However, since the Riemann tensor and the curvature scalars do not diverge at $x = 0$, there must exist a coordinate transformation which brings the metric into a non-singular form. Unfortunately a systematic approach to the problem of finding the "right" coordinates to extend the metric does not exist. We shall describe a procedure which is based on the behaviour of null geodesics. In two dimension the situation is easier, since null geodesics belong, *at least locally* to two classes: ingoing and outgoing. Two geodesics belonging to the same class cannot cross, because the two tangent vectors should coincide at that point, and consequently the two geodesic should coincide everywhere (remember that geodesics parallel-transport their own tangent vector). If

$$\vec{K} \to \{\frac{\delta x^\mu}{d\lambda}\}, \tag{11.51}$$

is the vector tangent to the null geodesic whose affine parameter is λ, we must have that

$$g_{\mu\nu}K^\mu K^\nu = 0. \tag{11.52}$$

In the case we are considering it becomes

$$0 = g_{\mu\nu}K^\mu K^\nu = -x^2\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2, \tag{11.53}$$

from which we find

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2}. \tag{11.54}$$

Therefore along the null geodesic

$$t = \pm\log x + const, \tag{11.55}$$

where the + identifies the outgoing geodesics and the - the ingoing geodesics. Accordingly, we define the null ingoing and outgoing coordinates as

$$u = t - \log x \qquad \text{and} \qquad v = t + \log x \tag{11.56}$$

and the metric in the new coordinates becomes

$$ds^2 = -e^{v-u} du dv. \tag{11.57}$$

The coordinates u and v vary in the range $(-\infty, +\infty)$, and they cover the original region $x > 0$, (they do not extend the spacetime yet!), thus we haven't solved the problem of eliminating the singularity. An extension of the spacetime can be accomplished if we reparametrize the null geodesics with new coordinates

$$\begin{aligned} U &= U(u) \\ V &= V(v). \end{aligned} \tag{11.58}$$

The form of the metric is so simple that we may define U and V immediately. But to have a feeling on what one should do in general we proceed in a more systematic way. From eqs. (11.46) and (11.51) it follows that for a massless particle

$$\xi_{(t)\alpha} K^\alpha = g_{\alpha\beta} \xi^\alpha_{(t)} K^\beta = const = -E, \qquad \rightarrow \qquad d\lambda = \frac{x^2}{E} dt. \tag{11.59}$$

Since $dt = \frac{1}{2} d(u+v)$, if we put $u = const$ and move along a null direction parallel to the $v-$axis, i.e. along an outgoing null geodesic, eq. (11.59) becomes

$$\begin{aligned} d\lambda &= \frac{x^2}{2E} dv, \qquad \text{or, since} \qquad 2\log x = v - u \rightarrow x = e^{\frac{v-u}{2}}, \\ \lambda &= \frac{1}{2E} \int e^{(v-u)} dv = C + \left(\frac{e^{-u}}{2E}\right) e^v, \end{aligned} \tag{11.60}$$

where C is a constant. If we shift $\lambda \rightarrow \frac{\lambda - C}{\frac{e^{-u}}{2E}}$, then the affine parameter along outgoing null geodesics becomes

$$\lambda_{out} = e^v. \tag{11.61}$$

Proceeding in a similar way we find that the affine parameter along ingoing null geodesics is

$$\lambda_{in} = -e^{-u}. \tag{11.62}$$

If we now choose

$$\begin{aligned} U &= -e^{-u} \\ V &= e^v, \end{aligned} \tag{11.63}$$

the metric becomes

$$\begin{aligned} ds^2 &= -dU dV, \qquad \text{or if we put} \qquad T = \frac{(U+V)}{2}, \quad X = \frac{(V-U)}{2} \\ ds^2 &= -dT^2 + dX^2, \end{aligned} \tag{11.64}$$

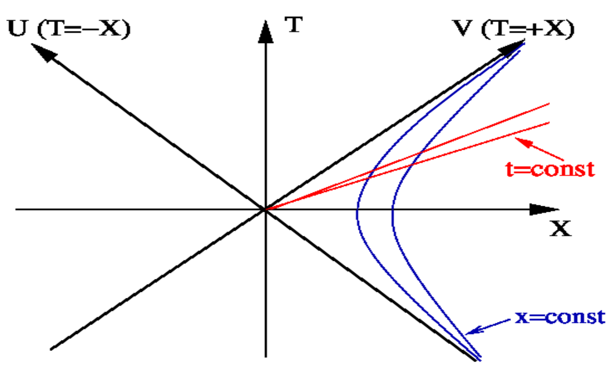
which is again a flat spacetime.

Summaryzing: 1) we find the equations for the ingoing and outgoing null geodesics, 2) we choose the affine parameters along these geodesics as coordinates, then we introduce X and T.

U and V range between $-\infty$ and $+\infty$. The original spacetime (x, t) coincides with the quadrant $[U < 0, V > 0]$, but since everything is regular at $[U = 0, V = 0]$, the metric is extended to the regions $U > 0$, and $V < 0$, which were not included before. The relation bewtween the old and the new coordinates is

$$x = (X^2 - T^2)^{\frac{1}{2}}$$
$$t = tanh^{-1}(\frac{T}{X}) = \frac{1}{2} \log \left(\frac{X + T}{X - T} \right) \tag{11.65}$$

A picture of the spacetime is given in the following figure



The singularity $x = 0$ corresponds to the lines $X = \pm T$, where the metric in the new coordinates is perfectly well behaved. From the second of eqs. (11.65)

$$X = -T \qquad \text{corresponds to} \qquad t \to -\infty$$
$$X = T \qquad \text{corresponds to} \qquad t \to +\infty. \tag{11.66}$$

The curves $x = const$ are now mapped onto the hyperbolae $X^2 - T^2 = const$, while the curves $t = const$ are mapped onto $T = constX$. The original Rindler space corresponds to the dashed region in the figure. Therefore we have finally extended the spacetime across the barrier $x = 0$.

If we now go back to Rindler's metric and consider the following coordinate transformation

$$y = \frac{1}{4} x^2, \tag{11.67}$$

the metric becomes

$$ds^2 = -4ydt^2 + \frac{1}{y}dy^2, \tag{11.68}$$

and rescaling the time coordinate $t \to 2t$

$$ds^2 = -ydt^2 + \frac{1}{y}dy^2. \tag{11.69}$$

This is similar to the form of the two-dimensional (t, r) part of the Schwarzschild metric.

11.7 The Kruskal extension

First we compute the null geodesics of the two-dimensional Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 \tag{11.70}$$

by imposing

$$0 = g_{\mu\nu} K^\mu K^\nu = -\left(1 - \frac{2m}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2. \tag{11.71}$$

Hence

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2m}{r}\right)^2 \quad \to \quad \frac{dt}{dr} = \pm\frac{r}{r - 2m}, \tag{11.72}$$

whose solution is

$$t = \pm r_* + const \tag{11.73}$$

where

$$r_* = r + 2m \log\left(\frac{r}{2m} - 1\right), \qquad \text{and} \qquad \frac{dr}{dr_*} = \left(1 - \frac{2m}{r}\right). \tag{11.74}$$

The coordinate r_* is called the "tortoise" coordinate, since if $r \to +\infty$ then $r_* \sim r$, but if $r \to 2m$ then $r \to -\infty$, thus as $r \to 2m$ r_* pushes the horizon to $-\infty$. We now define the null ingoing and outgoing coordinates

$$\begin{aligned} u &= t - r_* & & -\infty < u < +\infty, \\ v &= t + r_* & \to r_* = \tfrac{v-u}{2} \quad & -\infty < v < +\infty \end{aligned} \tag{11.75}$$

and the two-dimensional metric becomes

$$\begin{aligned} ds^2 = -\left(1 - \frac{2m}{r}\right)\left[dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)^2}\right] = \\ -\left(1 - \frac{2m}{r}\right)\left[dt^2 - dr_*^2\right] = -\left(1 - \frac{2m}{r}\right) dudv. \end{aligned} \tag{11.76}$$

hence [1]

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du dv = -\frac{2m}{r} e^{-\frac{r}{2m}} e^{\frac{v-u}{4m}} du dv. \tag{11.80}$$

Since $r = 2m$ corresponds to $u \to \infty$ and $v \to -\infty$, the metric (11.80) is regular everywhere. A comparison with the Rindler case shows that a convenient choice for U and V is

$$\begin{aligned} U &= -e^{-\frac{u}{4m}}, \quad \rightarrow \quad -\infty < U < 0 \\ V &= e^{\frac{v}{4m}}, \quad \rightarrow \quad 0 < V < +\infty \end{aligned} \tag{11.81}$$

The metric becomes

$$ds^2 = -\frac{32m^3 e^{-\frac{r}{2m}}}{r} dU dV. \tag{11.82}$$

The surface $r = 2m$ now corresponds to $U = 0$ **or** $V = 0$ where the metric (11.82) is non-singular. Therefore it can be extended across these two hypersurfaces to cover the whole two-dimensional spacetime. By introducing the coordinates T and X

$$T = \frac{V+U}{2} \qquad X = \frac{V-U}{2} \tag{11.83}$$

the four-dimensional metric finally becomes

$$ds^2 = \frac{32m^3 e^{-\frac{r}{2m}}}{r}[-dT^2 + dX^2] + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{11.84}$$

This extension was independently found by Kruskal and Szekeres in 1960. The relation between the old and the new coordinates is [2]

[1] From the definition of r_* we find

$$\frac{r_* - r}{2m} = \log\left(\frac{r-2m}{2m}\right) \quad \rightarrow \quad e^{\frac{r_*}{2m}} e^{-\frac{r}{2m}} = \frac{r-2m}{2m}. \tag{11.77}$$

$$(1 - \frac{2m}{r}) = \frac{r-2m}{2m}\frac{2m}{r} = \frac{2m}{r} e^{-\frac{r}{2m}} e^{\frac{r_*}{2m}}. \tag{11.78}$$

Since $r_* = (v-u)/2$, it follows

$$(1 - \frac{2m}{r}) = \frac{2m}{r} e^{-\frac{r}{2m}} e^{\frac{(v-u)}{4m}} \tag{11.79}$$

[2] The derivation of eqs. (11.85) and (11.86):

$$(X^2 - T^2) = \left(\frac{V-U}{2}\right)^2 - \left(\frac{V+U}{2}\right)^2 = -UV = +e^{-\frac{v-u}{4m}} = \frac{r}{2m}\left(1 - \frac{2m}{r}\right) e^{\frac{r}{2m}},$$

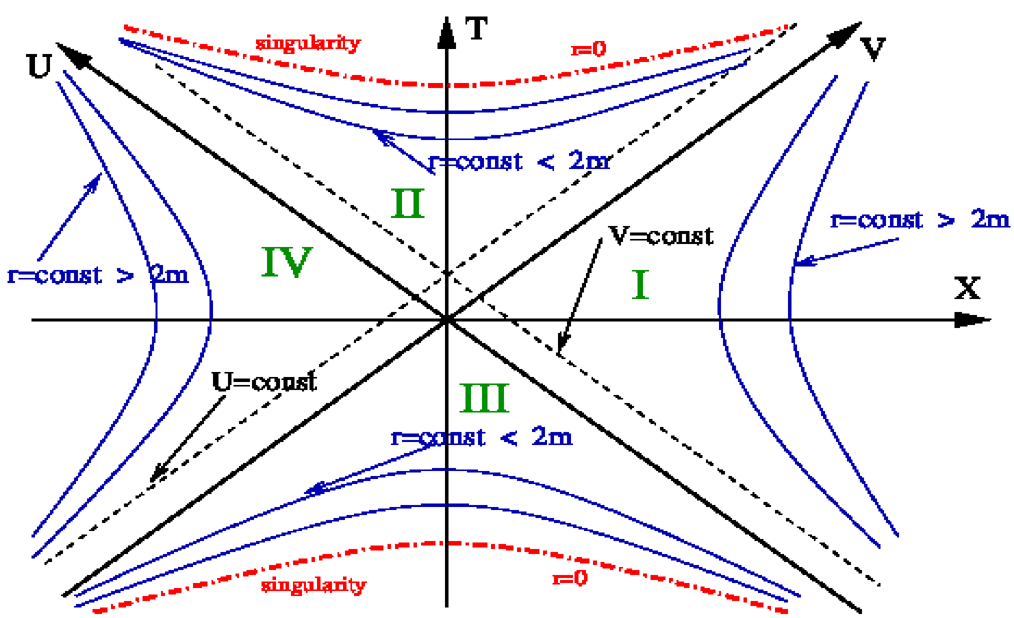
and

$$\log\left(\frac{X+T}{X-T}\right) = \log - \left(\frac{V}{U}\right) = \log e^{\frac{v-u}{4m}} = \frac{v+u}{4m} = \frac{t}{2m}.$$

$$(X^2 - T^2) = \left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}} \tag{11.85}$$

$$\frac{t}{2m} = \log\left(\frac{X+T}{X-T}\right) = 2tanh^{-1}\left(\frac{T}{X}\right) \tag{11.86}$$

The extended two-dimensional spacetime is shown in the following figure

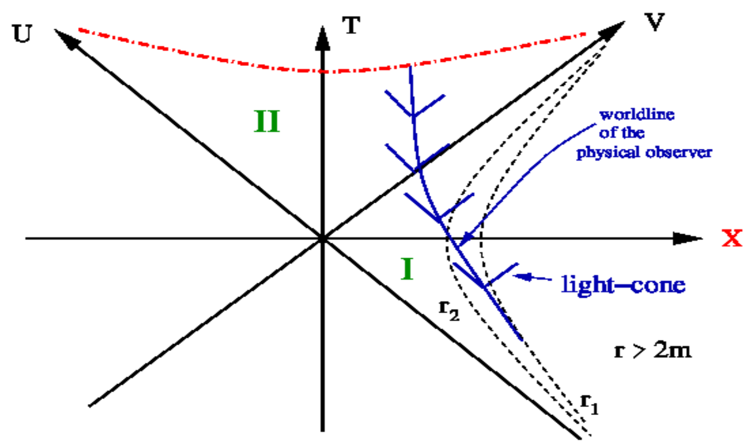


If $r = const > 2m$, from eq. (11.85) it follows that $X^2 - T^2 > 0$ and constant, and consequently $X = \pm\sqrt{T^2 + k}$, where $k = \left[\left(1 - \frac{2m}{r}\right) e^{\frac{r}{2m}}\right]_{r=const}$. These curves are indicated as continuous lines in the quadrants I and IV of the preceeding figure.

If $r = const < 2m$, $X^2 - T^2 < 0$ and constant, and $X = \pm\sqrt{T^2 - |k|}$. These curves are the dashed lines in the quadrants II and III. The curvature singularity $r = 0$

corresponds to the curves $X^2 - T^2 = -1$ and $X = \pm\sqrt{T^2 - 1}$ also represented in regions II and III. Radial null geodesics correspond to $U = const$ (ingoing) or to $V = const$ (outgoing). This diagram has the remarkable property that null geodesics are 45^0-straight lines. The curves $t = const$ are straight lines passing through the origin.

The original spacetime ($r > 0$), i.e.the Schwarzschild spacetime in the exterior of the horizon, corresponds to the quadrant $[-\infty < U < 0, 0 < V < \infty]$, labeled as region I. What is the meaning of the other regions? Consider a physical observer which starts at some point r in the exterior of the horizon, i.e. in region I, as indicated in the next figure



He can move only in the interior of the light-cones, which, at every point are 45^0-straight lines. As one can see from the figure, as long as the observer is outside the horizon, he can still invert its direction of motion and escape free at infinity. But as soon as he crosses the surface $U = 0$ and enters in region II, this is no longer possible, and he gets captured by the singularity $r = 0$, (compare with the discussion on the nature of the hypersurfaces $r = const$ in section 10.5. The singularity $r = 0$ is a spacelike singularity). Thus region II represents the spacetime in the interior of the horizon. Regions III and IV have the same characteristics as regions I and II, but they are time-reversed with respect to them: a particle in region III must necessarily have been emitted by the singularity sitting in that region. Then it will cross the surface $r = 2m$ ($U = 0$ or $V = 0$) and will escape free at infinity either in region I, or in its mirrow image region IV. It should be noted that region I and IV are causally unrelated, since a signal emitted by an observer in region I will never reach region IV and viceversa. It is interesting to ask whether regions IV and III do exist or not. Suppose that a black hole has formed, and we really have a singularity concealed by a horizon. We live in the exterior of the horizon (we can move inward and outward). We can send signals to region II, but no signal emitted by us will reach regions III and IV for the reasons explained above. On the other hand, no signal coming from region IV can reach us. A signal emitted in region III (the white hole region) might, in principle, reach region I. However it is reasonable to assume that the black hole has formed at some time as the result of some physical process (the collapse of a massive star, as we shall soon see), and since any signal emitted in region III would take an infinite time t to reach region I, region III cannot communicate with us. If we want to take a pragmatical point of view, we can conclude that since we cannot communicate with regions III and IV (and viceversa), they do not exist for us. To speculate on the existence of 'other universes', although intriguing, is outside the scope of this course.

The Kruskal extension is very useful to investigate the causal structure of the spacetime in the vicinity of the horizon. However it is unappropriate to describe the spacetime at infinity, due to the exponential behaviour of g_{TT} and g_{XX}.

Chapter 12

Experimental Tests of General Relativity

12.1 Gravitational redsfhift of spectral lines

Time intervals are measured using clocks, which are instruments whose functioning is based on the repetition of a periodic phenomenon, such as atomic oscillations or the oscillations of a quartz crystal. We choose as time unit the interval of proper time between two successive repetitions of the periodic phenomenon. The definition of proper time in general relativity is

$$dT = \frac{1}{c}\sqrt{-ds^2} \equiv \frac{1}{c}\sqrt{-g_{\mu\nu}(x^\mu)dx^\mu dx^\nu}. \tag{12.1}$$

In this expression $g_{\mu\nu}$ must be evaluated at the (spacetime) position of the body to which it refers; in the case under consideration it has to be evaluated at the clock position. Thus, if the clock is at rest with respect to the reference frame, $dx^i = 0, i = 1,3$ and the proper time interval between two ticks is

$$dT = \frac{1}{c}\sqrt{-g_{00}(x^\mu)}dx^0 = \sqrt{-g_{00}(x^\mu)}dt, \tag{12.2}$$

were dt is the interval of *coordinate time* between two ticks. Note that we are assuming that dT is very small, so that we can use the infinithesimal expression of the proper time without integrating over the clock worldline.

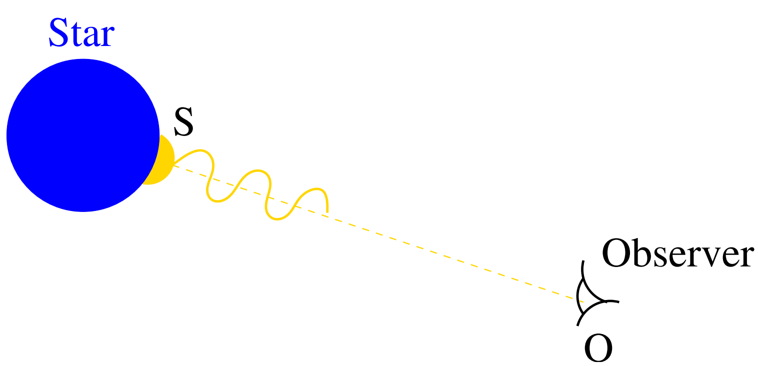
By dividing the proper time interval by dt we find

$$\frac{dT}{dt} = \frac{1}{c}\sqrt{-g_{\nu\mu}(x^\mu)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}. \tag{12.3}$$

$\frac{dT}{dt}$ is called *time dilation factor*; it gives the ratio between the interval of proper time between two events and the corresponding interval of coordinate time, and depends both on the metric and on the clock velocity. If the clock is at rest with respect to the reference frame it becomes

$$\frac{dT}{dt} = \sqrt{-g_{00}(x^\mu)}. \tag{12.4}$$

We shall now show that, in a gravitational field, the frequency of a signal detected at a point different from the emission point is different from the emission frequency. Let us assume that the gravitational field is stationary, which implies that there exists a timelike Killing vector and that, by a suitable coordinates choice, the metric can be made independent of time. In this case, the coordinate x^0 will be referred to as the *universal time*. This choice is not univocal, because we can always shift the origin of time, and rescale x^0 by an arbitrary constant. Be S a light source and O an observer, located at two different points.



The source S emits a wave crest which reaches O after an interval of coordinate time Δx^0. Since for a light signal $ds^2 = 0$, we can compute Δx^0 by solving this equation with respect to dx^0, and by integrating over the light path as follows:

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0i}dx^0dx^i + g_{ik}dx^idx^k = 0, \qquad i,k = 1,\ldots,3$$

$$\Delta x^0 = \int_{light\ path} dx^0 = \int_{light\ path} \frac{-g_{0i}dx^i \pm \sqrt{(g_{0i}dx^i)^2 - g_{00}g_{ik}dx^idx^k}}{g_{00}}.$$

The physical solution is that corresponding to the $-$ sign. [1] Since the metric is independent of time, if S and O are at rest the interval of coordinate time the light takes to go from S to O is the same for all signals; therefore if two wave crests are emitted with a time separation Δx^0_{em} by S, they will reach O with a time separation $\Delta x^0_{obs} = \Delta x^0_{em}$.

The period of the emitted wave, ΔT_{em}, is the interval of *proper time* of the source S, which elapses between the emission of two successive wave crests, i.e.

$$\Delta T_{em} = \sqrt{-g_{00}(x^\mu_{em})}\Delta t_{em},$$

and the emission frequency is

$$\nu_{em} = \frac{1}{\Delta T_{em}} = \frac{1}{\sqrt{-g_{00}(x^\mu_{em})}\Delta t_{em}}.$$

[1]Why do we have two solutions for Δx^0 corresponding to the $\pm$ sign? Firstly note that since g_{00} is negative and g_{ik} are positive $\sqrt{(g_{0i}dx^i)^2 - g_{00}g_{ik}dx^idx^k} > g_{0i}dx^i$; consequently the solution with the $+$ sign is negative and that with the $-$ is positive, i.e. $(\Delta x^0)_+ < 0$ and $(\Delta x^0)_- > 0$. Clearly the physical solution is $(\Delta x^0)_- > 0$, whereas $(\Delta x^0)_+ < 0$ would correspond to a signal that being emitted by O would reach S at $x^0 = 0$.

Similarly, the period measured by the observer is the interval of its own *proper time*, which elapses between the detection of two wave crests, i.e.

$$\Delta T_{obs} = \sqrt{-g_{00}(x^\mu_{obs})}\Delta t_{obs},$$

and the observed frequency is

$$\nu_{obs} = \frac{1}{\Delta T_{obs}} = \frac{1}{\sqrt{-g_{00}(x^\mu_{obs})}\Delta t_{obs}}.$$

Using the fact that $\Delta t_{em} = \Delta t_{obs}$ we finally find

$$\frac{\nu_{obs}}{\nu_{em}} = \frac{\lambda_{em}}{\lambda_{obs}} = \sqrt{\frac{g_{00}(x^\mu_{em})}{g_{00}(x^\mu_{obs})}}. \tag{12.5}$$

Thus, in general the frequency of a signal emitted in a gravitational field at a given point, is different from that detected at a different point, since the metric in the two points is different.

12.1.1 Some useful numbers

We now want to establish when a gravitational field can be considered as weak. Let us consider the Sun first. Its mass and radius are

$$M_\odot = 1.989 \cdot 10^{33}\ g, \qquad R_\odot = 6.9599 \cdot 10^5\ km; \tag{12.6}$$

moreover, being $G = 6.67 \times 10^{-8}\ \frac{cm^3}{gs^2}$ and $c = 2.998 \times 10^{10}\ cm/s$,

$$\frac{GM_\odot}{c^2} = \frac{1.989 \cdot 10^{33} \times 6.673 \cdot 10^{-8}}{(2.998 \cdot 10^{10})^2} \sim 1.4768\ km, \quad \text{and} \qquad \frac{GM_\odot}{R_\odot c^2} \sim 0.21 \cdot 10^{-5}. \tag{12.7}$$

The quantity $\frac{GM}{Rc^2}$ is said *surface gravity*, and it is a measure of how strong are the effects of general relativity. The surface gravity of the Sun is much smaller than unity, therefore we can say that its gravitational field is weak.

The Earth has mass $M_\oplus = 5.98 \cdot 10^{27}$ g and equatorial radius $R_\oplus = 6.378 \cdot 10^3$ Km. Since

$$M_\odot/M_\oplus \simeq 3 \cdot 10^5, \qquad \text{and} \qquad R_\odot/R_\oplus \simeq 10^2,$$

$$\frac{GM_\odot}{R_\odot c^2} / \frac{GM_\oplus}{R_\oplus c^2} \sim 3 \cdot 10^3 \tag{12.8}$$

i.e. the surface gravity of the Sun is about 3000 times larger than that of the Earth.

Conversely, if we consider a neutron star with typical mass and radius

$$M_{NS} \sim 1.4\ M_\odot, \qquad R_{NS} \sim 10\ km, \tag{12.9}$$

the surface gravity is

$$\frac{GM_{NS}}{R_{NS}c^2} \sim 0.21, \tag{12.10}$$

which is close to unity and much larger than that of the Sun. Thus, the effects of general relativity will be much more important for a neutron star than for the Sun.

12.1.2 Redshift of spectral lines in the weak field limit

Let us now consider eq. (12.5) in the weak field limit. In section 9.1 we have seen that if we assume that the gravitational field is stationary and weak, the geodesic equations show that the 00-component of the metric tensor is related to the Newtonian potential Φ, solution of the equation $\nabla^2\Phi = 4\pi G\rho$, by the equation

$$g_{00} \simeq -\left(1+\frac{2\Phi}{c^2}\right).$$

Consequently, if the gravitational field is weak and stationary eq. (12.5) becomes

$$\frac{\nu_{obs}-\nu_{em}}{\nu_{em}} = \frac{\lambda_{em}-\lambda_{obs}}{\lambda_{obs}} = \sqrt{\frac{1+\frac{2\Phi_{em}}{c^2}}{1+\frac{2\Phi_{obs}}{c^2}}} - 1 \simeq$$
$$\sqrt{\left(1+\frac{2\Phi_{em}}{c^2}\right)\left(1-\frac{2\Phi_{obs}}{c^2}\right)} - 1 \simeq \sqrt{1+\frac{2}{c^2}\left(\Phi_{em}-\Phi_{obs}\right)} - 1$$
$$\simeq \frac{1}{c^2}\left(\Phi_{em}-\Phi_{obs}\right)$$

and finally

$$\frac{\Delta\nu}{\nu} \simeq \frac{1}{c^2}\left(\Phi_{em}-\Phi_{obs}\right). \tag{12.11}$$

Let us suppose that the source of light is on the Sun, whose gravitational field is weak, and that the observer is on the Earth. We shall neglect the gravitational field of the Earth since it is much smaller than that of the Sun. In this case, $\Phi = -\frac{GM_\odot}{r}$, where r is the distance from the Sun center, $r_{em} = R_\odot$ and $r_{obs} = r_{Sun-Earth}$. Thus eq. (12.11) becomes

$$\frac{\Delta\nu}{\nu} \simeq \frac{GM_\odot}{c^2}\left(-\frac{1}{R_\odot}+\frac{1}{r_{Sun-Earth}}\right);$$

Since the average distance between the Sun and the Earth is $r_{Sun-Earth} = 149.6 \cdot 10^6$ km, which is about 210 times the Sun radius, we can assume $r_{Sun-Earth} \gg R_\odot$, so that

$$\frac{\Delta\nu}{\nu} \simeq -\frac{GM_\odot}{R_\odot c^2} \simeq -0.21 \cdot 10^{-5}. \tag{12.12}$$

Note the following:

- $\Delta\nu < 0$, i.e. the observed spectral lines are shifted toward lower frequencies, i.e. the light *reddened*.
- The redshift of spectral lines produced by the Sun is of the order of its surface gravity.

12.1.3 Redshift of spectral lines in a strong gravitational field

Let us now consider the case when the source emitting light and the observer are located in the gravitational field of a neutron star or of a black hole. The metric appropriate to describe the exterior of a neutron star, from $r = R_{NS}$ up to radial infinity, and a black hole is the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where $m = GM/c^2$ is the mass, either of the star or of the black hole, in geometric units. If we assume that the observer is located very far from the source emitting light, i.e. $r_{obs} \gg r_{em}$, eq. (12.5) gives

$$\frac{\nu_{obs}}{\nu_{em}} = \sqrt{\frac{-g_{00}(x^\mu_{em})}{-g_{00}(x^\mu_{obs})}} = \sqrt{\frac{1 - \frac{2m}{r_{em}}}{1 - \frac{2m}{r_{obs}}}} \sim \sqrt{1 - \frac{2m}{r_{em}}}. \tag{12.13}$$

If the light source is located on a neutron star surface, i.e. $r_{em} = R_{NS}$, this equation gives

$$\frac{\nu_{obs}}{\nu_{em}} \sim \sqrt{1 - \frac{2GM_{NS}}{R_{NS}c^2}} \sim \sqrt{1 - 2 \times 0.21} \sim 0.76 \quad \rightarrow \quad \frac{\Delta\nu}{\nu} = \frac{\nu_{obs} - \nu_{em}}{\nu_{em}} \sim -0.24,$$

where we have used eq. (12.10). This means that an observer located at infinity with respect to the neutron star will see the emitted ligth reddened ($\Delta\nu < 0$) by quite a large amount, much larger than that produced by the Sun which we computed in eq. (12.12).

Let us now suppose that the source of the gravitational field is a black hole, and that the source emitting light is on a spacecraft orbiting around it. From eq. (12.13) we see that as the light source approaches the horizon $r = 2m$,

$$\nu_{obs} \sim \sqrt{1 - \frac{2m}{r_{em}}}\, \nu_{em} \rightarrow 0,$$

i.e. the observed signal will fade away since the observed frequency tends to zero. Thus, the signal emitted by a source falling into a black hole has a distinctive feature, i.e. its frequency will progressively decrease tending to zero near the horizon.

NOTE THAT: to derive the gravitational redshift, we have used only the fact that the effects of the gravitational field are described by the metric tensor, i.e. we have used basically only the Equivalence Principle.

12.2 The geodesic equations in the Schwarzschild background

The geodesic equations can be derived not only from the Equivalence Principle as shown in previous chapters, but also from a variational principle, as we shall now show.

12.2.1 A variational principle for geodesic motion

Let us define the Lagrangian of a free particle as

$$\mathcal{L}\left(x^\alpha, \frac{dx^\alpha}{d\lambda}\right) = \frac{1}{2} g_{\mu\nu}(x^\alpha) \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \equiv \frac{1}{2} g_{\mu\nu}(x^\alpha)\dot{x}^\mu \dot{x}^\nu, \tag{12.14}$$

in the space of the curves $\{x^\mu(\lambda),\, \lambda \in [\lambda_0, \lambda_1]\}$, and the action

$$S = \int \mathcal{L}\left(x^\alpha, \dot{x}^\alpha\right) d\lambda = \frac{1}{2}\int g_{\mu\nu}(x^\alpha)\dot{x}^\mu \dot{x}^\nu d\lambda,$$

where we have set

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}. \tag{12.15}$$

λ can be the proper time if we consider massive particles, or an affine parameter which parametrizes the geodesic, if we consider massless particles. The Euler-Lagrange equations are obtained, as usual, by varying the action with respect to the coordinates, and by setting the variation equal to zero. By varying a curve $x^\mu(\lambda)$

$$x^\mu(\lambda) \longrightarrow x^\mu(\lambda) + \delta x^\mu(\lambda)$$

with $\delta x^\mu(\lambda_0) = \delta x^\mu(\lambda_1) = 0$, the action variation is

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial x^\sigma}\delta x^\sigma + \frac{\partial \mathcal{L}}{\partial \dot{x}^\sigma}\delta(\dot{x}^\sigma)\right) d\lambda. \tag{12.16}$$

Since

$$\delta(\dot{x}^\sigma) = \delta\left(\frac{dx^\sigma}{d\lambda}\right) = \frac{d\delta x^\sigma}{d\lambda},$$

the last term in eq. (12.16) can be written as

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\sigma}\delta(\dot{x}^\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{x}^\sigma}\frac{d\delta x^\sigma}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\sigma}\delta x^\sigma\right) - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\sigma}\right)\delta x^\sigma.$$

When integrated between λ_0 and λ_1 the first term on the RHS vanishes because $\delta x^\mu(\lambda_0) = \delta x^\mu(\lambda_1) = 0$, therefore eq. (12.16) becomes

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial x^\sigma}\delta x^\sigma - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\sigma}\right)\delta x^\sigma\right] d\lambda\,, \tag{12.17}$$

which vanishes for all δx^σ if and only if

$$\frac{\partial \mathcal{L}}{\partial x^\sigma} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial(\dot{x}^\sigma)} = 0. \tag{12.18}$$

These are the Euler–Lagrange equations. We shall now show that these equations, when written for the action (12.14), are the geodesic equations

$$\ddot{x}^\gamma + \Gamma^\gamma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{12.19}$$

By substituting the Lagrangian (12.14) in the Euler–Lagrange equations (12.18), and remembering that $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$ and $\dot{x}^\mu = \dot{x}^\mu(\lambda)$, we find

$$\begin{aligned}
& \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - \frac{d}{d\lambda}\left[\frac{1}{2}g_{\mu\nu}(\delta^\mu_\alpha\dot{x}^\nu + \dot{x}^\mu\delta^\nu_\alpha)\right] && (12.20)\\
= \;& g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - \frac{d}{d\lambda}\left[g_{\alpha\nu}\dot{x}^\nu + g_{\alpha\mu}\dot{x}^\mu\right] \\
= \;& g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - \frac{d}{d\lambda}\left[2g_{\alpha\nu}\dot{x}^\nu\right] \\
= \;& g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - 2g_{\alpha\nu,\beta}\dot{x}^\beta\dot{x}^\nu - 2g_{\alpha\nu}\ddot{x}^\nu \\
\equiv \;& g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - g_{\alpha\mu,\nu}\dot{x}^\nu\dot{x}^\mu - g_{\alpha\nu,\mu}\dot{x}^\mu\dot{x}^\nu - 2g_{\alpha\nu}\ddot{x}^\nu = 0
\end{aligned}$$

By contracting this equation with $g^{\alpha\gamma}$ we find

$$\delta^\gamma_\nu\ddot{x}^\nu + \frac{1}{2}g^{\alpha\gamma}\left[-g_{\mu\nu,\alpha} + g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu}\right]\dot{x}^\mu\dot{x}^\nu = 0$$

i.e.

$$\ddot{x}^\gamma + \frac{1}{2}g^{\alpha\gamma}\left[g_{\alpha\mu,\nu}g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right]\dot{x}^\mu\dot{x}^\nu = 0 \tag{12.21}$$

which coincides with eq. (12.19).

12.2.2 Geodesics in the Schwarzschild metric

For the Schwarzschild metric, the Lagrangian of a free particle is

$$\mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2m}{r}\right)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right],$$

(we put $G = c = 1$), and a dot indicates differentiation with respect to λ. The equations of motion for $\dot{t}$, $\dot{\phi}$ and $\dot{\theta}$ are:

1) Equation for $\dot{t}$:

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial(\dot{t})} = 0 \qquad \to \qquad \frac{d}{d\lambda}\left[\left(1 - \frac{2m}{r}\right)2\dot{t}\right] = 0$$

i.e.

$$\dot{t} = \frac{const}{\left(1 - \frac{2m}{r}\right)}. \tag{12.22}$$

It should be reminded that, since the Schwarzschild metric admits a timelike Killing vector $\frac{\vec{\partial}}{\partial t} \rightarrow \xi^{\alpha}_{(t)} = (1,0,0,0)$, there exists an associated conserved quantity for the geodesic motion

$$g_{\alpha\beta}\xi^{\alpha}_{(t)}u^{\beta} = const \qquad \rightarrow \qquad g_{00}u^0 = const \qquad \rightarrow \qquad \left(1 - \frac{2m}{r}\right)\dot{t} = const \tag{12.23}$$

where $u^0 = \dot{t} = \frac{dx^0}{d\lambda}$. Note that this equation coincides with eq. (12.22). As discussed in section 10.3, at radial infinity, where the Schwarzschild metric tends to Minkowski's metric in spherical coordinates, g_{00} becomes η_{00} and the equation $g_{00}u^0 = const$ reduces to $u^0 = const$. In flat spacetime (putting $G = c = 1$) the energy-momentum vector of a massive particle is $p^{\alpha} = mu^{\alpha} = \{\mathcal{E}, mv^i\gamma\}$; therefore $u^0 = const$ means $\mathcal{E}/m = const$. Therefore we are entitled to interpret the constant in eqs. (12.22) and (12.23) as the energy per unit mass of the particle at infinity. In this case the parameter λ is the particle proper time. If the particle is massless λ must be an affine parameter which parametrizes the null geodesic, ad it can be chosen in such a way that the constant is the particle energy at infinity. In the following we shall put $const = E$ and write eq. (12.22) as

$$\dot{t} = \frac{E}{\left(1 - \frac{2m}{r}\right)}. \tag{12.24}$$

2) Equation for $\dot{\phi}$:
since the Lagrangian does not depend on ϕ it is easy to show that

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial(\dot{\phi})} = 0 \qquad \rightarrow \qquad \dot{\phi} = \frac{const}{r^2 \sin^2\theta}. \tag{12.25}$$

Due to its spherical symmetry, the Schwarzschild metric admits the spacelike Killing vector $\frac{\vec{\partial}}{\partial \phi} \rightarrow \xi^{\alpha}_{(\phi)} = (0,0,0,1)$, which is associate to the conserved quantity

$$g_{\alpha\beta}\xi^{\alpha}_{(\phi)}u^{\beta} = const \qquad \rightarrow \qquad r^2 \sin^2\theta\dot{\phi} = const; \tag{12.26}$$

again eqs. (12.25) and (12.26) coincide. To understand the meaning of the constant, let us consider the simple case of a particle in circular orbit on the equatorial plane; in this case the conservation equation becomes

$$r^2\dot{\phi} = const;$$

from Newtonian mechanics we know that the particle angular momentum $\vec{\ell} = \vec{r} \wedge m\vec{v}$ is conserved so that, being $v = r\dot{\phi}$, it follows that $|\ell| = mr^2\dot{\phi} = const$. Thus we can interpret the constant as the particle angular momentum per unit mass (or as the particle angular momentum if it is a massless particle) at infinity. In the following we shall put $const = L$ and write eq. (12.25) as

$$\dot{\phi} = \frac{L}{r^2 \sin^2\theta}. \tag{12.27}$$

3) Equation for $\dot\theta$:

$$\frac{\partial\mathcal{L}}{\partial\theta}-\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial(\dot\theta)}=0 \qquad \rightarrow \qquad \frac{d}{d\lambda}(r^2\dot\theta)=r^2\sin\theta\cos\theta\dot\phi^2 .$$

Therefore the equation for θ is

$$\ddot\theta=-\frac{2}{r}\dot r\dot\theta+\sin\theta\cos\theta\dot\phi^2 \,. \tag{12.28}$$

We will prove that this equations implies that, as in Newtonian theory, orbits are planar.

Due to the spherical symmetry, the metric is invariant under rotations of the polar axes. Using this freedom, we choose them such that, for a given value of the affine parameter, say $\lambda=0$, the particle is on the equatorial plane $\theta=\frac{\pi}{2}$ and its three-velocity $(\dot r,\dot\theta,\dot\phi)$ lays on the same plane, i.e. $\theta(\lambda=0)=\frac{\pi}{2}$ and $\dot\theta(\lambda=0)=0$. Thus, we have to solve the following Cauchy problem

$$\begin{aligned}&\ddot\theta=-\frac{2}{r}\dot r\dot\theta+\sin\theta\cos\theta\dot\phi^2\\&\theta(\lambda=0)=\frac{\pi}{2}\\&\dot\theta(\lambda=0)=0\end{aligned} \tag{12.29}$$

which admits a unique solution. Since

$$\theta(\lambda)\equiv\frac{\pi}{2} \tag{12.30}$$

satisfies the differential equation and the initial conditions, it must be *the* solution. Thus, the orbit is plane and to hereafter we shall assume $\theta=\frac{\pi}{2}$ and $\dot\theta=0$.

4) Equation for $\dot r$:
it is convenient to derive this equation from the condition $u_\alpha u^\alpha=-1$, or $u_\alpha u^\alpha=0$, respectively valid for massive and massless particles.
A) massive particles:

$$g_{\alpha\beta}u^\alpha u^\beta=-\left(1-\frac{2m}{r}\right)\dot t^2+\frac{\dot r^2}{\left(1-\frac{2m}{r}\right)}+r^2\dot\theta^2+r^2\sin^2\theta\dot\phi^2=-1 \tag{12.31}$$

which becomes, by substituting the equations for $\dot t$ and $\dot\Phi$

$$\dot r^2+\left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)=E^2 \tag{12.32}$$

B) massless particles:

$$g_{\alpha\beta}u^\alpha u^\beta=-\left(1-\frac{2m}{r}\right)\dot t^2+\frac{\dot r^2}{\left(1-\frac{2m}{r}\right)}+r^2\dot\theta^2+r^2\sin^2\theta\dot\phi^2=0 \tag{12.33}$$

which becomes

$$\dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right) = E^2\,. \tag{12.34}$$

Finally, the geodesic equations are:
A) For massive particles:

$$\theta = \frac{\pi}{2}, \quad \dot{t} = \frac{E}{\left(1 - \frac{2m}{r}\right)} \tag{12.35}$$
$$\dot{\phi} = \frac{L}{r^2}, \quad \dot{r}^2 = E^2 - \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

B) For massless particles:

$$\theta = \frac{\pi}{2}, \quad \dot{t} = \frac{E}{\left(1 - \frac{2m}{r}\right)} \tag{12.36}$$
$$\dot{\phi} = \frac{L}{r^2}, \quad \dot{r}^2 = E^2 - \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right)$$

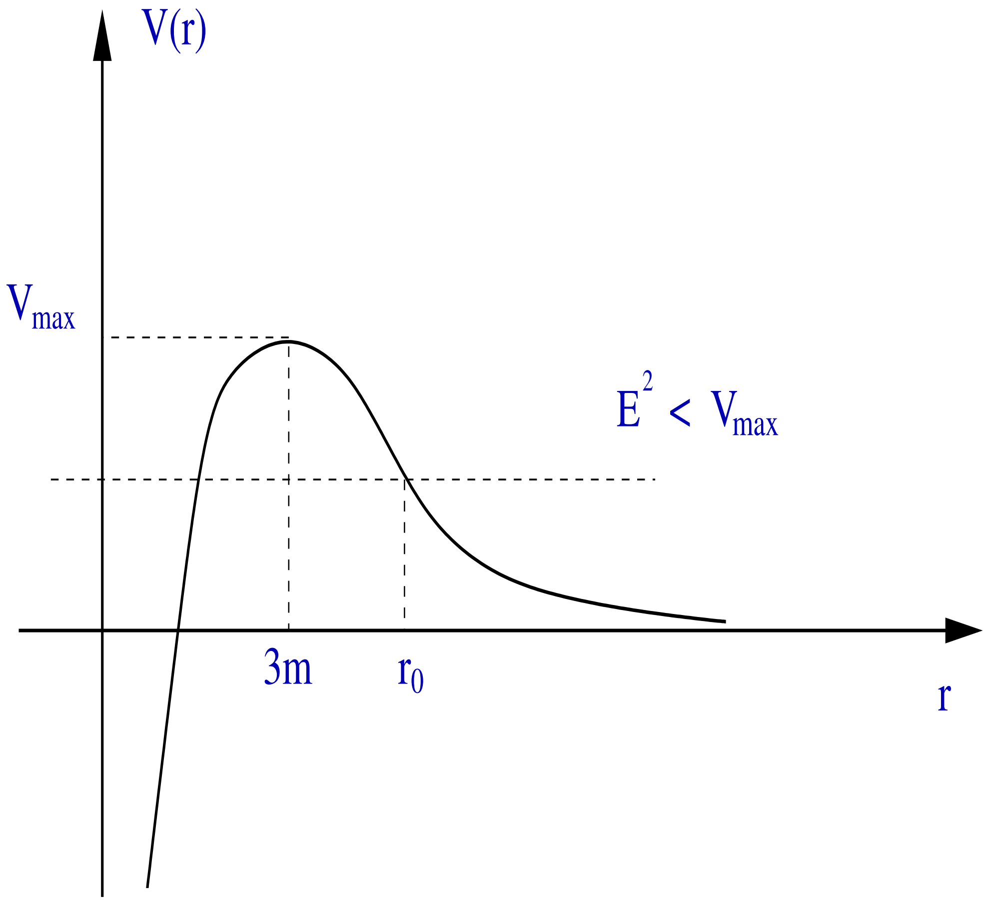
12.3 The orbits of a massless particle

Let us write the radial equation (12.34) in the following form

$$\dot{r}^2 = E^2 - V(r) \tag{12.37}$$

where

$$V(r) = \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right). \tag{12.38}$$



Note that:
- For massless particles the angular momentum L acts as a scale factor for the potential
- $V(r)$ tends to $-\infty$ as $r \to 0$, and approaches zero at $r \to \infty$
- $V(r)$ has only one maximum at $r_{max} = 3m$, where it takes the value

$$V_{max} = \frac{L^2}{27m^2} \tag{12.39}$$

It is useful to consider also the radial acceleration, obtained by differentiating eq. (12.37) with respect to λ

$$2\dot{r}\ddot{r} = -\frac{dV(r)}{dr}\dot{r} \qquad \to \qquad \ddot{r} = -\frac{1}{2}\frac{dV(r)}{dr}. \tag{12.40}$$

Let us assume that the particle, say a photon, starts its path from $+\infty$ with $\dot{r} < 0$. The energy of the particle can be:

1) $\mathbf{E^2 > V_{max}}$

according to eq. (12.37) $\dot{r}^2 > 0$ always, and the particle falls into the central body with increasing radial velocity, possibly making several revolutions around the central body before falling in.

2) $\mathbf{E^2 = V_{max}}$

as the particle approaches r_{max}, $|\dot{r}|$ decreases and tends to zero as $r = r_{max}$. Since at $r = r_{max}$ the radial acceleration is zero (see eq. 12.40), if a particle, at a given time, has $r = r_{max}$ and $\dot{r} = 0$ (i.e. $E = V_{max}$), it remains at the same r at later times, i.e. its orbit is circular. This is, however, an unstable orbit; indeed if the position is perturbed, the particle will

- either fall into the central body; this happens when the radial coordinate of the particle is displaced to $r < r_{max}$, since there the radial acceleration is negative
- or escape toward infinity; this happens when the radial coordinate is displaced to $r > r_{max}$, since there the radial acceleration is positive.

Thus, for massless particles, there exists only one *circular, unstable* orbit, and for this orbit

$$E^2 = \frac{L^2}{27m^2}. \tag{12.41}$$

3) $\mathbf{E^2 < V_{max}}$
be r_0 the abscissa of the point where $E^2 = V(r)$ (see figure); for $r > r_0$, $\dot{r}^2$ is always positive and becomes zero at $r = r_0$. This is a *turning point*: the particle cannot penetrate the potential barrier and reach values of $r < r_0$ because $\dot{r}$ would become imaginary; since at $r = r_0$ the radial acceleration is positive, the particle is forced to invert its radial velocity, and it escapes toward infinity on an open trajectory.
Thus, according to General relativity a light ray is deflected by the gravitational field of a massive body, provided its energy satisfies the following condition

$$E^2 < \frac{L^2}{27m^2}. \tag{12.42}$$

12.3.1 The deflection of light

We shall now compute the deflection angle that a massive body induces on the trajectory of a massless particle, say a photon. Referring to the figure 12.1, we shall use the following notation:
r is the radial coordinate of the particle in a frame centered in the center of attraction; r forms an angle ϕ with the y-axis.
b is the *impact parameter*, i.e. the distance between the direction of the incoming particle (dashed vertical line) and the center of attraction.
δ is the deflection angle which we are going to evaluate: it is the angle between the incoming direction and the outgoing direction (dashed, green line)
Note that, since the Schwarzschild metric is invariant under time reflection, the particle can go through the red trajectory on the figure either in the direction indicated by the red arrow, or in the opposite one. Thus, the trajectory must be simmetric. The periastron is indicated in the figure as r_0.

We choose the orientation of the frame axes such that the initial value of ϕ when the particle starts its motion at radial infinity be

$$\phi^{in} = 0\,. \tag{12.43}$$

The outgoing particle will escape to $r \to \infty$ at

$$\phi^{out} = \pi + \delta\,. \tag{12.44}$$

Our only assumption will be that, for all values of r reached by the particle,

$$\frac{m}{r} \ll 1\,. \tag{12.45}$$

This condition is satisfied, for instance, in the case of a photon deflected by the Sun; indeed, if R_s is the radius of the Sun, then $r \geq R_s$, and

$$\frac{m}{r} \leq \frac{m}{R_s} \sim 10^{-6} \,. \tag{12.46}$$

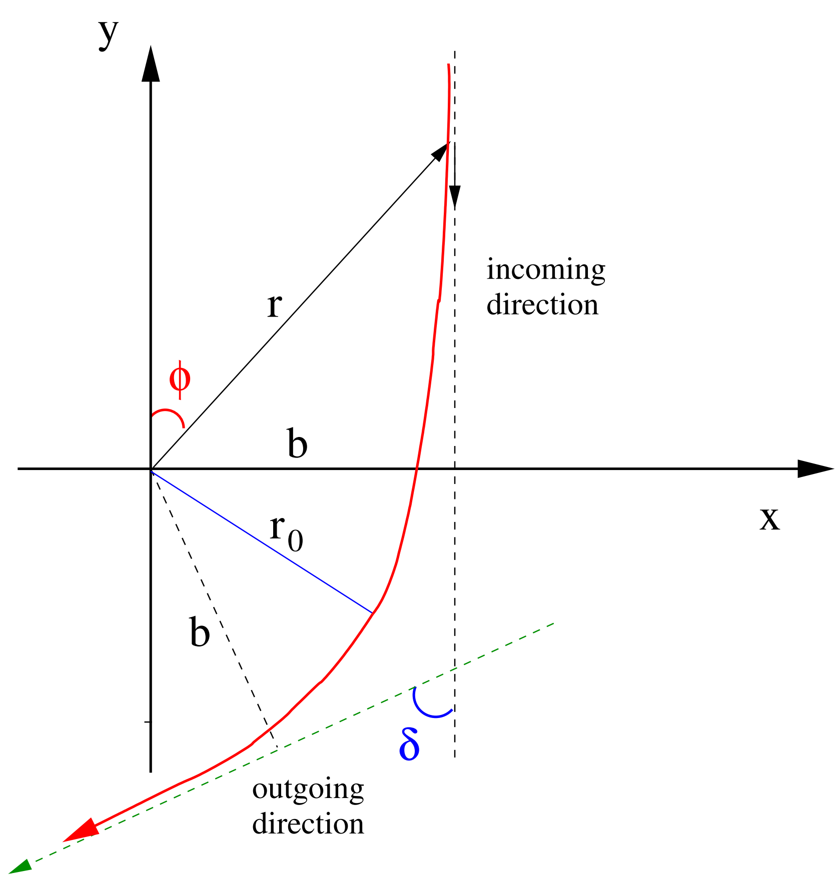


Figure 12.1:

From the figure we see that

$$b = \lim_{\phi \to 0} \; r \sin \phi \,. \tag{12.47}$$

We shall now express the impact parameter b in terms of the energy and the angular momentum of the particle.
When the particle arrives from infinity, r is large, $\phi \simeq 0$ and

$$b \simeq r\phi \qquad \to \qquad \frac{d\phi}{dr} \simeq -\frac{b}{r^2} \,. \tag{12.48}$$

$\frac{d\phi}{dr}$ can also be derived combining toghether the third and the fourth eqs. (12.36)

$$\frac{d\phi}{dr} = \pm \frac{L}{r^2 \sqrt{E^2 - V(r)}}; \tag{12.49}$$

taking the limit for $r \to \infty$ it gives

$$\frac{d\phi}{dr} \sim \pm \frac{L}{r^2 E}, \tag{12.50}$$

thus, combining together eqs. (12.48) and (12.50), we find that b can be written as

$$b = \frac{L}{E}. \tag{12.51}$$

In order the particle being deflected its energy must satisfy eq. (12.42), and this imposes a constraint on b, i.e.

$$b \geq \sqrt{27}m \equiv b_{crit}; \tag{12.52}$$

if b is smaller than this critical value, the particle is captured by the central body. Note that if the central body is not a black hole but a star, its radius R is in general larger than $\sqrt{27}m$, so the critical value of the impact parameter is R: if the particle reaches the stellar surface, it is not deflected.

To find the deflection angle, let us consider the third and the fourth eqs. (12.36); we introduce a new variable

$$u \equiv \frac{1}{r}; \tag{12.53}$$

by construction, it must be

$$u(\phi = 0) = 0\,. \tag{12.54}$$

Furthermore, u must also vanish when $\phi = \pi + \delta$, because this value of ϕ corresponds to the particle escaping to infinity.

In terms of the variable u, the third equation (12.36) for $\dot{\phi}$ becomes

$$\dot{\phi} = Lu^2\,.$$

By indicating with a prime differentiation with respect to ϕ we find that

$$\dot{r} = r'\dot{\phi} = -\frac{1}{u^2}u'\dot{\phi} = -Lu'.$$

By substituting this expression in the fourth eq. (12.36), it becomes

$$L^2(u')^2 + u^2L^2 - 2mL^2u^3 = E^2\,,$$

and differentiating with respect to ϕ,

$$2L^2u''u' + 2uu'L^2 - 6ML^2u'u^2 = 0\,.$$

Dividing by $2L^2u'$, we finally find the equation u must satisfy

$$u'' + u - 3mu^2 = 0\,, \tag{12.55}$$

to which we associate the boundary condition

$$\begin{aligned} u(\phi = 0) &= 0 \\ u'(\phi = 0) &= \frac{1}{b}\,. \end{aligned} \tag{12.56}$$

The second condition is obtained by the relation

$$u(\phi \simeq 0) = \frac{1}{b}\sin\phi$$

which derives from eq. (12.47).

If the mass of the central body vanishes, equations (12.55) becomes

$$u'' + u = 0 \tag{12.57}$$

the solution of which

$$u(\phi) = \frac{1}{b}\sin\phi \qquad \rightarrow \qquad b = r\sin\phi \tag{12.58}$$

describes the trajectory of a particle which is not deflected.

If there is a central body with a finite mass m, the solution of (12.55) is different from (12.58), and the light ray is deflected. We note that equations (12.55) and (12.57) differ by a term, $3mu^2$, which is much smaller than, say, the term u by a factor

$$\frac{3mu^2}{u} = \frac{3m}{r} \ll 1\,. \tag{12.59}$$

Consequently, it is appropriate to solve eq. (12.55) using a perturbative approach; we shall proceed as follows. We put

$$u = u^{(0)} + u^{(1)} \tag{12.60}$$

where $u^{(0)}$ is the solution of equation (12.57),

$$u^{(0)} \equiv \frac{1}{b}\sin\phi \tag{12.61}$$

and we assume that

$$u^{(1)} \ll u^{(0)}\,. \tag{12.62}$$

By substituting (12.60) in eq. (12.55) we find

$$(u^{(0)})'' + u^{(0)} - 3m(u^{(0)})^2 + (u^{(1)})'' + u^{(1)} - 3m(u^{(1)})^2 - 6mu^{(0)}u^{(1)} = 0\,. \tag{12.63}$$

Since $u^{(0)}$ satisfies (12.57), eq. (12.63) becomes

$$(u^{(1)})'' + u^{(1)} - 3m(u^{(0)})^2 - 3m(u^{(1)})^2 - 6mu^{(0)}u^{(1)} = 0\,. \tag{12.64}$$

The terms $3m(u^{(1)})^2$ and $6mu^{(0)}u^{(1)}$ are of higher order with respect to $3m(u^{(0)})^2$, therefore the leading terms in equation (12.55) are

$$(u^{(1)})'' + u^{(1)} - 3m(u^{(0)})^2 = 0\,. \tag{12.65}$$

Consequently,

$$(u^{(1)})'' + u^{(1)} = \frac{3m}{b^2}\sin^2\phi = \frac{3m}{2b^2}\left(1 - \cos 2\phi\right)\,. \tag{12.66}$$

The solution of (12.66) which satisfies the boundary conditions (12.57) is

$$u^{(1)} = \frac{3m}{2b^2}\left(1 + \frac{1}{3}\cos 2\phi - \frac{4}{3}\cos\phi\right)\,, \tag{12.67}$$

as can be checked by direct substitution. It should be noticed that the boundary conditions (12.57) must be satisfied by the complete solution $u = u^{(0)} + u^{(1)}$. Therefore,

$$u = \frac{1}{b}\sin\phi + \frac{3m}{2b^2}\left(1 + \frac{1}{3}\cos 2\phi - \frac{4}{3}\cos\phi\right) . \qquad (12.68)$$

We now want to find the deflection angle, i.e., the small angle δ such that $u(\pi + \delta) = 0$. By substituting $\phi = \pi + \delta$ in (12.68) we finally find

$$u(\pi + \delta) \simeq -\frac{\delta}{b} + \frac{3m}{2b^2} \cdot \frac{8}{3} \qquad (12.69)$$

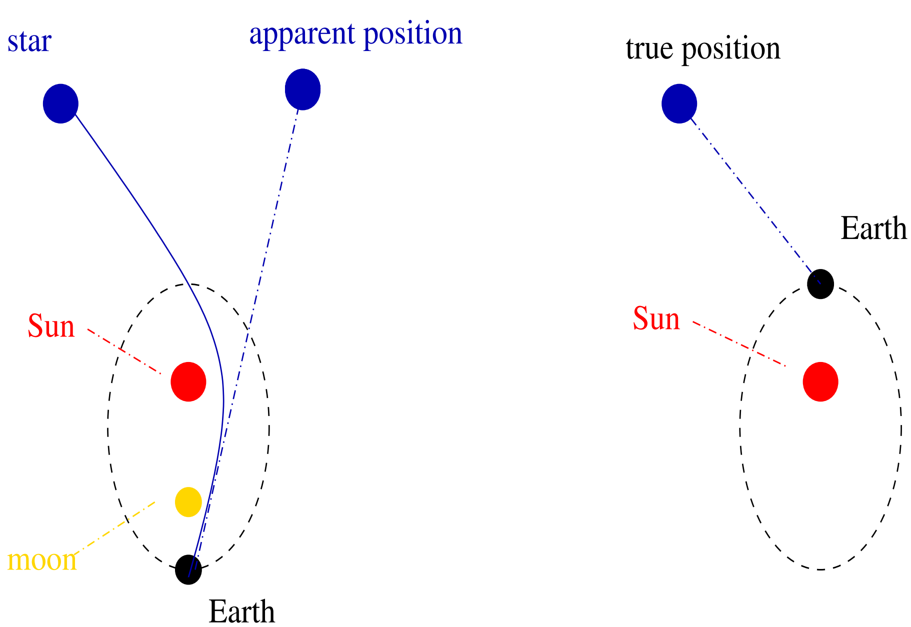
which vanishes for

$$\delta = \frac{4m}{b}. \qquad (12.70)$$

For a light ray which passes close to the surface of the Sun

$$\delta \sim 1.75 \text{ seconds of arc} \qquad (12.71)$$

The first measurement of the deflection of light was done by Eddington, Dayson and Davidson during the solar eclypse in 1919. What was measured was the apparent position of a star behind the Sun (see figure) during the eclypse, when some light coming from the star was able to reach the Earth because the luminosity of the Sun was obscured by the eclypse. Comparing this apparent position with the position of the star as measured when the Earth is on the opposite side of its orbit around the Sun, one finds $\delta\phi$. The deflection was measured with an accuracy of about 10% at that time. Today, the bending of radio waves by quasars has been measured with an accuracy of 1%.



12.4 The orbits of a massive particle

Let us first discuss the orbits that a massive particle is allowed to move on. The equations of motion are

$$\theta = \frac{\pi}{2}, \quad \dot{t} = \frac{E}{\left(1 - \frac{2m}{r}\right)} \tag{12.72}$$
$$\dot{\phi} = \frac{L}{r^2}, \quad \dot{r}^2 = E^2 - \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right).$$

Let us study the radial equation

$$\dot{r}^2 = E^2 - V(r), \tag{12.73}$$

where

$$V(r) = \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right). \tag{12.74}$$

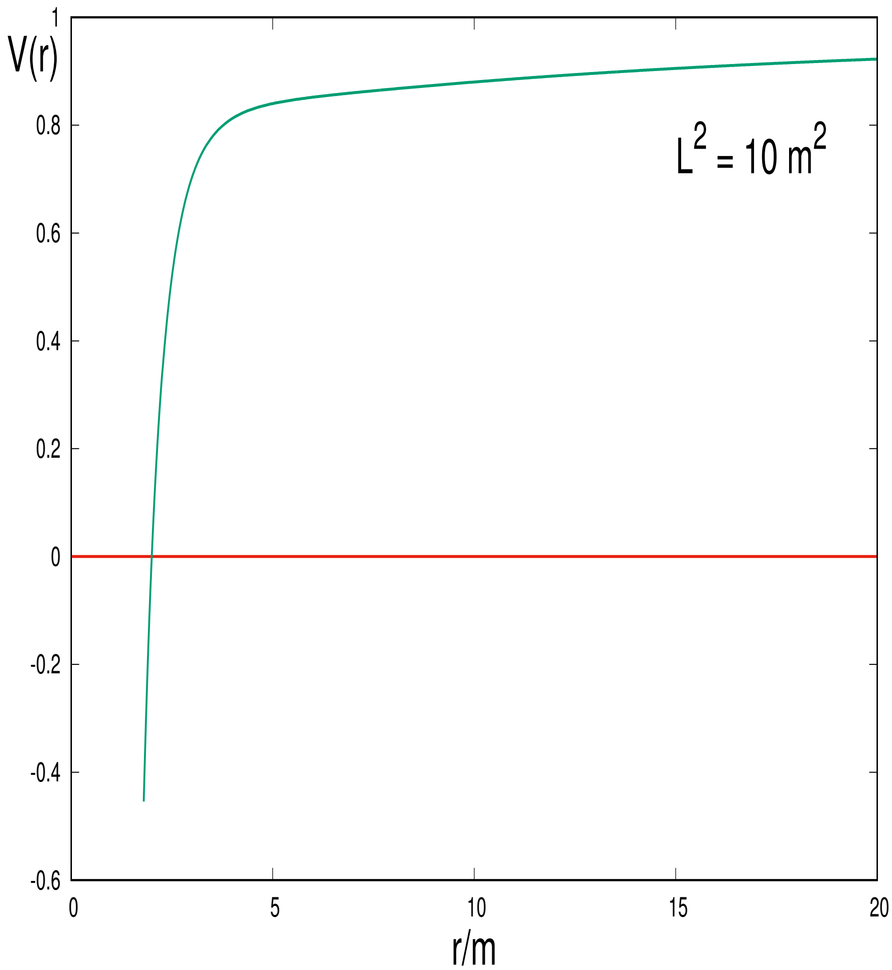
First of all we note that, contrary to the massless case, the potential does not scale with the angular momentum and that $V(r) \to 1$ when $r \to \infty$. To plot the potential, let us first see if it admits a minimum or a maximum by solving

$$\frac{\partial V}{\partial r} = 2\frac{mr^2 - L^2 r + 3mL^2}{r^4} = 0;$$

this equation has two roots

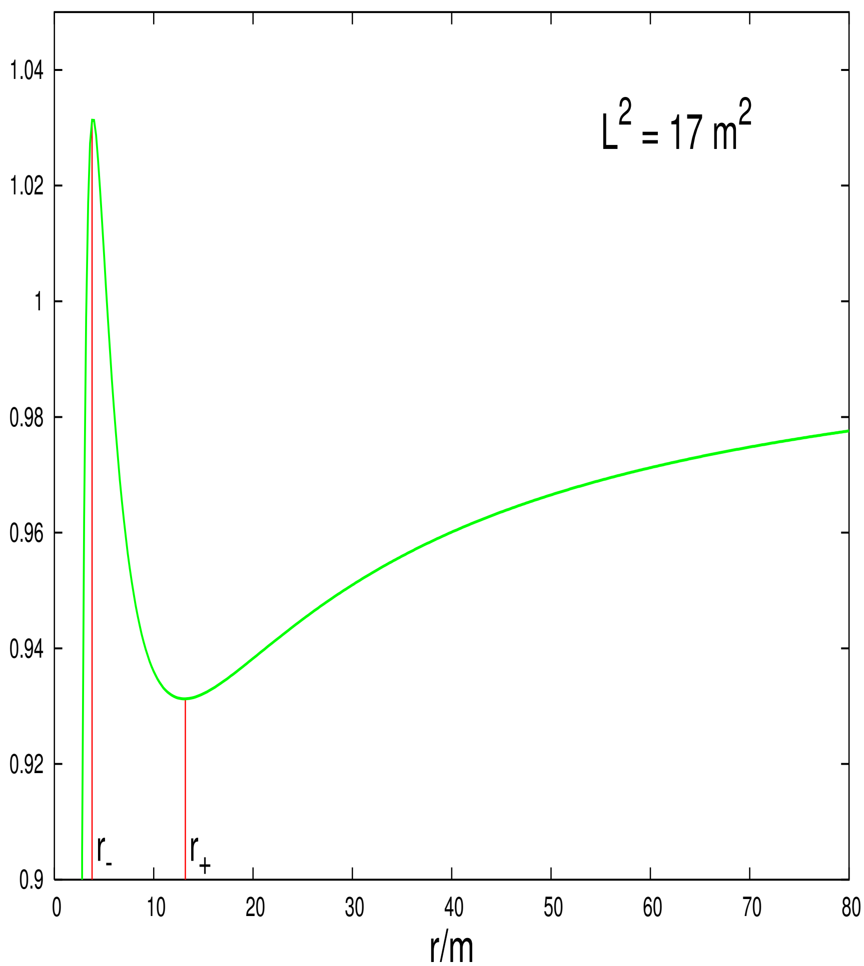
$$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12m^2L^2}}{2m}. \tag{12.75}$$

If $L^2 < 12m^2$ the roots are complex and there are no extrema; the potential will have the shape shown in the figure



from which is clear that a particle arriving from infinity with $\dot{r} \leq 0$ and having $L^2 < 12m^2$ will be captured by the black hole.

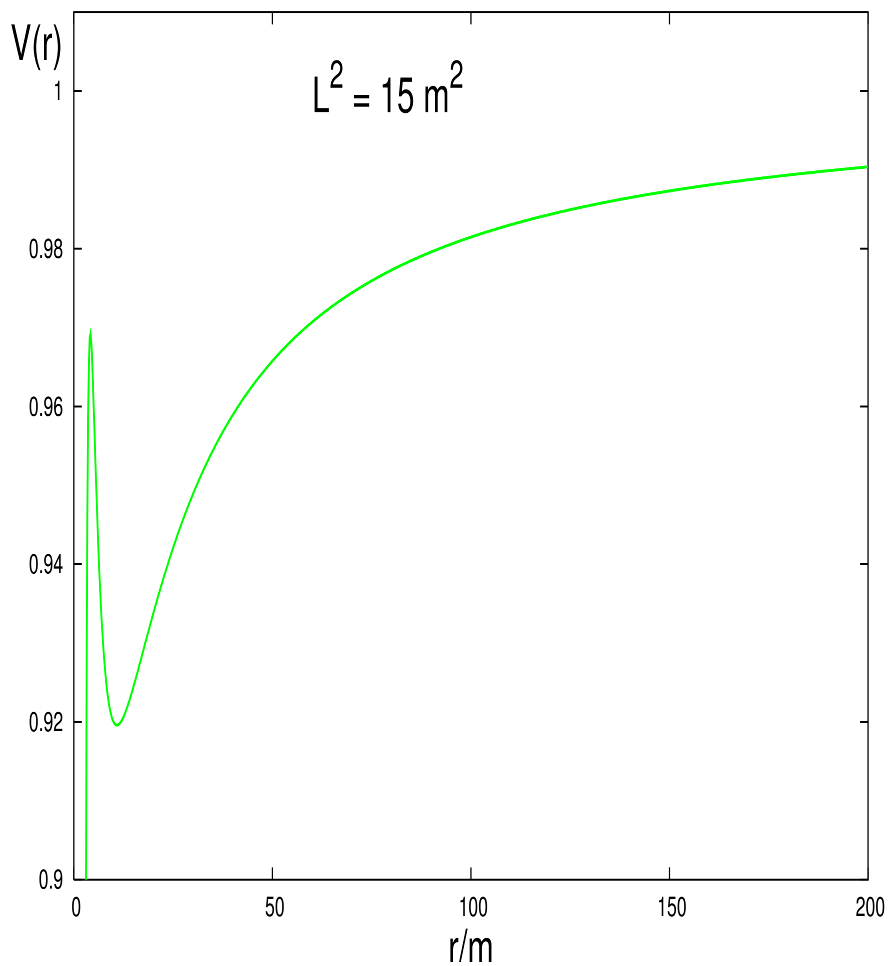
If $L^2 > 12m^2$ the potential has the following form



$V(r)$ has a maximum in $r = r_-$ followed by a minimum in $r = r_+$; thus, a particle with energy $E^2 = V(r_-) \equiv V_{max}$ will move on an **unstable** circular orbit at $r = r_-$, whereas if $E^2 = V(r_+) \equiv V_{min}$ it will move on a **stable** circular orbit at $r = r_+$. (See the discussion for $\mathbf{E^2} = \mathbf{V_{max}}$ in section 12.3)
Depending on the value of L the maximum of the potential can be greater or smaller than 1, i.e.

$$\begin{array}{lll} \textbf{a)} & L^2 > 16m^2 & V_{max} > 1, \\ \textbf{b)} & 12m^2 < L^2 < 16m^2 & V_{max} < 1. \end{array}$$

Case **b)** is shown in the following figure



Therefore:
in case **a)** a particle with $V_{min} < E^2 < 1$ will move on an **ellipse** (actually, an approximate ellipse as we will see below), if $1 < E^2 < V_{max}$ and $\dot{r} \leq 0$ it will approach the black hole, reach a turning point r_0 where $E^2 = V(r_p)$ and $\dot{r} = 0$ then, since it cannot penetrate the barrier, it will invert its radial velocity and escape free at infinity. (See the discussion for $\mathbf{E^2} < \mathbf{V_{max}}$ in section 12.3).
Conversely, if $E^2 > V_{max}$ and $\dot{r} \leq 0$ it will fall in the black hole.

In case **b)** a particle with $V_{min} < E^2 < V_{max}$ will move on an elliptic orbit, whereas if $E^2 > 1$ and $\dot{r} \leq 0$, since $\dot{r}^2 = E^2 - V(r)$, it will approach the black hole horizon with increasing velocity and finally fall in.

From the expression of $r_\pm$ given in eq. (12.75) we see that if $L^2 = 12m^2$ the two roots coincide and

$$r_- = r_+ = 6m\,;$$

furthermore, r_+ is an increasing function of L and, as $L \to \infty$, $r_+ \to \infty$. This means that there cannot exist stable circular orbits with radius smaller than $6m$. In addition, it is easy to show that r_- is a decreasing function of L and, as $L \to \infty$,

$$r_- = \frac{L^2}{2m}\left(1-\sqrt{1-\frac{12m^2}{L^2}}\right) \to \frac{L^2}{2m}\left(1-(1-\frac{6m^2}{L^2}+...)\right) = 3m + O(\frac{m}{L})\,;$$

therefore, unstable circular orbits exist only bewteen

$$3m < r_- < 6m.$$

12.4.1 The radial fall of a massive particle

Let us consider a massive particle falling radially into a Schwarzschild black hole.

In this case $d\phi/d\tau = 0$, therefore $L = 0$; moreover, since the particle is moving inwards, $\dot{r} < 0$. Equations (12.72) become

$$\frac{dt}{d\tau} = \frac{E}{1-\frac{2m}{r}} \qquad \frac{dr}{d\tau} = -\sqrt{E^2 - 1 + \frac{2m}{r}} \tag{12.76}$$

$$\frac{d\theta}{d\tau} = 0 \qquad \frac{d\phi}{d\tau} = 0\,. \tag{12.77}$$

If we consider a particle which is at rest at infinity, i.e. such that

$$\lim_{r\to\infty} \frac{dr}{d\tau} = 0 \tag{12.78}$$

from (12.76) it follows that

$$E = 1 \tag{12.79}$$

and the equations for t and r reduce to

$$\frac{dt}{d\tau} = \frac{1}{1-\frac{2m}{r}} \tag{12.80}$$

$$\frac{dr}{d\tau} = -\sqrt{\frac{2m}{r}}\,. \tag{12.81}$$

We shall now integrate these equations.

- Putting $r_0 \equiv r(\tau = 0)$, eq. (12.81) gives

$$\tau(r) = -\int_{r_0}^{r} dr'\sqrt{\frac{r'}{2m}} = -\frac{1}{\sqrt{2m}}\int_{r_0}^{r} dr'(r')^{1/2}$$
$$= \frac{2}{3}\frac{1}{\sqrt{2m}}\left(r_0^{3/2} - r^{3/2}\right)\,. \tag{12.82}$$

- To find $t(r)$, we combine equations (12.80) and (12.81):

$$\frac{dt}{dr} = -\frac{1}{1-\frac{2m}{r}}\sqrt{\frac{r}{2m}}\,. \tag{12.83}$$

If we set $t = 0$ when $\tau = 0$ we find

$$t(r) = \int_0^t dt' = -\int_{r_0}^{r} \frac{1}{1 - \frac{2m}{r'}} \sqrt{\frac{r'}{2m}}\,; \tag{12.84}$$

by solving the integral in (12.84) we get (we omit the explicit computation and give only the result):

$$\begin{aligned} t(r) &= \frac{2}{3}\frac{1}{\sqrt{2m}}\left[r_0^{3/2} - r^{3/2} + 6mr_0^{1/2} - 6mr^{1/2}\right] \\ &+2m\ln\left[\frac{\sqrt{r_0}-\sqrt{2m}}{\sqrt{r_0}+\sqrt{2m}}\frac{\sqrt{r}+\sqrt{2m}}{\sqrt{r}-\sqrt{2m}}\right]\,; \end{aligned} \tag{12.85}$$

$r(t)$ is the inverse function of $t(r)$ and, as $r(\tau)$, is not known analytically.

In figure 12.2 we plot $t(r)$ and $\tau(r)$.

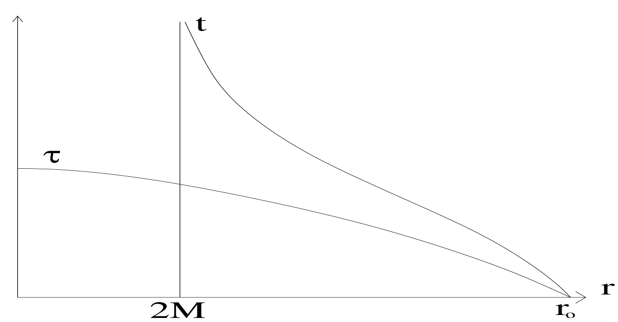


Figure 12.2:

Assuming for simplicity $r_0 \gg 2m$, the behaviour of $t(r)$ for $r \to 2m$ and $r \gg 2m$ is:

- for $r \simeq 2m$

$$t \simeq -2m\ln(\sqrt{r} - \sqrt{2m}) + const. \to \infty \tag{12.86}$$

- for $r \gg 2m$

$$t \simeq \frac{2}{3}\frac{1}{\sqrt{2m}}(r_0^{3/2} - r^{3/2}) \equiv \tau\,. \tag{12.87}$$

From eq. (12.86) we see that for $r \to 2m$, $t(r)$ diverges[2] while eq. (12.82) shows that $\tau(r)$ is *regular* at $r = 2m$. The inverse functions $r(\tau)$ and $r(t)$ are plotted in figure 12.3. From figure 12.3 we also see that $r(\tau)$, which is the radial trajectory as a function of the *proper time*, i.e. as seen by an observer moving with the particle, for $r = 2m$ has a regular behaviour: this observer does not feel anything strange in crossing the horizon, and after crossing it he reaches the singularity in a finite amount of proper time.

[2]We also note that even if the coordinate frame $\{t, r, \theta, \phi\}$ is defined in $\{0 < r < 2m\} \cup \{r > 2m\}$, namely, outside and inside the horizon, these coordinates are really meaningful (i.e., they are useful to describe physical processes) only for $r > 2m$, i.e. outside the horizon.

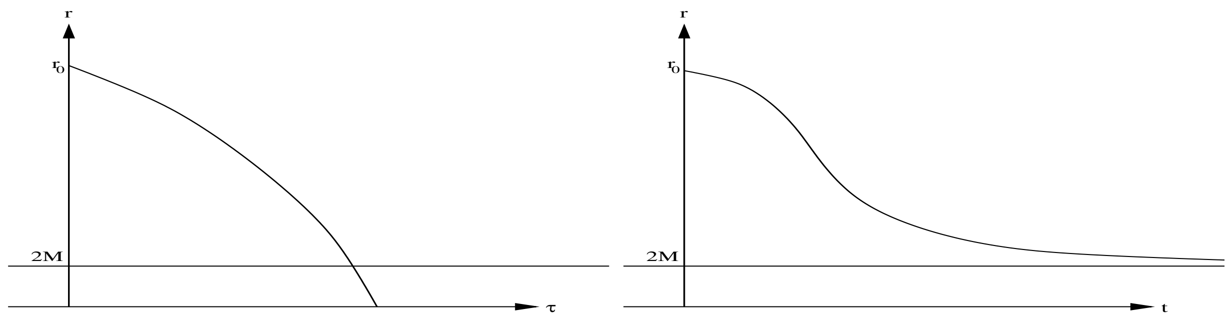


Figure 12.3:

The function $r(t)$, instead, approaches $r = 2m$ only asymptotically. In order to understand what is the meaning of this behaviour, let us consider a spaceship which, while falling radially into the black hole, sends an SOS in the form of a sequence of equally spaced electromagnetic pulses; these signals are received by an observer at radial infinity (the spaceship and the observer have the same $\phi = const$), located at $r = r^{obs}$. The SOS travels along null geodesics $t = t(\lambda)$, $r = r(\lambda)$, with θ, ϕ constants and $L = 0$; λ is the affine parameter along the geodesic. Therefore, from eqs. (12.36) we find

$$\theta = \frac{\pi}{2}, \quad \phi = const, \quad \frac{dt}{d\lambda} = \frac{E}{1 - \frac{2m}{r}}, \quad \frac{dr}{d\lambda} = \pm E \tag{12.88}$$

hence

$$\frac{dt}{dr} = \pm \frac{r}{r - 2m}, \tag{12.89}$$

and the solution is

$$t = \pm r_* + const, \tag{12.90}$$

where r_* is the *tortoise coordinate* already introduced in eq. (11.74)

$$r_* \equiv r + 2m \log\left(\frac{r}{2m} - 1\right), \tag{12.91}$$

so that

$$\frac{dr}{dr_*} = 1 - \frac{2m}{r}\,. \tag{12.92}$$

As in (11.75) we define the outgoing coordinate

$$u \equiv t - r_* \tag{12.93}$$

so that a given outgoing null geodesic is characterized by a constant value of u.

Let us consider two electromagnetic pulses sent from the spaceship as it approaches the horizon, the first at $\tau = \tau_1$, the second at $\tau = \tau_2$ (see figure 12.4.1). The two pulses correspond to $u = u_1$ and $u = u_2$, respectively. The observer at infinity detects the pulses at two values of its own proper time, which coincides with the coordinate time, i.e. at $t = t_1^{obs}$

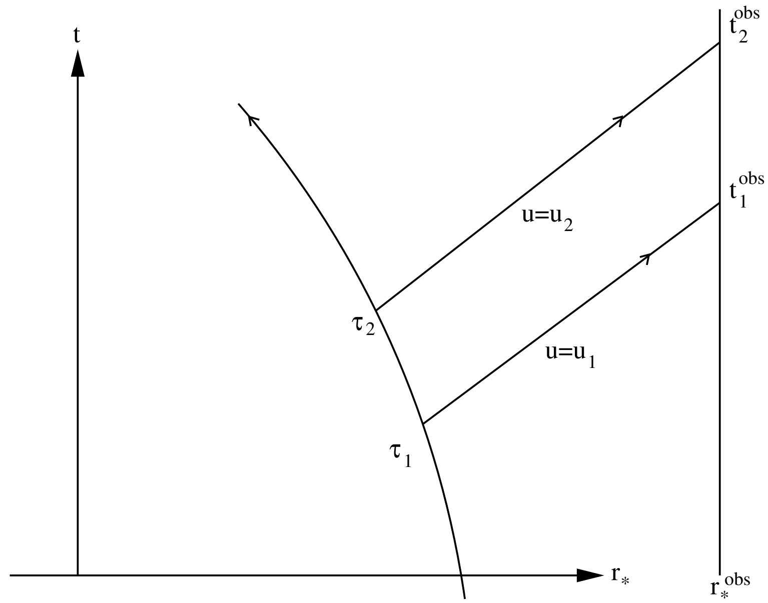


Figure 12.4: A spaceship radially falling into the black hole sends electromagnetic signals to a distant observer

and $t = t_2^{obs}$. Thus, while the person on the spaceship measures a proper time interval between the pulses

$$\Delta\tau = \tau_2 - \tau_1 \,, \tag{12.94}$$

the observer at infinity measures a corresponding coordinate time interval

$$\Delta t^{obs} = t_2^{obs} - t_1^{obs} = (u_2 + r_*^{obs}) - (u_1 + r_*^{obs}) = u_2 - u_1 = \Delta u \,. \tag{12.95}$$

Since u is constant along the two null geodesics, u_1 and u_2 can also be evaluated in terms of points along the spaceship geodesic:

$$\begin{aligned} u_1 &= t(\tau_1) - r_*(\tau_1) \\ u_2 &= t(\tau_2) - r_*(\tau_2) \end{aligned} \tag{12.96}$$

thus $\Delta u = \Delta t(\tau) - \Delta r_*(\tau)$. Therefore, assuming that the pulses are emitted at very short time intervals, we can write (12.93), we find

$$\frac{\Delta t^{obs}}{\Delta\tau} = \frac{\Delta u}{\Delta\tau} \simeq \frac{dt}{d\tau} - \frac{dr_*}{d\tau} = \frac{dt}{d\tau} - \frac{dr_*}{dr}\frac{dr}{d\tau} = \frac{1}{1 - \frac{2m}{r}}\left(1 + \sqrt{\frac{2m}{r}}\right) \,. \tag{12.97}$$

This equation shows that as $r \to 2m$, $\dfrac{\Delta t^{obs}}{\Delta\tau} \to \infty$, which means that the time interval between pulses as detected by the observer at infinity increases, and finally diverges, as the spaceship approaches the horizon.

It is interesting to note that the right hand side of eq. (12.97) has two terms: the first is the square of the gravitational redshift, the second is a Doppler contribution due to the fact that, while sending the pulses, the ship is moving away from the observer.

12.4.2 The motion of a planet around the Sun

Let us now use the geodesic equations (12.72) to study the motion of a planet around the Sun. We can consider the limit

$$\frac{m}{r} \ll 1 \,, \tag{12.98}$$

indeed, if we consider Mercury, which is the closest planet to the Sun, since the Mercury-Sun distance is $r \sim 5.8 \cdot 10^7$ km, we find

$$\frac{GM_\odot}{rc^2} = \frac{1.4768}{5.8 \cdot 10^7} \sim 2.5 \cdot 10^{-8} \,.$$

In what follows, we shall indicate with a prime differentiation with respect to ϕ, and use the variable $u \equiv \frac{1}{r}$, as we did in Section 12.3.1. In therms of u, the equation for $\dot{\phi}$ becomes

$$\dot{\phi} = Lu^2 \,,$$

and

$$\dot{r} = r'\dot{\phi} = -\frac{1}{u^2}u'\dot{\phi} = -Lu' .$$

By substituting in eq. (12.72), it becomes

$$L^2(u')^2 + 1 - 2mu + u^2L^2 - 2mL^2u^3 = E \,,$$

and differentiating with respect to ϕ,

$$2L^2u''u' - 2mu' + 2uu'L^2 - 6mL^2u'u^2 = 0 \,.$$

Dividing by $2L^2u'$, we find the equation for u

$$u'' + u - \frac{m}{L^2} - 3mu^2 = 0 \,. \tag{12.99}$$

The Newtonian equation

The Newtonian equation which corresponds to the third eq. (12.72) is derived from the energy conservation law

$$\frac{1}{2}m_p\left[(\dot{r})^2 + r^2(\dot{\phi})^2\right] - \frac{m_p\, m}{r} = const \qquad \Rightarrow \qquad (\dot{r})^2 - \frac{2m}{r} + \frac{L^2}{r^2} = const \,.$$

where m_p is the particle mass and we have set $G = 1$. By expressing (12.4.2) in terms of u and differentiating with respect to ϕ, we find

$$2L^2u''u' - 2mu' + 2uu'L^2 = 0 \,,$$

which becomes

$$u'' + u - \frac{m}{L^2} = 0 \,. \tag{12.100}$$

Equation (12.100) differs from equation (12.99) only by the term $3mu^2$, which is smaller than, say, u by a factor

$$3mu = \frac{3m}{r} \ll 1 \,.$$

Equation (12.100) can be written as

$$\left(u - \frac{m}{L^2}\right)'' + \left(u - \frac{m}{L^2}\right) = 0 \ ,$$

the solution of which is

$$u - \frac{m}{L^2} = A\cos(\phi - \phi_0) \qquad \Rightarrow \qquad u = \frac{m}{L^2}\left[1 + \frac{L^2 A}{m}\cos(\phi - \phi_0)\right] \ ,$$

where ϕ_0 and A are integration constants. In terms of r the solution is

$$r = \frac{L^2}{m}\frac{1}{1 + \frac{L^2 A}{m}\cos(\phi - \phi_0)} \ .$$

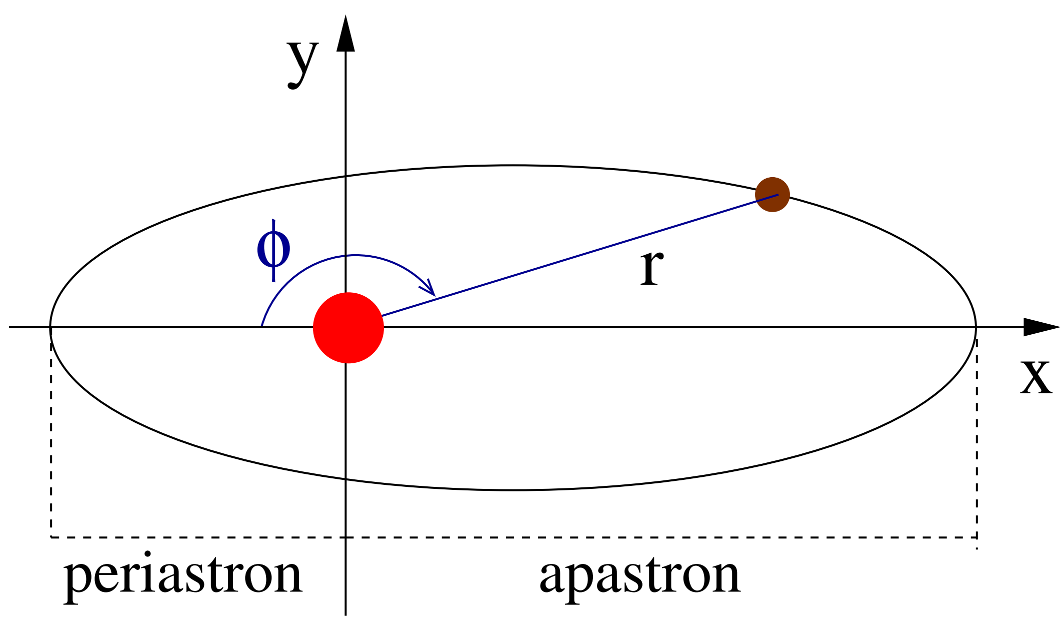
If we set

$$e = \frac{L^2 A}{m} \ , \tag{12.101}$$

the previous equation becomes

$$r = \frac{L^2}{m}\frac{1}{1 + e\cos(\phi - \phi_0)} \ , \tag{12.102}$$

which describes an ellipse with eccentricity e in polar coordinates (r, ϕ). If we set for example



$\phi_0 = 0$, we see that the periastron, i.e. the minimum distance the planet reaches in its motion around the central body (perihelion if the central body is the Sun) occurs when $\phi = 0$, i.e.

$$r_{periastron} = \frac{L^2}{m}\frac{1}{1 + e} \ . \tag{12.103}$$

The apastron (the maximum distance from the central body, aphelion in the case of the Sun) is

$$r_{apastron} = \frac{L^2}{m}\frac{1}{1-e} \ . \tag{12.104}$$

It is worth noting that, since

$$\frac{m}{L^2} = \frac{1}{r_{periastron}(1+e)} \qquad \Rightarrow \qquad \frac{m^2}{L^2} = \frac{m}{r_{periastron}(1+e)} \ ,$$

and since $m/r \ll 1$, it follows that

$$\frac{m^2}{L^2} \ll 1 \ . \tag{12.105}$$

The relativistic equations

In order to solve equation (12.99)

$$u'' + u - \frac{m}{L^2} - 3mu^2 = 0 \tag{12.106}$$

we adopt the same perturbative approach used in Section 12.3.1 to study the deflection of light by a massive body. We search for a solution in the form

$$u = u^{(0)} + u^{(1)}$$

where $u^{(0)}$ is the solution of the Newtonian equation, i.e.

$$u^{(0)} = \frac{m}{L^2}\left(1 + e\cos\phi\right) \ ,$$

and

$$u^{(1)} \ll u^{(0)} \, .$$

Proceeding as for eq. (12.55) we find

$$(u^{(0)})'' + u^{(0)} - \frac{m}{L^2} - 3m(u^{(0)})^2 + (u^{(1)})'' + u^{(1)} - 3m(u^{(1)})^2 - 6mu^{(0)}u^{(1)} = 0 \, . \tag{12.107}$$

Since $u^{(0)}$ satisfies (12.100), eq. (12.107) becomes

$$(u^{(1)})'' + u^{(1)} - 3m(u^{(0)})^2 - 3m(u^{(1)})^2 - 6mu^{(0)}u^{(1)} = 0 \, . \tag{12.108}$$

The terms $3m(u^{(1)})^2$ and $6mu^{(0)}u^{(1)}$ are of higher order with respect to $3m(u^{(0)})^2$, therefore the leading terms in equation (12.55) are

$$(u^{(1)})'' + u^{(1)} = 3m(u^{(0)})^2 \, , \tag{12.109}$$

i.e.

$$(u^{(1)})'' + u^{(1)} = 3\frac{m^3}{L^4}\left(1 + e\cos\phi\right)^2 \ . \tag{12.110}$$

Let us expand the right-hand side

$$
\begin{aligned}
(u^{(1)})'' + u^{(1)} &= 3\frac{m^3}{L^4}\left[1 + e^2\cos^2\phi + 2e\cos\phi\right] \\
&= 3\frac{m^3}{L^4}\left[1 + \frac{1}{2}e^2(1+\cos 2\phi) + 2e\cos\phi\right] \\
&= 3\frac{m^3}{L^4}\left[cost + \frac{1}{2}e^2\cos 2\phi + 2e\cos\phi\right] .
\end{aligned}
$$

This is the equation of a harmonic oscillator with three forcing terms. They are all very small, because, as shown in eq. (12.105), $\frac{m^2}{L^2} \ll 1$. However, the term

$$2e\cos(\phi) ,$$

is in resonance with the free oscillations of the harmonic oscillator, therefore, even if its amplitude is comparable to that of the other terms, it determines a secular perturbation of the planet motion which, after a long time, becomes relevant. For this reason, we will neglect the constant term and the term $\frac{1}{2}e^2\cos 2\phi$ and look for the solution of the resulting equation

$$(u^{(1)})'' + u^{(1)} = 6e\frac{m^3}{L^4}\cos(\phi) . \tag{12.111}$$

As can be checked by direct substitution, the solution of this equation is

$$u^{(1)} = \frac{3em^3}{L^4}\phi\sin\phi ,$$

therefore, the complete solution is

$$u = \frac{m}{L^2}\left[1 + e\left(\cos\phi + 3\frac{m^2}{L^2}\phi\sin\phi\right)\right] .$$

At first order in m^2/L^2,

$$
\begin{aligned}
\cos\left(\frac{3m^2}{L^2}\phi\right) &\simeq 1 \\
\sin\left(\frac{3m^2}{L^2}\phi\right) &\simeq \frac{3m^2}{L^2}\phi
\end{aligned}
$$

therefore we can write

$$u \sim \frac{m}{L^2}\left[1 + e\cos\phi\left(1 - 3\frac{m^2}{L^2}\right)\right] . \tag{12.112}$$

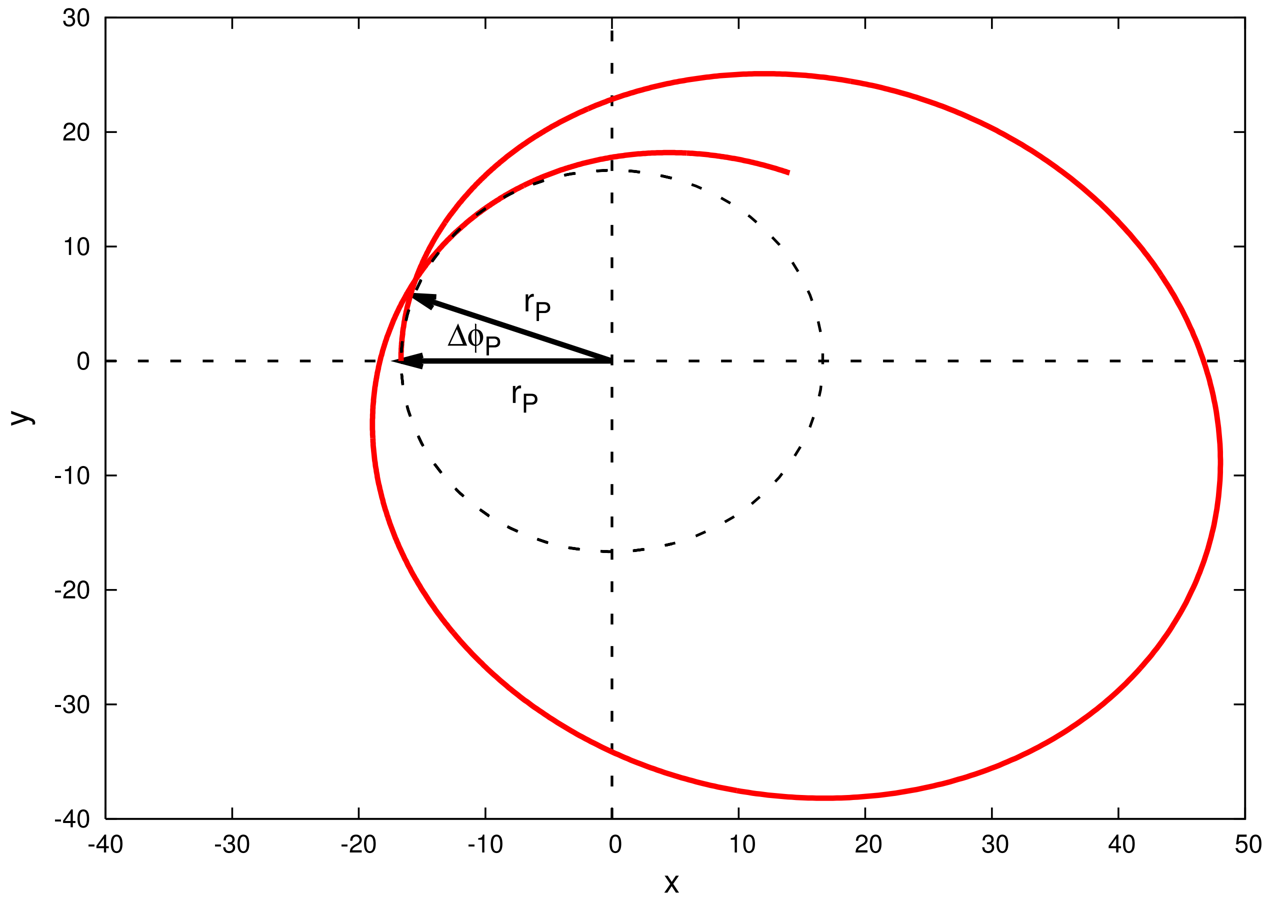
A comparison with the corresponding newtonian equation shows that the term $\frac{3m^2}{L^2}\phi$ determines a secular precession of the periastron. When the argument of the sinusoidal function in eq. (12.112) goes from zero to 2π, i.e. when the planet reaches again the radial distance $r = r_{periastron}$, ϕ changes by

$$\Delta\phi = \frac{2\pi}{1 - \frac{3m^2}{L^2}} \simeq 2\pi\left(1 + \frac{3m^2}{L^2}\right) .$$

Consequently, in one period the periastron is shifted by

$$\Delta\phi_P = \frac{6\pi m^2}{L^2} \ , \tag{12.113}$$

as shown in the following figure.



Thus, in general relativity the orbit of a planet around a central object is not an ellipse; it is an open orbit, and the periastron shifts by $\Delta\phi_P$ at each revolution.

For example, for Mercury equation (12.113) gives a precession of 42.98 arcsec/century. The observed value, after all effects which can be explained with newtonian theory (precession of the equinoxes, perturbations of other planets on Mercury's orbit, etc) is 42.98 ± 0.04 arcsec/century.

Chapter 13

Gravitational Waves

One of the most interesting predictions of the theory of General Relativity is the existence of gravitational waves. The idea that a perturbation of the gravitational field should propagate as a wave is, in some sense, intuitive. For example electromagnetic waves were introduced when the Coulomb theory of electrostatics was replaced by the theory of electrodynamics, and it was shown that they transport through space the information about the evolution of charged systems. In a similar way when a mass-energy distribution changes in time, the information about this change should propagate in the form of waves. However, gravitational waves have a distinctive feature: due to the twofold nature of $g_{\mu\nu}$, which is the metric tensor and the gravitational potential, gravitational waves are **metric waves**. Thus when they propagate the geometry, and consequently the proper distance between spacetime points, change in time.

Gravitational waves can be studied by following two different approaches, one based on perturbative methods, the second on the solution of the non linear Einstein equations.

The perturbative approach

Be $g^0_{\mu\nu}$ a known exact solution of Einstein's equations; it can be, for instance, the metric of flat spacetime $\eta_{\mu\nu}$, or the metric generated by a Schwarzschild black hole. Let us consider a small perturbation of $g^0_{\mu\nu}$ caused by some source described by a stress-energy tensor $T^{\mu\nu}_{pert}$. We shall write the metric tensor of the perturbed spacetime, $g_{\mu\nu}$, as follows

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}, \tag{13.1}$$

where $h_{\mu\nu}$ is the small perturbation

$$|h_{\mu\nu}| << |g^0_{\mu\nu}|.$$

It is clear that this assumption is ambiguous, because we should specify in which reference frame this is true; however we shall assume that this frame does exists.

The inverse metric can be written as

$$g^{\mu\nu} = g^{0\,\mu\nu} - h^{\mu\nu} + O(h^2)\,, \tag{13.2}$$

where the indices of $h^{\mu\nu}$ have been raised with the *unperturbed* metric

$$h^{\mu\nu} \equiv g^{0\,\mu\alpha} g^{0\,\nu\beta} h_{\alpha\beta}\,. \tag{13.3}$$

Indeed, with this definition,

$$\left(g^{0\,\mu\nu} - h^{\mu\nu}\right)\left(g^0_{\nu\alpha} + h_{\nu\alpha}\right) = \delta^\mu_\alpha + O(h^2)\,. \tag{13.4}$$

In order to find the equations that describe $h_{\mu\nu}$, we shall write Einstein's equations for the metric (13.1) in the form

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\,, \tag{13.5}$$

where $T_{\mu\nu}$ is the sum of two terms, one associate to the source that generates the background geometry $g^0_{\mu\nu}$, say $T^0_{\mu\nu}$, and one associate to the source of the perturbation $\delta T_{\mu\nu}$, which is of order h. We remind that the Ricci tensor $R_{\mu\nu}$ is

$$R_{\mu\nu} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha{}_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha{}_{\mu\alpha} + \Gamma^\alpha{}_{\sigma\alpha}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\alpha}, \tag{13.6}$$

and that the affine connections $\Gamma^\gamma_{\beta\mu}$ are

$$\Gamma^\gamma_{\beta\mu} = \frac{1}{2}g^{\gamma\alpha}\left[g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right]. \tag{13.7}$$

The $\Gamma^\gamma_{\beta\mu}$ computed for the perturbed metric (13.1) are

$$\begin{aligned}
\Gamma^\gamma_{\beta\mu}(g_{\mu\nu}) &= \frac{1}{2}\left[g^{0\alpha\gamma} - h^{\alpha\gamma}\right]\left[\left(g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right) + \left(h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right)\right] \\
&= \frac{1}{2}g^{0\alpha\gamma}\left[g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right] + \frac{1}{2}g^{0\alpha\gamma}\left[h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right] \\
&- \frac{1}{2}h^{\alpha\gamma}\left[g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right] + O(h^2) \\
&\equiv \Gamma^\gamma_{\beta\mu}\left(g^0\right) + \delta\Gamma^\gamma_{\beta\mu}(h) + O(h^2),
\end{aligned} \tag{13.8}$$

where $\delta\Gamma^\gamma_{\beta\mu}(h)$ are of first order in $h_{\mu\nu}$

$$\delta\Gamma^\gamma_{\beta\mu}(h) = \frac{1}{2}g^{0\alpha\gamma}\left[h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right] - \frac{1}{2}h^{\alpha\gamma}\left[g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right]. \tag{13.9}$$

When we substitute eq. (13.9) in the Ricci tensor we get

$$\begin{aligned}
R_{\mu\nu}(g_{\mu\nu}) &= R^0_{\mu\nu}\left(g^0\right) \\
&+ \frac{\partial}{\partial x^\alpha}\delta\Gamma^\alpha{}_{\mu\nu}(h) - \frac{\partial}{\partial x^\nu}\delta\Gamma^\alpha{}_{\mu\alpha}(h) \\
&+ \Gamma^\alpha{}_{\sigma\alpha}\left(g^0\right)\delta\Gamma^\sigma{}_{\mu\nu}(h) + \delta\Gamma^\alpha{}_{\sigma\alpha}(h)\,\Gamma^\sigma{}_{\mu\nu}\left(g^0\right) \\
&- \Gamma^\alpha{}_{\sigma\nu}\left(g^0\right)\delta\Gamma^\sigma{}_{\mu\alpha}(h) - \delta\Gamma^\alpha{}_{\sigma\nu}(h)\,\Gamma^\sigma{}_{\mu\alpha}\left(g^0\right) + O(h^2) \equiv R^0_{\mu\nu}\left(g^0\right) + \delta R_{\mu\nu}(h) + O(h^2)
\end{aligned} \tag{13.10}$$

We now have to work out the right-hand side of the Einstein equations (13.5), i.e. $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$, and separate the terms which are of order h. Since $T_{\mu\nu} = T^0_{\mu\nu} + \delta T_{\mu\nu}$

$$\begin{aligned}
T &= g^{\mu\nu}T_{\mu\nu} = \left(g^{0\mu\nu} - h^{\mu\nu}\right)\left(T^0_{\mu\nu} + \delta T_{\mu\nu}\right) \\
&= g^{0\mu\nu}T^0_{\mu\nu} - h^{\mu\nu}T^0_{\mu\nu} - g^{0\mu\nu}\delta T_{\mu\nu} + O(h^2) \equiv T^0 + \delta T + O(h^2).
\end{aligned} \tag{13.11}$$

Consequently

$$\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right) = T^0_{\mu\nu}+\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}+h_{\mu\nu}\right)\left(T^0+\delta T\right)+O(h^2) \tag{13.12}$$
$$= \left(T^0_{\mu\nu}-\frac{1}{2}g^0_{\mu\nu}T^0\right)+\left[\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}\delta T+h_{\mu\nu}T^0\right)\right]+O(h^2).$$

Combining eqs. (13.10) and (13.11), and reminding that $g^0_{\mu\nu}$ is, by assumption, the exact solution of Einstein's equations in vacuum $R_{\mu\nu}\left(g^0\right)=\frac{8\pi G}{c^4}\left(T^0_{\mu\nu}-\frac{1}{2}g^0_{\mu\nu}T^0\right)$, Einstein's equations for the perturbations $h_{\mu\nu}$ reduce to

$$\begin{aligned}
&\frac{\partial}{\partial x^\alpha}\Gamma^\alpha{}_{\mu\nu}(h)-\frac{\partial}{\partial x^\nu}\Gamma^\alpha{}_{\mu\alpha}(h) \\
+\;&\Gamma^\alpha{}_{\sigma\alpha}\left(g^0\right)\Gamma^\sigma{}_{\mu\nu}(h)+\Gamma^\alpha{}_{\sigma\alpha}(h)\,\Gamma^\sigma{}_{\mu\nu}\left(g^0\right) \\
-\;&\Gamma^\alpha{}_{\sigma\nu}\left(g^0\right)\Gamma^\sigma{}_{\mu\alpha}(h)-\Gamma^\alpha{}_{\sigma\nu}(h)\,\Gamma^\sigma{}_{\mu\alpha}\left(g^0\right)=\frac{8\pi G}{c^4}\left[\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}\delta T+h_{\mu\nu}T^0\right)\right]+O(h^2),
\end{aligned} \tag{13.13}$$

that are linear in $h_{\mu\nu}$. Their solution describes the propagation of gravitational waves in the considered background.[1] This approximation works sufficiently well in a variety of physical situations because gravitational waves are very weak. This point will be better understood in the next chapter, when we will discuss the generation of gravitational waves.

The "exact" approach

The second approach to the study of gravitational waves seeks for exact solutions of Einstein's equations which describe both the source and the emitted wave, but no solution of this kind has been found so far. Of course the non-linearity of the equations makes the problem very difficult; however, it may be noted that also in electrodynamics an exact solution of Maxwell's equations appropriate to describe the electromagnetic field produced by a current which decreases in an electric oscillator due to the emission of electromagnetic waves has never been found, although Maxwell's equations are linear.

Exact solutions of Einstein's equations describing gravitational waves can be found only if one imposes some particular symmetry as for example plane, spherical, or cylindrical symmetry. The interaction of plane waves can also be described in terms of exact solutions, and due to the non-linearity of the equations of gravity it is very different from the interaction of electromagnetic waves.

In the following we shall use the perturbative approach to show that a weak perturbation of the flat spacetime satisfies the wave equation.

[1] Notice that the right-hand side of eq.(13.13) is a particular case of the Palatini identity.

13.1 A perturbation of the flat spacetime propagates as a wave

Let us consider the flat spacetime described by the metric tensor $\eta_{\mu\nu}$ and a small perturbation $h_{\mu\nu}$, such that the resulting metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1. \tag{13.14}$$

The affine connections (13.8) computed for the metric (13.14) give

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}\left[\frac{\partial}{\partial x^{\mu}}h_{\rho\nu} + \frac{\partial}{\partial x^{\nu}}h_{\rho\mu} - \frac{\partial}{\partial x^{\rho}}h_{\mu\nu}\right] + O(h^2). \tag{13.15}$$

Since the metric $g^0_{\mu\nu} \equiv \eta_{\mu\nu}$ is constant, $\Gamma^{\lambda}{}_{\mu\nu}(g^0) = 0$ and the right-hand side of eq. (13.13) simply reduces to

$$\begin{aligned} \frac{\partial \Gamma^{\alpha}{}_{\mu\nu}}{\partial x^{\alpha}} &- \frac{\partial \Gamma^{\alpha}{}_{\mu\alpha}}{\partial x^{\nu}} + O(h^2) \\ &= \frac{1}{2}\left\{-\Box_F h_{\mu\nu} + \left[\frac{\partial^2}{\partial x^{\lambda}\partial x^{\mu}}h^{\lambda}_{\nu} + \frac{\partial^2}{\partial x^{\lambda}\partial x^{\nu}}h^{\lambda}_{\mu} - \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}h^{\lambda}_{\lambda}\right]\right\} + O(h^2). \end{aligned} \tag{13.16}$$

The operator $\Box_F$ is the D'Alambertian in flat spacetime

$$\Box_F = \eta^{\alpha\beta}\frac{\partial}{\partial x^{\alpha}}\frac{\partial}{\partial x^{\beta}} = -\frac{\partial^2}{c^2\partial t^2} + \nabla^2. \tag{13.17}$$

Einstein's equations (13.5) for $h_{\mu\nu}$ finally become

$$\left\{\Box_F h_{\mu\nu} - \left[\frac{\partial^2}{\partial x^{\lambda}\partial x^{\mu}}h^{\lambda}_{\nu} + \frac{\partial^2}{\partial x^{\lambda}\partial x^{\nu}}h^{\lambda}_{\mu} - \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}h^{\lambda}_{\lambda}\right]\right\} = -\frac{16\pi G}{c^4}\left(\delta T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta T\right). \tag{13.18}$$

As already discussed in chapter 8, the solution of eqs. (13.18) is not uniquely determined. If we make a coordinate transformation, the transformed metric tensor is still a solution: it describes the same physical situation seen from a different frame. But since we are working in the weak field limit, we are entitled to make only those transformations which preserve the condition $|h'_{\mu\nu}| << 1$ (note that in this Section we denote the transformed tensor as $h'_{\mu\nu}$ rather than as $h_{\mu'\nu'}$, since this simplifies the discussion of infinitesimal coordinate transformations).

If we make an infinitesimal coordinate transformation

$$x^{\mu\prime} = x^{\mu} + \epsilon^{\mu}(x), \tag{13.19}$$

(the prime refers to the coordinate x^{μ}, not to the index μ) where ϵ^{μ} is an arbitrary vector such that $\frac{\partial \epsilon^{\mu}}{\partial x^{\nu}}$ is of the same order of $h_{\mu\nu}$, then

$$\frac{\partial x^{\alpha\prime}}{\partial x^{\mu}} = \delta^{\alpha}_{\mu} + \frac{\partial \epsilon^{\alpha}}{\partial x^{\mu}}\,. \tag{13.20}$$

Since

$$g_{\mu\nu} = g'_{\alpha\beta}\frac{\partial x^{\alpha\prime}}{\partial x^\mu}\frac{\partial x^{\beta\prime}}{\partial x^\nu} = \left(\eta_{\alpha\beta} + h'_{\alpha\beta}\right)\left(\delta^\alpha_\mu + \frac{\partial \epsilon^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu + \frac{\partial \epsilon^\beta}{\partial x^\nu}\right)$$
$$= \eta_{\mu\nu} + h'_{\mu\nu} + \epsilon_{\nu,\mu} + \epsilon_{\mu,\nu} + O(h^2)\,, \tag{13.21}$$

and $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then (up to $O(h^2)$)

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \epsilon_\mu}{\partial x^\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu}. \tag{13.22}$$

In order to simplify eq. (13.18) it appears convenient to choose a coordinate system in which the harmonic gauge condition is satisfied, i.e.

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0. \tag{13.23}$$

Let us see why. This condition is equivalent to say that, up to terms that are first order in $h_{\mu\nu}$, the following equation is satisfied [2]

$$\frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu. \tag{13.24}$$

Using this condition the term in square brackets in eq. (13.18) vanishes, and Einstein's equations reduce to a simple wave equation supplemented by the condition (13.24)

$$\begin{cases} \Box_F h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(\delta T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta T\right) \\ \frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu, \end{cases} \tag{13.25}$$

(to hereafter, we omit the superscript 'pert' to indicate the stress-energy tensor associated to the source of the perturbation). If we introduce the tensor

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\,, \tag{13.26}$$

where $h = \eta^{\mu\nu} h_{\mu\nu} \equiv h^\mu{}_\mu$, eqs. (13.25) become

$$\begin{cases} \Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\delta T_{\mu\nu} \\ \frac{\partial}{\partial x^\mu}\bar{h}^\mu{}_\nu = 0\,, \end{cases} \tag{13.27}$$

and outside the source where $\delta T_{\mu\nu} = 0$

$$\begin{cases} \Box_F \bar{h}_{\mu\nu} = 0 \\ \frac{\partial}{\partial x^\mu}\bar{h}^\mu{}_\nu = 0\,. \end{cases} \tag{13.28}$$

---

[2]

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\kappa}\left\{\frac{\partial h_{\kappa\mu}}{\partial x^\nu} + \frac{\partial h_{\kappa\nu}}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x^\kappa}\right\} = \frac{1}{2}\eta^{\lambda\kappa}\left\{h^\nu{}_{\kappa,\nu} + h^\mu{}_{\kappa,\mu} - h^\nu{}_{\nu,\kappa}\right\}$$

Since the first two terms are equal we find

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = \eta^{\lambda\kappa}\left\{h^\mu{}_{\kappa,\mu} - \frac{1}{2}h^\nu{}_{\nu,\kappa}\right\}$$

q.e.d.

Thus, we have shown that a perturbation of a flat spacetime propagates as a wave travelling at the speed of light, and that Einstein's theory of gravity predicts the existence of gravitational waves.

As in electrodynamics, the solution of eqs. (13.27) can be written in terms of retarded potentials

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int \frac{T_{\mu\nu}(t - \frac{|\mathbf{x}\text{-}\mathbf{x}'|}{c}, \mathbf{x}')}{|\mathbf{x}\text{-}\mathbf{x}' |} \, d^3x', \tag{13.29}$$

and the integral extends over the past light-cone of the event $(t, \mathbf{x})$. In eq. (13.29) we have removed the 'δ' in front of the stress energy tensor which, to hereafter, will be considered as a quantity of order h. Equation (13.29) describes the gravitational waves generated by the source $T_{\mu\nu}$.

We may now ask how eqs. (13.28) and (13.27) should be modified if, instead of considering the perturbation of a flat spacetime, we would consider the perturbation of a curved background. For example, suppose $g^0_{\mu\nu}$ is the Schwarzshild solution for a non rotating black hole. In this case, it is possible to show that, by a suitable choice of the gauge, the Einstein equations written for certain combinations of the components of the metric tensor, can be reduced to a form similar to eqs. (13.27). However, since the background spacetime is now curved, the propagation of the waves will be modified with respect to the flat case. The curvature will act as a potential barrier by which waves are scattered and the final equation will have the form

$$\Box_F \Phi - V(x^\mu)\Phi = -\frac{16\pi G}{c^4} T \tag{13.30}$$

where Φ is the appropriate combination of metric functions, T is a combination of the stress-energy tensor components, $\Box_F$ is the d'Alambertian of the flat spacetime and V is the potential barrier generated by the spacetime curvature. In other words, the perturbations of a sperically symmetric, stationary gravitational field would be described by a Schroedinger-like equation! A complete account on the theory of perturbations of black holes can be found in the book *The Mathematical Theory of Black Holes* by S. Chandrasekhar, Oxford: Claredon Press, (1984).

13.2 How to choose the harmonic gauge

We shall now show that if the harmonic-gauge condition is not satisfied in a reference frame, we can always find a new frame where it is, by making an infinitesimal coordinate transformation

$$x^{\lambda'} = x^\lambda + \epsilon^\lambda, \tag{13.31}$$

provided

$$\Box_F \epsilon_\rho = \frac{\partial h^\beta_\rho}{\partial x^\beta} - \frac{1}{2}\frac{\partial h^\beta_\beta}{\partial x^\rho}. \tag{13.32}$$

Indeed, when we change the coordinate system $\Gamma^\lambda = g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu}$ transforms according to equation (9.63), i.e.

$$\Gamma^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^\rho}\Gamma^\rho - g^{\rho\sigma}\frac{\partial^2 x^{\lambda'}}{\partial x^\rho \partial x^\sigma}, \tag{13.33}$$

where, from eq. (13.31)

$$\frac{\partial x^{\lambda'}}{\partial x^\rho} = \delta^\lambda_\rho + \frac{\partial \epsilon^\lambda}{\partial x^\rho}.$$

If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (see footnote after eq. (13.23))

$$\Gamma^\rho = \eta^{\rho\kappa}\left\{h^\mu{}_{\kappa,\mu} - \frac{1}{2}h^\nu{}_{\nu,\kappa}\right\}; \tag{13.34}$$

moreover

$$\begin{aligned} g^{\rho\sigma}\frac{\partial^2 x^{\lambda'}}{\partial x^\rho \partial x^\sigma} &= g^{\rho\sigma}\left[\frac{\partial}{\partial x^\rho}\left(\frac{\partial x^\lambda}{\partial x^\sigma} + \frac{\partial \epsilon^\lambda}{\partial x^\sigma}\right)\right] = \\ & g^{\rho\sigma}\left[\frac{\partial}{\partial x^\rho}\left(\delta^\lambda_\sigma + \frac{\partial \epsilon^\lambda}{\partial x^\sigma}\right)\right] \simeq \eta^{\rho\sigma}\left[\frac{\partial^2 \epsilon^\lambda}{\partial x^\rho \partial x^\sigma}\right] = \Box_F \epsilon^\lambda, \end{aligned} \tag{13.35}$$

therefore in the new gauge the condition $\Gamma^{\lambda'} = 0$ becomes

$$\Gamma^{\lambda'} = \left[\delta^\lambda_\rho + \frac{\partial \epsilon^\lambda}{\partial x^\rho}\right]\eta^{\rho\kappa}\left[\frac{\partial h^\mu_\kappa}{\partial x^\mu} - \frac{1}{2}\frac{\partial h^\nu{}_\nu}{\partial x^\kappa}\right] - \Box_F \epsilon^\lambda = 0. \tag{13.36}$$

If we neglect second order terms in h eq.(13.36) becomes

$$\Gamma^{\lambda'} = \eta^{\lambda\kappa}\left[\frac{\partial h^\mu_\kappa}{\partial x^\mu} - \frac{1}{2}\frac{\partial h^\nu{}_\nu}{\partial x^\kappa}\right] - \Box_F \epsilon^\lambda = 0.$$

Contracting with $\eta_{\lambda\alpha}$ and remembering that $\eta_{\lambda\alpha}\eta^{\lambda\kappa} = \delta^\kappa_\alpha$ we finally find

$$\Box_F \epsilon_\alpha = \left(\frac{\partial h^\mu{}_\alpha}{\partial x^\mu} - \frac{1}{2}\frac{\partial h^\nu{}_\nu}{\partial x^\alpha}\right).$$

This equation can in principle be solved to find the components of ϵ_α, which identify the coordinate system in which the harmonic gauge condition is satisfied.

13.3 Plane gravitational waves

The simplest solution of the wave equation in vacuum (13.28) is a monocromatic plane wave

$$\bar{h}_{\mu\nu} = \Re\left\{A_{\mu\nu}e^{ik_\alpha x^\alpha}\right\}, \tag{13.37}$$

where $A_{\mu\nu}$ is the polarization tensor, i.e. the wave amplitude and $\vec{k}$ is the wave vector. By direct substitution of (13.37) into the first equation we find

$$\begin{aligned} \Box_F \bar{h}_{\mu\nu} = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\left(e^{ik_\gamma x^\gamma}\right) &= \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left[ik_\gamma \frac{\partial x^\gamma}{\partial x^\beta}e^{ik_\gamma x^\gamma}\right] = \\ \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left[ik_\gamma \delta^\gamma{}_\beta e^{ik_\gamma x^\gamma}\right] &= \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left[ik_\beta e^{ik_\gamma x^\gamma}\right] = \\ = -\eta^{\alpha\beta}k_\alpha k_\beta \; e^{ik_\gamma x^\gamma} = 0, \qquad &\rightarrow \qquad \eta^{\alpha\beta}k_\alpha k_\beta = 0, \end{aligned} \tag{13.38}$$

thus, (13.37) is a solution of (13.28) if $\vec{k}$ is a null vector. In addition the harmonic gauge condition requires that

$$\frac{\partial}{\partial x^\mu}\bar{h}^\mu{}_\nu = 0\,, \tag{13.39}$$

which can be written as

$$\eta^{\mu\alpha}\frac{\partial}{\partial x^\mu}\bar{h}_{\alpha\nu} = 0\,. \tag{13.40}$$

Using eq. (13.37) it gives

$$\eta^{\mu\alpha}\frac{\partial}{\partial x^\mu}A_{\alpha\nu}e^{ik_\gamma x^\gamma} = 0 \quad \rightarrow \quad \eta^{\mu\alpha}A_{\alpha\nu}k_\mu = 0 \quad \rightarrow \quad k_\mu A^\mu{}_\nu = 0\,. \tag{13.41}$$

This further condition expresses the orthogonality of the wave vector and of the polarization tensor.

Since $\bar{h}_{\mu\nu}$ is constant on those surfaces where

$$k_\alpha x^\alpha = const, \tag{13.42}$$

these are the equations of the **wavefront**. It is conventional to refer to k^0 as $\frac{\omega}{c}$, where ω is the frequency of the waves. Consequently

$$\vec{k} = (\frac{\omega}{c}, \mathbf{k}). \tag{13.43}$$

Since $\vec{k}$ is a null vector

$$-(k_0)^2 + (k_x)^2 + (k_y)^2 + (k_z)^2 = 0, \qquad \text{i.e.} \tag{13.44}$$

$$\omega = ck_0 = c\sqrt{(k_x)^2 + (k_y)^2 + (k_z)^2}, \tag{13.45}$$

where (k_x, k_y, k_z) are the components of the unit 3-vector $\mathbf{k}$.

13.4 The TT-gauge

We now want to see how many of the ten components of $h_{\mu\nu}$ have a real physical meaning, i.e. what are the degrees of freedom of a gravitational plane wave. Let us consider a wave propagating in flat spacetime along the $x^1 = x$-direction. Since $h_{\mu\nu}$ is independent of y and z, eqs. (13.28) become (as before we raise and lower indices with $\eta_{\mu\nu}$)

$$\left(-\frac{\partial^2}{c^2\partial t^2} + \frac{\partial^2}{\partial x^2}\right)\bar{h}^\mu{}_\nu = 0, \tag{13.46}$$

i.e. $\bar{h}^\mu{}_\nu$ is an arbitrary function of $t \pm \frac{x}{c}$, and

$$\frac{\partial}{\partial x^\mu}\bar{h}^\mu{}_\nu = 0\,. \tag{13.47}$$

Let us consider, for example, a progressive wave $\bar{h}^{\mu}{}_{\nu} = \bar{h}^{\mu}{}_{\nu}\left[\chi(t,x)\right]$, where $\chi(t,x) = t - \frac{x}{c}$. Being

$$\begin{cases} \frac{\partial}{\partial t}\bar{h}^{\mu}{}_{\nu} = \frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial \chi}\frac{\partial \chi}{\partial t} = \frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial \chi}, \\ \frac{\partial}{\partial x}\bar{h}^{\mu}{}_{\nu} = \frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial \chi}\frac{\partial \chi}{\partial x} = -\frac{1}{c}\frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial \chi}, \end{cases} \tag{13.48}$$

eq. (13.47) gives

$$\frac{\partial}{\partial x^{\mu}}\bar{h}^{\mu}{}_{\nu} = \frac{1}{c}\frac{\partial \bar{h}^{t}{}_{\nu}}{\partial t} + \frac{\partial \bar{h}^{x}{}_{\nu}}{\partial x} = \frac{1}{c}\frac{\partial}{\partial \chi}\left[\bar{h}^{t}{}_{\nu} - \bar{h}^{x}{}_{\nu}\right] = 0. \tag{13.49}$$

This equation can be integrated, and the constants of integration can be set equal to zero because we are interested only in the time-dependent part of the solution. The result is

$$\begin{array}{ll} \bar{h}^{t}{}_{t} = \bar{h}^{x}{}_{t}, & \bar{h}^{t}{}_{y} = \bar{h}^{x}{}_{y}, \\ \bar{h}^{t}{}_{x} = \bar{h}^{x}{}_{x}, & \bar{h}^{t}{}_{z} = \bar{h}^{x}{}_{z}. \end{array} \tag{13.50}$$

We now observe that the harmonic gauge condition does not determine the gauge uniquely. Indeed, if we make an infinitesimal coordinate transformation

$$x^{\mu\prime} = x^{\mu} + \epsilon^{\mu}, \tag{13.51}$$

from eq. (13.33) we find that, if in the old frame $\Gamma^{\rho} = 0$, in the new frame $\Gamma^{\lambda\prime} = 0$, provided

$$\eta^{\rho\sigma}\frac{\partial^2 x^{\lambda\prime}}{\partial x^{\rho}\partial x^{\sigma}} = 0, \tag{13.52}$$

namely, if ϵ^{μ} satisfies the wave equation

$$\Box_F \epsilon^{\mu} = 0. \tag{13.53}$$

If we have a solution of the wave equation,

$$\Box_F \bar{h}_{\mu\nu} = 0 \tag{13.54}$$

and we perform a gauge transformation, the perturbations in the new gauge

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} \tag{13.55}$$

give

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} + \eta_{\mu\nu}\partial^{\alpha}\epsilon_{\alpha} \tag{13.56}$$

and, due to (13.53), the new tensor is solution of the wave equation,

$$\Box_F \bar{h}'_{\mu\nu} = 0\,. \tag{13.57}$$

It can be shown that the converse is also true: it is always possible to find a vector ϵ_{μ} satisfying (13.53) to set to zero four components of $\bar{h}_{\mu\nu}$ solution of (13.54).

Thus, we can use the four functions ϵ^{μ} to set to zero the following four quantities

$$\bar{h}^{t}{}_{x} = \bar{h}^{t}{}_{y} = \bar{h}^{t}{}_{z} = \bar{h}^{y}{}_{y} + \bar{h}^{z}{}_{z} = 0. \tag{13.58}$$

From eq. (13.50) it then follows that

$$\bar{h}^x{}_x = \bar{h}^x{}_y = \bar{h}^x{}_z = \bar{h}^t{}_t = 0. \tag{13.59}$$

The remaining non-vanishing components are $\bar{h}^z{}_y$ and $\bar{h}^y{}_y - \bar{h}^z{}_z$. These components cannot be set equal to zero, because we have exhausted our gauge freedom.

From eqs. (13.58) and (13.59) it follows that

$$\bar{h}^\mu{}_\mu = \bar{h}^t{}_t + \bar{h}^x{}_x + \bar{h}^y{}_y + \bar{h}^z{}_z = 0, \tag{13.60}$$

and since

$$\bar{h}^\mu{}_\mu = h^\mu{}_\mu - 2h^\mu{}_\mu = -h^\mu{}_\mu, \tag{13.61}$$

it follows that

$$h^\mu{}_\mu = 0, \qquad \to \qquad \bar{h}^\mu{}_\nu \equiv h^\mu{}_\nu, \tag{13.62}$$

i.e. in this gauge $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ coincide and are traceless. Thus, a plane gravitational wave propagating along the x-axis is characterized by two functions h_{xy} and $h_{yy} = -h_{zz}$, while the remaining components can be set to zero by choosing the gauge as we have shown:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{yy} & h_{yz} \\ 0 & 0 & h_{yz} & -h_{yy} \end{pmatrix}. \tag{13.63}$$

In conclusion, a gravitational wave has only two physical degrees of freedom which correspond to the two possible polarization states. The gauge in which this is clearly manifested is called the TT-gauge, where 'TT-' indicates that the components of the metric tensor $h_{\mu\nu}$ are different from zero only on the plane orthogonal to the direction of propagation (transverse), and that $h_{\mu\nu}$ is traceless.

13.5 How does a gravitational wave affect the motion of a single particle

Consider a particle at rest in flat spacetime before the passage of the wave. We set an inertial frame attached to this particle, and take the x-axis coincident with the direction of propagation of an incoming TT-gravitational wave. The particle will follow a geodesic of the curved spacetime generated by the wave

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \equiv \frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu}U^\mu U^\nu = 0. \tag{13.64}$$

At $t = 0$ the particle is at rest ($U^\alpha = (1,0,0,0)$) and the acceleration impressed by the wave will be

$$\left(\frac{dU^\alpha}{d\tau}\right)_{(t=0)} = -\Gamma^\alpha{}_{00} = -\frac{1}{2}\eta^{\alpha\beta}\left[h_{\beta0,0} + h_{0\beta,0} - h_{00,\beta}\right], \tag{13.65}$$

but since we are in the TT-gauge it follows that

$$\left(\frac{dU^\alpha}{d\tau}\right)_{(t=0)} = 0. \tag{13.66}$$

Thus, U^α remains constant also at later times, which means that the particle is not accelerated neither at $t = 0$ nor later! It remains at a **constant coordinate position**, regardeless of the wave. We conclude that **the study of the motion of a single particle is not sufficient to detect a gravitational wave.**

13.6 Geodesic deviation induced by a gravitational wave

We shall now study the relative motion of particles induced by a gravitational wave. Consider two neighbouring particles A and B, with coordinates x^μ_A, x^μ_B. We shall assume that the two particles are initially at rest, and that a plane-fronted gravitational wave reaches them at some time $t = 0$, propagating along the x-axis. We shall also assume that we are in the TT-gauge, so that the only non-vanishing components of the wave are those on the (y, z)-plane. In this frame, the metric is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\eta_{\mu\nu} + h^{TT}_{\mu\nu})dx^\mu dx^\nu\,. \tag{13.67}$$

Since $g_{00} = \eta_{00} = -1$, we can assume that both particles have proper time $\tau = ct$. Since the two particles are initially at rest, they will remain at a constant coordinate position even later, when the wave arrives, and their *coordinate* separation

$$\delta x^\mu = x^\mu_B - x^\mu_A \tag{13.68}$$

remains constant. However, since the metric changes, the proper distance between them will change. For example if the particles are on the y-axis,

$$\Delta l = \int ds = \int_{y_A}^{y_B} |g_{yy}|^{\frac{1}{2}}dy = \int_{y_A}^{y_B} |1 + h^{TT}{}_{yy}(t - x/c)|^{\frac{1}{2}}dy \neq \text{constant}. \tag{13.69}$$

We now want to study the effect of the wave by using the equation of geodesic deviation. To this purpose, it is convenient to change coordinate system and use a locally inertial frame $\{x^{\alpha\prime}\}$ centered on the geodesic of one of the two particles, say the particle A; in the neighborhood of A the metric is

$$ds^2 = \eta_{\alpha'\beta'}dx^{\alpha\prime}dx^{\beta\prime} + O(|\delta x|^2)\,. \tag{13.70}$$

i.e. it differs from Minkowski's metric by terms of order $|\delta x|^2$. It may be reminded that, as discussed in Chapter 1, it is always possible to define such a frame.

In this frame the particle A has space coordinates $x^{i'}_A = 0$ $(i = 1, 2, 3)$, and

$$t_A = \tau/c\,, \qquad \frac{dx^{\mu'}}{d\tau}_{|A} = (1,0,0,0)\,, \qquad g_{\mu'\nu'\,|A} = \eta_{\mu'\nu'}\,, \qquad g_{\mu'\nu',\alpha'\,|A} = 0 \ \ (\text{i.e. } \Gamma^{\alpha'}_{\mu'\nu'\,|A} = 0)\,, \tag{13.71}$$

where the subscript $_{|A}$ means that the quantity is computed along the geodesic of the particle A. Moreover, the space components of the vector $\delta x^{\mu'}$ which separates A and B are the coordinates of the particle B:

$$x^{i'}_B = \delta x^{i'} \,. \tag{13.72}$$

To simplify the notation, in the following we will rename the coordinates of this locally inertial frame attached to A as $\{x^\mu\}$, and we will drop all the primes.

The separation vector δx^μ satisfies the equation of geodesic deviation (see Chapter 7):

$$\frac{D^2 \delta x^\mu}{d\tau^2} = R^\mu_{\alpha\beta\gamma} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \delta x^\gamma \,. \tag{13.73}$$

If we evaluate this equation along the geodesic of the particle A, using eqs. (13.71) (removing the primes) we find

$$\frac{1}{c^2}\frac{d^2 \delta x^i}{dt^2} = R^i_{00j}\delta x^j \,. \tag{13.74}$$

If the gravitational wave is due to a perturbation of the flat metric, as discussed in this chapter, the metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and the Riemann tensor

$$\begin{aligned} R_{\alpha\kappa\lambda\mu} &= \frac{1}{2}\left(\frac{\partial^2 g_{\alpha\mu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 g_{\kappa\lambda}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 g_{\alpha\lambda}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\kappa\mu}}{\partial x^\alpha \partial x^\lambda}\right) + \\ &+ g_{\nu\sigma}\left(\Gamma^\nu{}_{\kappa\lambda}\Gamma^\sigma{}_{\alpha\mu} - \Gamma^\nu{}_{\kappa\mu}\Gamma^\sigma{}_{\alpha\lambda}\right) , \end{aligned} \tag{13.75}$$

after neglecting terms which are second order in $h_{\mu\nu}$, becomes

$$R_{\alpha\kappa\lambda\mu} = \frac{1}{2}\left(\frac{\partial^2 h_{\alpha\mu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 h_{\kappa\lambda}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 h_{\alpha\lambda}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\kappa\mu}}{\partial x^\alpha \partial x^\lambda}\right) + O(h^2); \tag{13.76}$$

consequently

$$R_{i00m} = \frac{1}{2}\left(\frac{\partial^2 h_{im}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{00}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{i0}}{\partial x^0 \partial x^m} - \frac{\partial^2 h_{0m}}{\partial x^i \partial x^0}\right) = \frac{1}{2} h^{TT}_{im,00}, \tag{13.77}$$

because in the TT-gauge $h_{i0} = h_{00} = 0$. i and m can assume only the values 2 and 3, i.e. they refer to the y and z components. It follows that

$$R^\lambda{}_{00m} = \eta^{\lambda i} R_{i00m} = \frac{1}{2}\,\eta^{\lambda i}\,\frac{\partial^2 h^{TT}{}_{im}}{c^2 \partial t^2}, \tag{13.78}$$

and the equation of geodesic deviation (13.74) becomes

$$\frac{d^2}{dt^2}\delta x^\lambda = \frac{1}{2}\,\eta^{\lambda i}\,\frac{\partial^2 h^{TT}{}_{im}}{\partial t^2}\delta x^m. \tag{13.79}$$

For $t \le 0$ the two particles are at rest relative to each other, and consequently

$$\delta x^j = \delta x^j_0, \qquad \text{with} \qquad \delta x^j_0 = \quad \text{const}, \qquad t \le 0. \tag{13.80}$$

Since $h_{\mu\nu}$ is a small perturbation, when the wave arrives the relative position of the particles will change only by infinitesimal quantities, and therefore we put

$$\delta x^j(t) = \delta x_0^j + \delta x_1^j(t), \qquad t > 0, \tag{13.81}$$

where $\delta x_1^j(t)$ has to be considered as a small perturbation with respect to the initial position δx_0^j . Substituting (13.81) in (13.79), remembering that δx_0^j is a constant and retaining only terms of order $O(h)$, eq. (13.79) becomes

$$\frac{d^2}{dt^2}\delta x_1^j = \frac{1}{2}\,\eta^{ji}\frac{\partial^2 h^{TT}{}_{ik}}{\partial t^2}\delta x_0^k. \tag{13.82}$$

This equation can be integrated and the solution is

$$\delta x^j = \delta x_0^j + \frac{1}{2}\,\eta^{ji}\,h^{TT}{}_{ik}\,\delta x_0^k, \tag{13.83}$$

which clearly shows the tranverse nature of the gravitational wave; indeed, using the fact that if the wave propagates along x only the components $h_{22} = -h_{33}, h_{23} = h_{32}$ are different from zero, from eqs. (13.83) we find

$$\begin{aligned}
\delta x^1 &= \delta x_0^1 + \frac{1}{2}\,\eta^{11} h^{TT}{}_{1k}\,\delta x_0^k = \delta x_0^1 \\
\delta x^2 &= \delta x_0^2 + \frac{1}{2}\,\eta^{22} h^{TT}{}_{2k}\,\delta x_0^k = \delta x_0^2 + \frac{1}{2}\left(h^{TT}{}_{22}\,\delta x_0^2 + h^{TT}{}_{23}\,\delta x_0^3\right) \\
\delta x^3 &= \delta x_0^3 + \frac{1}{2}\,\eta^{33} h^{TT}{}_{3k}\,\delta x_0^k = \delta x_0^3 + \frac{1}{2}\left(h^{TT}{}_{32}\,\delta x_0^2 + h^{TT}{}_{33}\,\delta x_0^3\right).
\end{aligned} \tag{13.84}$$

Thus, the particles will be accelerated only in the plane orthogonal to the direction of propagation.

Let us now study the effect of the polarization of the wave. Consider a plane wave whose nonvanishing components are (we omit in the following the superscript TT)

$$\begin{aligned}
h_{yy} &= -h_{zz} = 2\Re\left\{A_+ e^{i\omega(t-\frac{x}{c})}\right\}, \\
h_{yz} &= h_{zy} = 2\Re\left\{A_\times e^{i\omega(t-\frac{x}{c})}\right\}.
\end{aligned} \tag{13.85}$$

Consider two particles located, as indicated in figure (13.1) at $(0, y_0, 0)$ and $(0, 0, z_0)$. Let us consider the polarization '+' first, i.e. let us assume

$$A_+ \neq 0 \qquad \text{and} \qquad A_\times = 0. \tag{13.86}$$

Assuming A_+ real eqs. (13.85) give

$$h_{yy} = -h_{zz} = 2A_+ \cos\omega(t - \frac{x}{c}), \qquad h_{yz} = h_{zy} = 0. \tag{13.87}$$

If at $t = 0$ $\omega(t - \frac{x}{c}) = \frac{\pi}{2}$, eqs. (13.84) written for the two particles for $t > 0$ give

$$\begin{aligned}
&1) \qquad z = 0, \quad y = y_0 + \frac{1}{2}h_{yy}\,y_0 = y_0\,[1 + A_+ \cos\omega(t - \frac{x}{c})], \\
&2) \qquad y = 0, \quad z = z_0 + \frac{1}{2}h_{zz}\,z_0 = z_0[1 - A_+ \cos\omega(t - \frac{x}{c})].
\end{aligned} \tag{13.88}$$

After a quarter of a period ($\cos\omega(t-\frac{x}{c})=-1$)

$$\begin{array}{lll} 1) & z=0, & y=y_0[1-A_+], \\ 2) & y=0, & z=z_0[1+A_+]. \end{array} \tag{13.89}$$

After half a period ($\cos\omega(t-\frac{x}{c})=0$)

$$\begin{array}{lll} 1) & z=0, & y=y_0, \\ 2) & y=0, & z=z_0. \end{array} \tag{13.90}$$

After three quarters of a period ($\cos\omega(t-\frac{x}{c})=1$)

$$\begin{array}{lll} 1) & z=0, & y=y_0[1+A_+], \\ 2) & y=0, & z=z_0[1-A_+]. \end{array} \tag{13.91}$$

Similarly, if we consider a small ring of particles centered at the origin, the effect produced by a gravitational wave with polarization '+' is shown in figure (13.2).

Let us now see what happens if $A_\times \neq 0$ and $A_+=0$:

$$h_{yy}=h_{zz}=0, \qquad h_{yz}=h_{zy}=2A_\times\cos\omega(t-\frac{x}{c}). \tag{13.92}$$

Comparing with eqs. (13.84) we see that a generic particle initially at $P=(y_0,z_0)$, when $t>0$ will move according to the equations

$$\begin{aligned} y &= y_0+\frac{1}{2}h_{yz}\ z_0 = y_0+z_0A_\times\cos\omega(t-\frac{x}{c}), \\ z &= z_0+\frac{1}{2}h_{zy}\ y_0 = z_0+y_0A_\times\cos\omega(t-\frac{x}{c}). \end{aligned} \tag{13.93}$$

Let us consider four particles disposed as indicated in figure (13.3)

$$\begin{array}{lll} 1) & y=r, & z=r, \\ 2) & y=-r, & z=r, \\ 3) & y=-r, & z=-r, \\ 4) & y=r, & z=-r. \end{array} \tag{13.94}$$

As before, we shall assume that the initial time $t=0$ corresponds to $\omega(t-\frac{x}{c})=\frac{\pi}{2}$. After a quarter of a period $(\cos\omega(t-\frac{x}{c})=-1)$, the particles will have the following positions

$$\begin{array}{lll} 1) & y=r[1-A_\times], & z=r[1-A_\times], \\ 2) & y=r[-1-A_\times], & z=r[1+A_\times], \\ 3) & y=r[-1+A_\times], & z=r[-1+A_\times], \\ 4) & y=r[1+A_\times], & z=r[-1-A_\times]. \end{array} \tag{13.95}$$

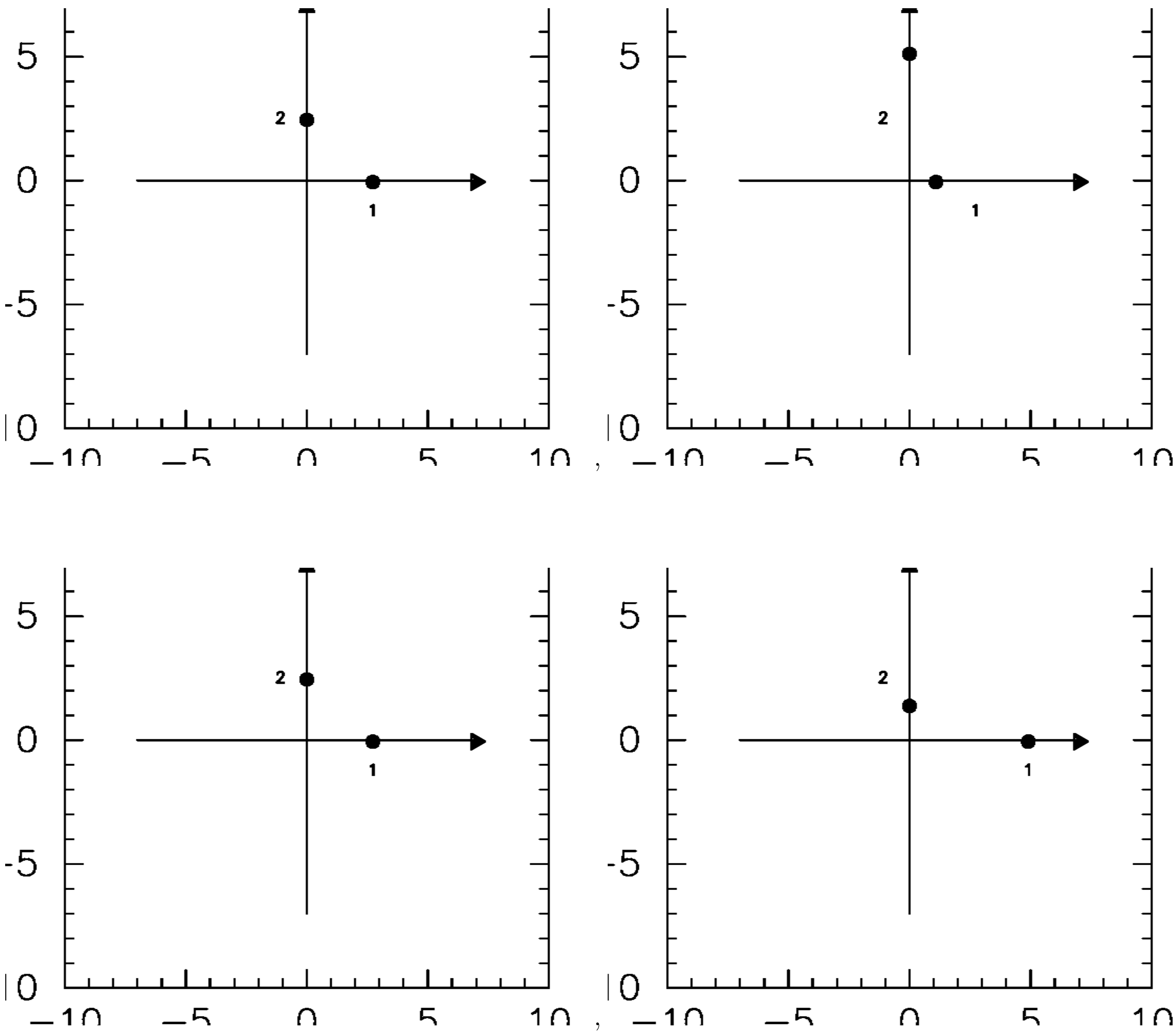


Figure 13.1:

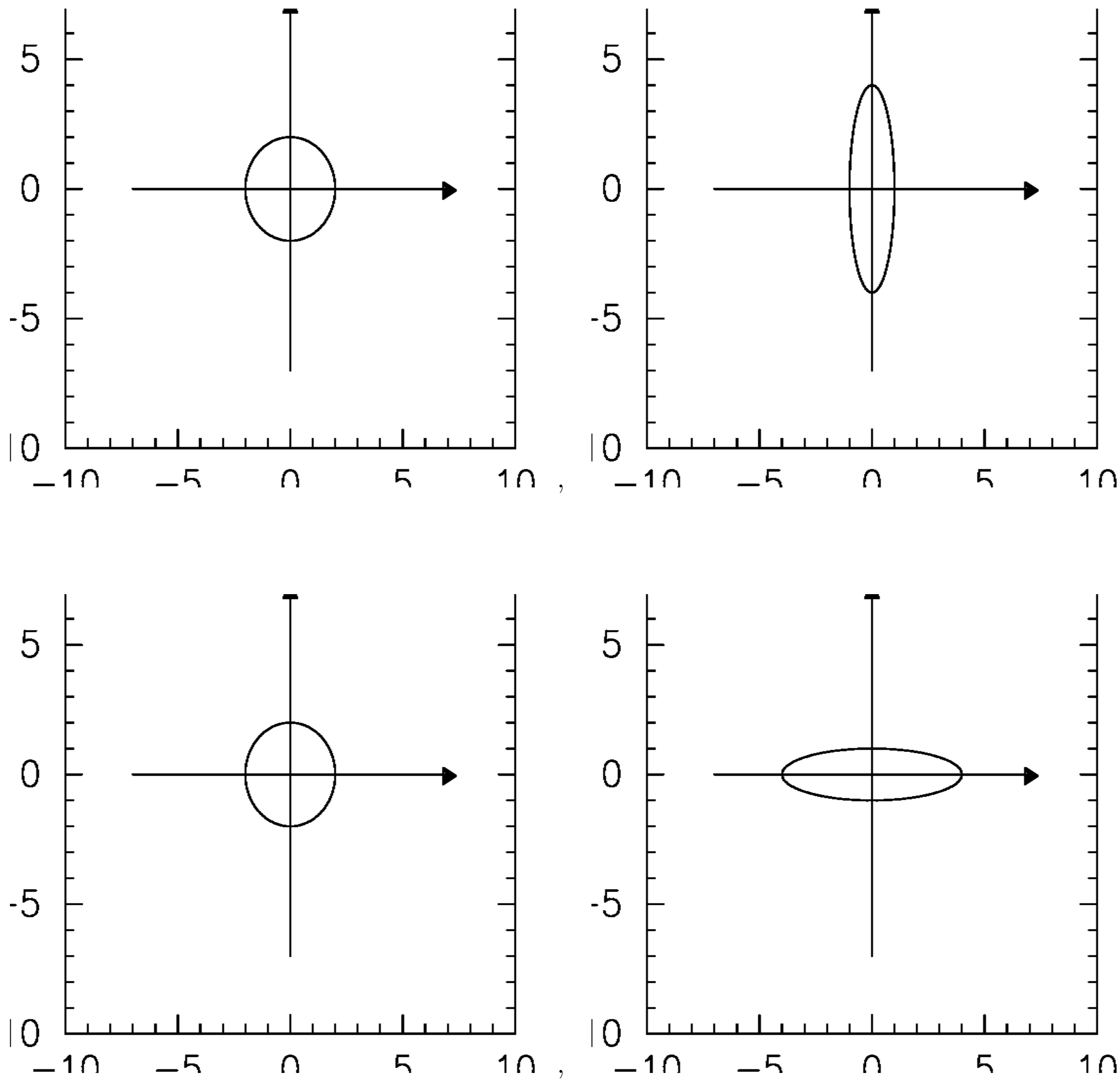


Figure 13.2:

After half a period $\cos\omega(t-\frac{x}{c})=0$, and the particles go back to the initial positions. After three quarters of a period, when $\cos\omega(t-\frac{x}{c})=1$

$$\begin{array}{lll} 1) & y=r[1+A_\times], & z=r[1+A_\times], \\ 2) & y=r[-1+A_\times], & z=r[1-A_\times], \\ 3) & y=r[-1-A_\times], & z=r[-1-A_\times], \\ 4) & y=r[1-A_\times], & z=r[-1+A_\times]. \end{array} \tag{13.96}$$

The motion of the particles is indicated in figure (13.3).

It follows that a small ring of particles centered at the origin, will again become an ellipse, but rotated at 45^0 (see figure (13.4)) with respect to the case previously analysed. In conclusion, we can define A_+ and $A_\times$ as the **polarization amplitudes** of the wave. The wave will be linearly polarized when only one of the two amplitudes is different from zero.

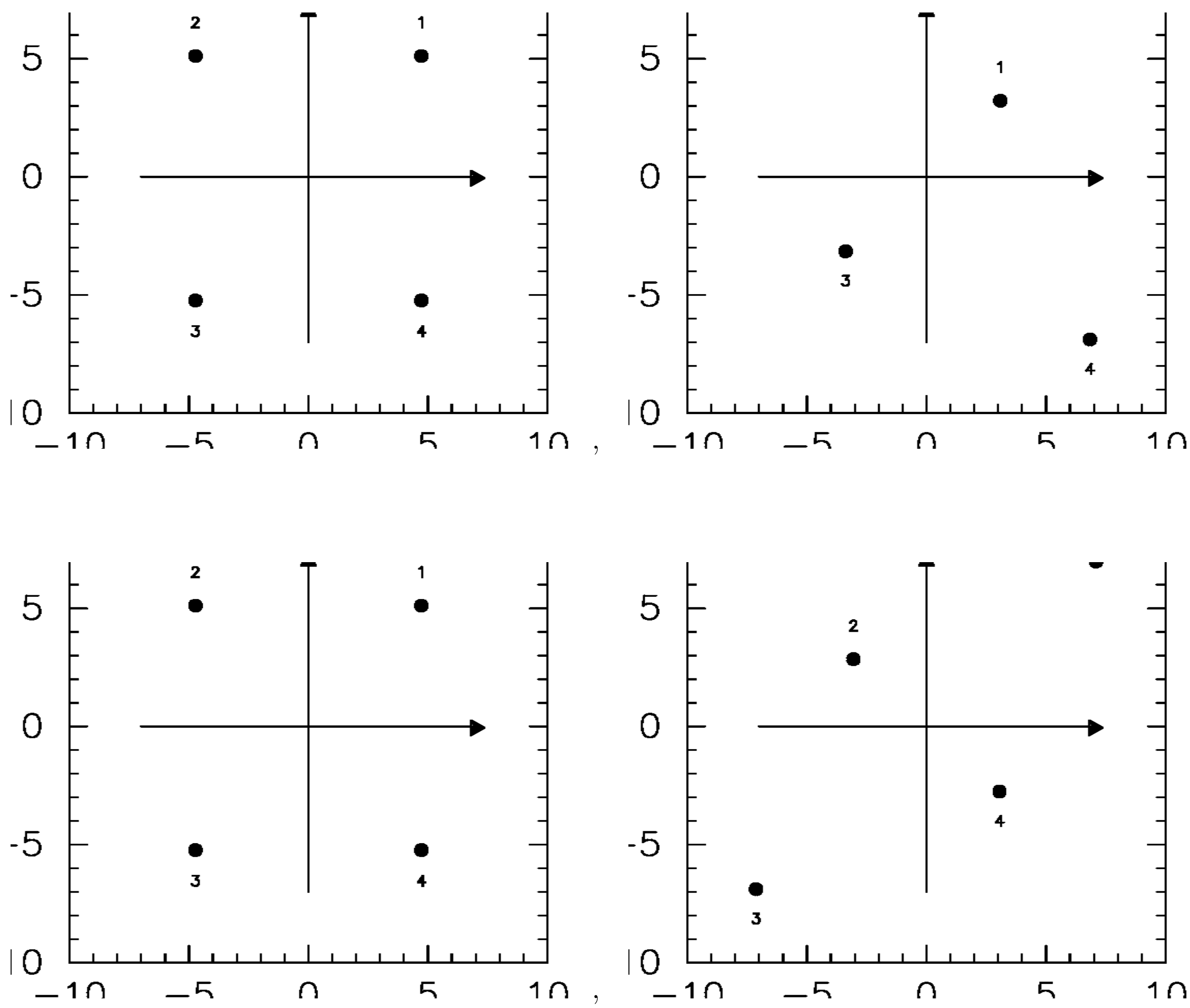

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Figure 13.3:

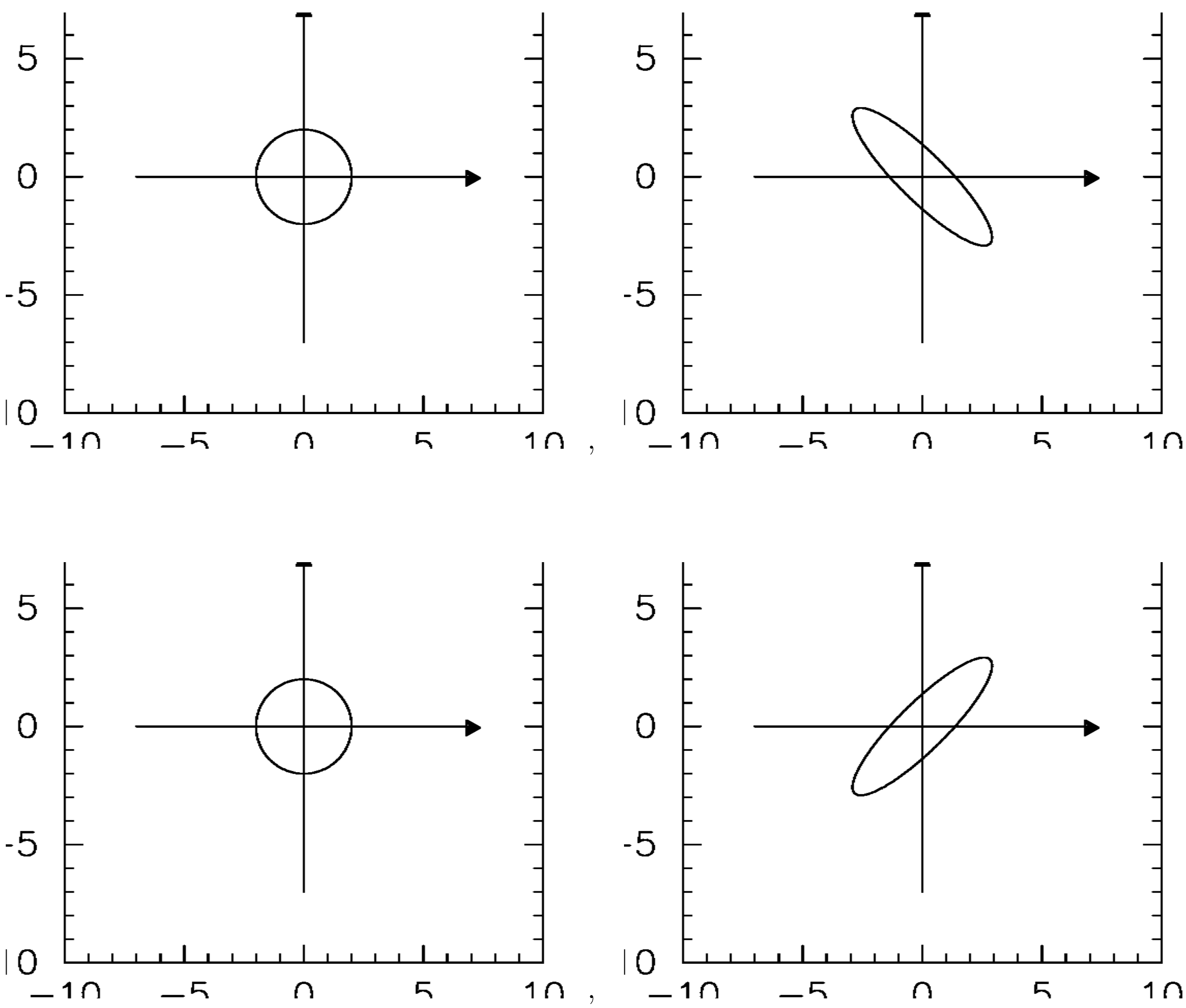

Figure 13.4: