Field Theory 1.

Introduction to Quantum Field Theory

(A Primer for a Basic Education)

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0.1 Prologue

The content of this manuscript is the first semester course in quantum field theory that I have delivered at the Physics Department of the Faculty of Sciences of the University of Bologna since the academic year 2007/2008. It is a pleasure for me to warmly thank all the students and in particular Angelo Giuseppe Ferrari, Pietro Longhi, Lorenzo Rossi, Fabrizio Sgrignuoli and Demetrio Vilardi for their invaluable help in finding many mistakes and misprints. In addition it is mandatory to me to express my deep and warm gratitude to Dr. Paola Giacconi for her concrete help in reading my notes and her continuous encouragement. To all of them I express my profound thankfulness.

Roberto Soldati

Distanze
Questo è il tempo in cui tutto
è a portata di mano
e niente a portata di cuore

Sono sempre di piú quelli
che aspettano uno sguardo.
I nostri occhi non sanno piú vedere.
Per questo, un giorno, qualcosa cambierá.

Maurizio Bacchilega

A revolution is not a dinner party, or writing an essay,
or painting a picture, or doing embroidery;
it cannot be so refined, so leisurely and gentle,
so temperate, kind, courteous, restrained and magnanimous.
A revolution is an insurrection, an act of violence...

Mao Tse Tung

Introduction : Some Notations

Here I like to say a few words with respect to the notation adopted. The components of all the tetra-vectors or 4-vectors have been chosen to be real. The metric is defined by means of the Minkowski constant symmetric tensor of rank two

$$g_{\mu\nu} = \begin{cases} 0 & \text{for} \quad \mu \neq \nu \\ 1 & \text{for} \quad \mu = \nu = 0 \\ -1 & \text{for} \quad \mu = \nu = 1,2,3 \end{cases} \qquad \mu, \nu = 0,1,2,3$$

i.e. the invariant product of two tetra-vectors a and b with components a^0, a^1, a^2, a^3 and b^0, b^1, b^2, b^3 is defined in the following manner

$$a \cdot b \equiv g_{\mu\nu}\, a^\mu b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} = a^0 b^0 - a^k b^k$$

Thus the Minkowski space will be denoted by $\mathcal{M} = \mathbb{R}^{1,3}$. Summation over repeated indexes is understood – Einstein's convention – unless differently stated, while bold type $\mathbf{a}, \mathbf{b}$ notation is used to denote ordinary vectors in the 3-dimensional Euclidean space. Indexes representing all four components 0,1,2,3 are usually denoted by Greek letters, while all the indexes representing the three space components 1,2,3 are denoted by Latin letters.

The upper indexes are, as usual, contravariant while the lower indexes are covariant. Raising and lowering indexes are accomplished with the aid of the Minkowski metric tensor , *e.g.*

$$a_\mu = g_{\mu\nu}\, a^\nu \qquad a^\mu = g^{\,\mu\nu}\, a_\nu \qquad g_{\mu\nu} = g^{\,\mu\nu}$$

in such a manner that the invariant product can also be written in the form

$$a \cdot b = g_{\mu\nu}\, a^\mu b^\nu = a^\mu b_\mu = a^0 b_0 + a^k b_k = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3$$

Throughout the notes the natural system of units is used

$$c = \hbar = 1$$

for the speed of light and the reduced Planck's constant, unless explicitly stated. In turn, for physical units the Heaviside–Lorentz C. G. S. system of electromagnetic units will be employed. In the natural system of units it appears that energy, momentum and mass have the dimensions of a reciprocal length or a wave number, while the time x_0 has the dimensions of a length. The Coulomb's potential created by a point charge q is

$$\varphi(\mathbf{x}) = \frac{q}{4\pi\,|\,\mathbf{x}\,|} = \left(\frac{q}{e^2}\right)\,\frac{\alpha}{r}$$

and the fine structure constant is

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} = 7.297\,352\,568(24) \times 10^{-3} \approx \frac{1}{137}$$

The symbol $-e$ $(e > 0)$ stands for the negative electron charge with $[\,e\,] =$ eV G^{-1} cm^{-1} . We generally work with the four dimensional Minkowski form of the Maxwell equations

$$\varepsilon^{\,\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \qquad \partial_\mu F^{\,\mu\nu} = \frac{1}{c}\,J^\nu$$

where

$$A^\mu = (\varphi, \mathbf{A}) \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad J^\mu = (c\rho, \mathbf{J})$$

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\varphi - \frac{1}{c}\,\dot{\mathbf{A}}$$

in which

$$\mathbf{E} = (E^1, E^2, E^3)\,, \qquad E^k = F^{k0} = F_{0k}\,, \quad (\,k = 1, 2, 3\,)$$

$$\mathbf{B} = (F_{32}, F_{13}, F_{21}) \qquad B^{\,\jmath} = \tfrac{1}{2}\,\varepsilon^{\,\jmath k\ell}\,F_{k\ell} \qquad (\,\varepsilon^{\,123} = \varepsilon^{\,1230} = -1\,)$$

are the electric and magnetic fields respectively while

$$\partial_\mu = \frac{\partial}{\partial x^{\,\mu}} = \left(\frac{1}{c}\,\frac{\partial}{\partial t}\,,\,\nabla\right) = (c^{-1}\partial_t\,,\,\nabla\,) = (c^{-1}\partial_t, \partial_x, \partial_y, \partial_z)$$

so that

$$\nabla\cdot\mathbf{E} = \rho \qquad c\,\nabla\times\mathbf{B} = \mathbf{J} + \frac{\partial\mathbf{E}}{\partial t}$$

Notice that in the Heaviside–Lorentz C. G. S. system of electromagnetic units we have $[\,\mathbf{E}\,] = [\,\mathbf{B}\,] = \mathrm{G} = \mathrm{eV}^{\frac{1}{2}}\ \mathrm{cm}^{-\frac{3}{2}}$ while $[\,J^\mu\,] = \mathrm{esu\ cm}^{-2}\ \mathrm{s}^{-1} = \mathrm{G\ s}^{-1}$.

We will often work with the relativistic generalizations of the Schrödinger wave functions of 1-particle quantum mechanical states. We represent the energy-momentum operators acting on such wave functions following the convention:

$$P^{\,\mu} = (P_0\,,\,\mathbf{P}) = i\,\partial^{\,\mu} = (\,i\partial_t\,,\,-i\nabla\,)$$

In so doing the plane wave $\exp\{-\,i\,p\cdot x\}$ has momentum $p^{\,\mu} = (\,p_0\,,\,\mathbf{p})$ and the Lorentz covariant coupling between a charged particle of charge q and the electromagnetic field is provided by the so called minimal substitution

$$P_\mu - q\,A_\mu(x) = \begin{cases} i\,\partial_t - q\,\varphi(x) & \text{for} \quad \mu = 0 \\ i\,\boldsymbol{\nabla} - q\,\mathbf{A}(x) & \text{for} \quad \mu = 1, 2, 3 \end{cases}$$

The Pauli spin matrices are the three Hermitean 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy

$$\sigma_i \, \sigma_j = \delta_{ij} \; + \; i \, \varepsilon_{\,ijk} \, \sigma_k$$

where $\varepsilon_{\,123} = 1 = -\,\varepsilon^{\,123}$ so that

$$[\,\sigma_i\,,\,\sigma_j\,] \;=\; 2i\,\varepsilon_{\,ijk}\,\sigma_k \qquad \{\sigma_i\,,\,\sigma_j\} \;=\; 2\,\delta_{\,ij}$$

The Dirac matrices in the Weyl, or spinorial, or even chiral representation are given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma^{\,k} = \begin{pmatrix} 0 & \sigma_k \\ -\,\sigma_k & 0 \end{pmatrix} \qquad (\,k = 1,2,3\,)$$

with the Hermitean conjugation property

$$\gamma^{\,\mu\,\dagger} = \gamma^{\,0}\,\gamma^{\,\mu}\,\gamma^{\,0} \qquad \{\gamma^{\,\mu}\,,\,\gamma^{\,\nu}\} = 2\,g^{\,\mu\nu}\,\mathbb{I}$$

so that

$$\beta = \gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \alpha^{\,k} = \begin{pmatrix} -\,\sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad (\,k = 1,2,3\,)$$

It follows that the Dirac operator in the Weyl representation of the Clifford algebra takes the form

$$i\partial\!\!\!/ - m = i\hbar \begin{pmatrix} imc/\hbar & 0 & \partial_0 + \partial_z & \partial_x - i\partial_y \\ 0 & imc/\hbar & \partial_x + i\partial_y & \partial_0 - \partial_z \\ -\partial_0 - \partial_z & -\partial_x + i\partial_y & imc/\hbar & 0 \\ -\partial_x - i\partial_y & -\partial_0 + \partial_z & 0 & imc/\hbar \end{pmatrix}$$

Concerning the Levi-Civita symbol $\varepsilon^{\,\mu\nu\rho\sigma}$, *i.e.* the totally anti-symmetric unit tensors, I follow the conventions in Lev D. Landau and Evgenij M. Lifšits, *Teoria dei campi*, Editori Riuniti, Roma (1976) §6., pp. 30-40: namely,

$$\varepsilon^{\,0123} = +1 = -\,\varepsilon_{\,0123} \qquad \varepsilon_{\,123} = +1 = -\,\varepsilon^{\,123}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \qquad \mathrm{C}_{\imath} = \tfrac{1}{2}\,\varepsilon_{\,\imath\jmath k}\,\mathrm{A}_{\jmath}\,\mathrm{B}_k = -\,\mathrm{C}^{\,\imath} = \varepsilon^{\,\imath\jmath k}\,\mathrm{A}^{\jmath}\,\mathrm{B}^{\,k}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \qquad \mathrm{B}_{\imath} = \tfrac{1}{2}\,\varepsilon_{\,\imath\jmath k}\,\partial_{\jmath}\,\mathrm{A}_k = -\,\mathrm{B}^{\,\imath} = -\,\tfrac{1}{2}\,\varepsilon_{\,\imath\jmath k}\,\partial_{\jmath}\,\mathrm{A}^{k}$$

Chapter 1

Basics in Group Theory

The aim of this Chapter is to briefly summarize the main definitions and key results of group theory, a huge and tough mathematical subject, which turns out to be absolutely necessary in order to understand and appreciate the crucial role of the symmetries in the development of field theory models to describe High Energy and Particle Physics. Needless to say this brief review is by no means exhaustive and accurate, so that I warmly recommend the students to approach one or many of the excellent textbooks in the subject available in the Literature.

1.1 Groups and Group Representations

1.1.1 Definitions

A set G of elements $\mathrm{e}, f, g, h, \ldots \in G$ that satisfies all the four conditions listed below is called an abstract group or simply a group.

1. A law of composition, or multiplication, is defined for the set, such that the multiplication of each pair of elements f and g gives an element h of the set : this is written as

   $$fg = h$$

   Element h is called the product of the elements f and g , which are called the factors. In general the product of two factors depends upon the order of the factors, so that the elements fg and gf can be different.

2. The multiplication is associative : if f , g and h are any three elements, the product of the element f with gh must be equal to the product of the element fg with h

   $$f(gh) = (fg)h$$

3. The set G contains a unit element e called the identity giving the relation
$$\mathrm{e}f = f\mathrm{e} = f \qquad \forall f \in G$$
4. For any element $f \in G$ there is an element $f^{-1} \in G$ called the inverse or reciprocal of f such that
$$f^{-1}f = ff^{-1} = \mathrm{e} \qquad \forall f \in G$$

If the number of elements in G is finite, then the group is said to be finite, otherwise the group is called infinite. The number of elements in a finite group is named its order.

If the multiplication is commutative, *i.e.* , if for any pair of elements f and g we have $fg = gf$, then the group is said to be commutative or Abelian.

Any subset of a group G , forming a group relative to the very same law of multiplication, is called a subgroup of G .

The 1:1 correspondence between the elements of two groups F and G
$$f \;\leftrightarrow\; g \qquad f \in F \quad g \in G$$
is said to be an isomorphism iff for any pair of relations
$$f_1 \;\leftrightarrow\; g_1 \quad f_2 \;\leftrightarrow\; g_2 \qquad f_1, f_2 \in F \quad g_1, g_2 \in G$$
then there follows the relation
$$f_1 f_2 \;\leftrightarrow\; g_1 g_2$$
Groups between the elements of which an isomorphism can be established are called isomorphic groups. As an example, consider the set of the $n-$th roots of unity in the complex plane
$$z_k = \exp\{2\pi i\, k/n\} \qquad (\, k = 0, 1, 2, \dots, n-1, \quad n \in \mathbb{N}\,)$$
which form the commutative group
$$\mathbb{Z}_n \equiv \left\{ z_k \in \mathbb{C} \,|\, z_k^n = z_0 = \mathrm{e}^{2\pi i \jmath}\,,\ \jmath \in \mathbb{Z}\,,\ k = 0, 1, 2, \dots, n-1\,,\ n \in \mathbb{N} \right\}$$
where the composition law is the multiplication
$$z_k \,\cdot\, z_h = z_{k+h} = z_{h+k} \qquad \forall\, k, h = 0, 1, 2, \dots, n-1$$

the identity is $z_0 = \mathrm{e}^{2\pi i \jmath}$ $(\,\jmath \in \mathbb{Z}\,)$ and the inverse $z_k^{-1} = \bar{z}_k$ is the complex conjugate. This finite group with n elements is isomorphic to the group of the counterclockwise rotations around the OZ axis $(\mathrm{mod}\, 2\pi)$, for instance, through the n angles

$$\varphi_k = \frac{2\pi k}{n} \quad (\,k = 0, 1, 2, \ldots, n-1, \quad n \in \mathbb{N}\,)$$

in such a manner that

$$R_k = \begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix} \qquad (\,k = 0, 1, 2, \ldots, n-1, \quad n \in \mathbb{N}\,)$$

The isomorphism of these groups follow from the correspondence

$$\varphi_k \;\leftrightarrow\; z_k = \mathrm{e}^{i\varphi_k} \qquad (\,k = 0, 1, 2, \ldots, n-1, \quad n \in \mathbb{N}\,)$$

Homomorphism between two groups differs from isomorphism only by the absence of the requirement of 1:1 correspondence, so that isomorphism is a particular case of homomorphism. For example, the group S_3 of permutations of 3 objects is homomorphic to $\mathbb{Z}_2 = \{1, -1\}$, the law of composition being multiplication. The following relationships establish the homomorphism of the two groups: namely,

$$\left\{ \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{matrix} \right\} \to 1 \qquad \left\{ \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix} \right\} \to 1 \qquad \left\{ \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix} \right\} \to 1$$

$$\left\{ \begin{matrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{matrix} \right\} \to -1 \qquad \left\{ \begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix} \right\} \to -1 \qquad \left\{ \begin{matrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{matrix} \right\} \to -1$$

The results of one branch of the theory of groups, namely the theory of group representations, are used in the overwhelming majority of important cases in which group theory is applied to Physics. The theory of the group representations studies the homomorphic mappings of an arbitrary abstract group on all possible groups of linear operators.

- We shall say that a representation T of a group G is given in a certain linear space L, iff to each element $g \in G$ there is a corresponding linear operator $T(g)$ acting in the space L, such that to each product of the elements of the group there is a corresponding product of the linear operators, *i.e.*

$$T(g_1)T(g_2) = T(g_1 g_2) \qquad \forall g_1, g_2 \in G$$

$$T(\mathrm{e}) = \mathbb{I} \qquad T(g^{-1}) = T^{-1}(g)$$

where $\mathbb{I}$ denotes the identity operator on L. The dimension of the space L is said to be the dimension of the representation. A group can have representations both of a finite and of an infinite number of dimensions. By the very definition, the set of all linear operators $T(g) : L \to L \quad (g \in G)$ is closed under the multiplication or composition law. Hence it will realize an algebra of linear operators over L that will be denoted by $\mathfrak{A}(L)$. If the mapping $T : G \to \mathfrak{A}(L)$ is an isomorphism, then the representation T is said faithful. In what follows, keeping in mind the utmost relevant applications in Physics, we shall always assume that the linear spaces upon which the representations act are equipped by an inner product [1], in such a manner that the concepts of orthonormality, adjointness and unitarity are well defined in the conventional way.

One of the problems in the theory of representations is to classify all the possible representations of a given group. In the study of this problem two concepts play a fundamental role: the concept of equivalence of representations and the concept of reducible representations.

- Knowing any representation T of a group G in the space L one can easily set up any number of new representations of the group. For this purpose, let us select any non-singular linear operator A, carrying vectors from L into a space L' with an equal number d of dimensions, and assign to each element $g \in G$ the linear operator

$$T_A(g) = A\, T(g) A^{-1} \qquad \forall g \in G$$

acting in the vector space L'. It can be readily verified that the map $g \mapsto T_A(g)$ is a representation of the group G that will be thereby said equivalent to the representation $T(g)$.

All the representations equivalent to a given one are equivalent among themselves. Hence, all the representations of a given group split into classes of mutually equivalent ones, so that, accordingly, the problem of classifying all representations of a group is reduced to the more limited one of finding all mutually nonequivalent representations.

- Consider a subspace L_1 of the linear vector space L. The subspace $L_1 \subseteq L$ will be said an *invariant subspace* with respect to some given

[1] The inner product between to vectors $a, b \in L$ is usually denoted by $(a\,,\,b)$ in the mathematical literature and by the Dirac notation $\langle\, a \,|\, b \,\rangle$ in quantum physics. Here I shall often employ both notations without loss of clarity.

linear operator A acting on L iff

$$A\ell_1 \in L_1 \qquad \forall \ell_1 \in L_1$$

Of course L itself and the empty set $\varnothing$ are trivial invariant sub-spaces.

- The representation T of the group G in the vector space L is said to be reducible iff there exists in L at least one nontrivial subspace L_1 invariant with respect to all operators $T(g)\,,\ g \in G$. Otherwise the representation is called irreducible. All one dimensional representations are evidently irreducible.

- A representation T of the group G in the vector space L is said to be *unitary* iff all the linear operators $T(g)\,,\ g \in G$, are unitary operators

$$T^{\dagger}(g) = T^{-1}(g) = T(g^{-1}) \qquad \forall g \in G$$

1.1.2 Theorems

There are two important theorems concerning unitary representations.

- Theorem 1. Let T a unitary reducible representation of the group G in the vector space L and let $L_1 \subseteq L$ an invariant subspace. Then the subspace $L_2 = \complement L_1$, the orthogonal complement of L_1, is also invariant.

  `Proof.`

  Let $\ell_1 \in L_1\,,\ \ell_2 \in L_2$, then $T^{-1}(g)\ell_1 \in L_1$ and $(\,T^{-1}(g)\ell_1, \ell_2\,) = 0$. On the other hand the unitarity of the representation T actually implies $(\,\ell_1, T(g)\ell_2\,) = 0$. Hence, $T(g)\ell_2$ is orthogonal to ℓ_1 and consequently $T(g)\ell_2 \in L_2\,,\ \forall g \in G$, so the theorem is proved. $\square$

  Hence, if the vector space L transforms according to a unitary reducible representation, it decomposes into two mutually orthogonal invariant sub-spaces L_1 and L_2 such that $L = L_1 \oplus L_2$. Iterating this process we inevitably arrive at the irreducible representations.

- Theorem 2. Each reducible unitary representation $T(g)$ of a group G on a vector space L decomposes, uniquely up to equivalence, into the direct sum of irreducible unitary representations $\boldsymbol{\tau}_a(g)\,,\ a = 1,2,3,\dots$, acting on the invariant vector spaces $L_a \subseteq L$ in such a way that

$$L = L_1 \oplus L_2 \oplus L_3 \oplus \ldots = \bigoplus_a L_a \qquad L_a \perp L_b \text{ for } a \neq b$$

$$\boldsymbol{\tau}_a(g)\,\ell_a \in L_a \quad \forall a = 1, 2, \ldots \qquad \forall g \in G$$

Conversely, each reducible unitary representation $T(g)$ of a group G can be always composed from the irreducible unitary representations $\boldsymbol{\tau}_a(g)\,,\ a = 1, 2, 3, \ldots$, of the group.

The significance of this theorem lies in the fact that it reduces the problem of classifying all the unitary representations of a group G, up to equivalent representations, to that of finding all its irreducible unitary representations.

As an example of the decomposition of the unitary representations of the rotation group, we recall the decomposition of the orbital angular momentum which is well known from quantum mechanics. The latter is characterized by an integer $\ell = 0, 1, 2, \ldots$ and consist of $(2\ell+1) \times (2\ell+1)$ square matrices acting on quantum states of the system with given eigenvalues

$$\lambda_\ell = \hbar^2 \ell(\ell+1) \qquad\qquad \ell + 1 \in \mathbb{N}$$

of the orbital angular momentum operator $\mathbf{L}^2 = [\,\mathbf{r} \times (-i\hbar\nabla)\,]^2$. The set of $2\ell + 1$ quantum states belonging to the subspace of the Hilbert space with a definite orbital angular momentum ℓ are labeled by the possible values of the projections of the orbital angular momentum along a certain axis, *e.g.* $L_z = -\hbar\ell, -\hbar(\ell-1), \ldots, \hbar(\ell-1), \hbar\ell$, in such a manner that we have the spectral decomposition

$$\mathbf{L}^2 = \sum_{\ell=0}^{\infty} \hbar^2 \ell(\ell+1)\,\widehat{P}_\ell \qquad\qquad \widehat{P}_\ell = \sum_{m=-\ell}^{\ell} |\,\ell\,m\rangle\langle \ell\,m\,|$$

with

$$\langle\,\ell\,m\,|\,\ell'\,m'\,\rangle = \delta_{\ell\ell'}\,\delta_{mm'} \qquad\qquad \mathrm{tr}\,\widehat{P}_\ell = 2\ell + 1$$

where $[\mathrm{tr}\,]$ denotes the trace (sum over the diagonal matrix elements) of the projectors $\widehat{P}_\ell$ over the finite dimensional spaces L_ℓ $(\ell = 0, 1, 2, \ldots)$ spanned by the basis $\{\,|\,\ell\,m\rangle\,|\,m = -\ell, -\ell+1, \ldots, \ell-1, \ell\,\}$ of the common eigenstates of $\mathbf{L}^2$ and L_z . Thus, for each rotation $g \in G$ around a fixed point in the ordinary three dimensional Euclidean space, there exists a $(2\ell+1) \times (2\ell+1)$ unitary matrix $\boldsymbol{\tau}_\ell(g)$ that specifies the way how the $(2\ell+1)$ quantum states transform among themselves as a result of the rotation $g \in G$, which actually realizes an irreducible unitary finite dimensional representation of the rotation group. The Hilbert space $\mathfrak{H}$ for a point-like spin-less particle is thereby decomposed according to

$$\mathfrak{H} = \bigoplus_{\ell=0}^{\infty} L_\ell \qquad L_\ell \;\perp\; L_m \text{ for } \ell \neq m \qquad \dim\,(L_\ell) = 2\ell + 1$$

$$\boldsymbol{\tau}_\ell \; : \; L_\ell \; \to \; L_\ell \qquad \ell = 0, 1, 2, \ldots$$

1.1.3 Direct Products of Group Representations

It is very useful to introduce the concepts of characters and the definition of product of group representations.

- Let $T(g)$ any linear operator corresponding to a given representation of the group G . We define the characters $\chi(g)$ of the representation to be the sum of the diagonal matrix elements of $T(g)$

  $$\chi(g) \equiv \mathrm{tr}\, T(g) = \sum_{\imath} T_{\imath\imath}(g) \qquad \forall\, g \in G \tag{1.1}$$

  In the case of infinite dimensional representations we have to suppose the linear operators $T(g)$ to be of the trace class.

  Two equivalent representations have the same characters, as the trace operation does not depend upon the choice of the basis in the vector space.

- **Definition**. Consider two representations $T_1(g)$ and $T_2(g)$ of the group G acting on the vector (Hilbert) spaces L_1 and L_2 of dimensions n_1 and n_2 respectively. Let

  $$\{ e_{1j} \in L_1 \,|\, j = 1, 2, \ldots, n_1 \} \qquad \{ e_{2r} \in L_2 \,|\, r = 1, 2, \ldots, n_2 \}$$

  any two bases so that

  $$e_{1j\,2r} \equiv \{ e_{1j} \otimes e_{2r} \,|\, j = 1, 2, \ldots, n_1 \, , \; r = 1, 2, \ldots, n_2 \}$$

  is a basis in the tensor product $L_1 \otimes L_2$ of the two vector spaces with $\dim(L_1 \otimes L_2) = n_1\, n_2$. Then the matrix elements of the linear operators $T_1(g)$ and $T_2(g)$ with respect to the above bases will be denoted by

  $$[\, T_1(g) \,]_{jk} \equiv (\, e_{1j}, T_1(g)\, e_{1k} \,) \qquad [\, T_2(g) \,]_{rs} \equiv (\, e_{2r}, T_2(g)\, e_{2s} \,)$$

  The direct product

  $$T(g) \equiv T_1(g) \times T_2(g) \qquad \forall\, g \in G$$

  of the two representations is a representation of dimension $n = n_1\, n_2$ the linear operators of which, acting upon the tensor product vector space $L_1 \otimes L_2$, have the matrix elements, with respect to the basis $e_{1j\,2r}$, which are defined by

$$
\begin{aligned}
(\, e_{1j\,2r} \,,\, T(g)\, e_{1k\,2s} \,) \;&\equiv\; [\, T_1(g) \,]_{jk} \,[\, T_2(g) \,]_{rs} \\
&=\; (\, e_{1j} \,,\, T_1(g)\, e_{1k} \,) \;(\, e_{2r} \,,\, T_2(g)\, e_{2s} \,)
\end{aligned}
\tag{1.2}
$$

where $j, k = 1, 2, \ldots, n_1$ and $r, s = 1, 2, \ldots, n_2$. From the definition (1.1) it is clear that we have

$$
\chi(g) = \chi_1(g)\,\chi_2(g) \qquad\qquad \forall\, g \in G
$$

because

$$
\begin{aligned}
\chi(g) \;&\equiv\; \sum_{j=1}^{n_1} \sum_{r=1}^{n_2} (\, e_{1j\,2r} \,,\, T(g)\, e_{1j\,2r} \,) \\
&=\; \sum_{j=1}^{n_1} (\, e_{1j} \,,\, T_1(g)\, e_{1j} \,) \sum_{r=1}^{n_2} (\, e_{2r} \,,\, T_2(g)\, e_{2r} \,) \\
&=\; \chi_1(g)\,\chi_2(g) \qquad\qquad \forall\, g \in G
\end{aligned}
\tag{1.3}
$$

1.2 Continuous Groups and Lie Groups

1.2.1 The Continuous Groups

A group G is continuous if the set of its elements forms a topological space. This means that each element $g \in G$ is in correspondence with an infinite number of subsets $U_g \subset G$, called a system of neighborhoods of any $g \in G$. This correspondence has to satisfy certain conditions that fully guarantee the compatibility between the structure of topological space and the group associative composition law - see the excellent monographs [17] for details.

To illustrate the concept of neighborhood we consider the group of the rotations around a fixed axis, which is an Abelian continuous group. Let $g = R(\varphi)$, $0 \leq \varphi \leq 2\pi$, a rotation through an angle φ around *e.g.* the OZ axis. By choosing arbitrarily a positive number $\varepsilon > 0$, we consider the set $U_g(\varepsilon)$ consisting of all the rotations $g' = R(\varphi')$ satisfying the inequality $|\varphi - \varphi'| < \varepsilon$. Every such set $U_g(\varepsilon)$ is a neighborhood of the rotation $g = R(\varphi)$ and giving ε all its possible real positive values we obtain the infinite manifold of the neighborhoods of the rotation $g = R(\varphi)$. Another well known example is provided by the set $\mathbb{R}$ of the real numbers, which is a topological space and a group, the composition law being the addition, the unit element being the zero and the inverse being the opposite.

The real functions $f : G \to \mathbb{R}$ over the group G is said to be *continuous for the element* $g_0 \in G$ if, for every positive number $\delta > 0$, there exists such a neighborhood U_0 of g_0 that $\forall g \in U_0$ the following inequality is satisfied

$$|f(g) - f(g_0)| < \delta \qquad \forall g \in U_0 \qquad g_0 \in U_0 \subset G$$

A continuous group G is called compact if and only if each real function $f(g)$, continuous for all the elements $g \in G$ of the group, is bounded. For example, the group of the rotations around a fixed axis is compact, the rotation group in the three dimensional space is also a compact group. On the other hand, the continuous Abelian group $\mathbb{R}$ of all the real numbers is not compact, since there exist continuous although not bounded functions, *e.g.* $f(x) = x$, $x \in \mathbb{R}$. The Lorentz group is not compact.

A continuous group G is called locally compact if and only if each real function $f(g)$, continuous for all the elements $g \in G$ of the group, is bounded in every neighborhood $U \subset G$ of the element $g \in G$. According to this definition, the group of all the real numbers is a locally compact group and the Lorentz group is also a locally compact group.

- Theorem : if a group G is locally compact, it always admits irreducible unitary representations in infinite dimensional Hilbert spaces.

In accordance with this important theorem, proved by Gel'fand and Raikov, Gel'fand and Naïmark succeeded in finding the classification of all the unitary irreducible representations of the Lorentz group and of certain other locally compact groups.

In general, if we consider all possible continuous functions defined over a continuous group G, we may find among them some multi-valued functions. These continuous multi-valued functions can not be made single-valued *by brute force* without violating continuity, that is, by rejecting the exceeding values for each element $g \in G$. As an example we have the function

$$f(\varphi) = \mathrm{e}^{\frac{1}{2} i\varphi}$$

over the rotation group around a fixed axis. Since each rotation $g = R(\varphi)$ through an angle φ can also be considered as a rotation through an angle $\varphi + 2\pi$, this function must have two values for a rotation through the same angles: namely,

$$f_+(\varphi) = \mathrm{e}^{\frac{1}{2} i\varphi} \qquad f_-(\varphi) = \mathrm{e}^{\frac{1}{2} i\varphi + i\pi} = -\,\mathrm{e}^{\frac{1}{2} i\varphi} \;=\; -\,f_+(\varphi)$$

Had we rejected the second of these two values, then the function $f(\varphi)$ would become discontinuous at the point $\varphi = 0 = 2\pi$.

The continuous groups which admit continuous many-valued functions are called multiply connected. As the above example shows, it turns out that the rotation group around a fixed axis is multiply connected. Also the rotation group in the three dimensional space is multiply-connected, as I will show below in some detail. The presence of many-valued continuous functions in certain continuous groups leads us to expect that some of the continuous representations of these groups will be many-valued. On the one hand, these many-valued representations can not be ignored just because of its importance in many physical applications. On the other hand, it is not always possible, in general, to apply to those ones all the theorems valid for single-valued continuous representations.

To overcome this difficulty, we use the fact that every multiply connected group G is an homomorphic image of a certain simply connected group $\widetilde{G}$. It turn out that the simply connected group $\widetilde{G}$ can always be chosen in such a manner that none of its simply connected subgroups would have the group G as its homomorphic image. When the simply connected group $\widetilde{G}$ is selected in this way, it is called the universal covering group or the universal enveloping group of the multiply connected group G – see [17] for the proof. Consider once again, as an example, the multiply connected abelian group of the rotations around a fixed axis. The universal covering group for it is the simply connected commutative group $\mathbb{R}$ of all the real numbers. The homomorphism is provided by the relationship [2]

$$x \to \varphi = x - 2\pi\,[\,x/2\pi\,] \qquad -\infty < x < \infty \qquad 0 \le \varphi \le 2\pi$$

where $x = [\,x\,] + \{x\}$. It turns out that every continuous representation of the group G , including any multi-valued one, can always be considered as a single-valued continuous representation of the universal enveloping group $\widetilde{G}$. The representations of the group G obtained in this manner do exhaust all the continuous representations of the group G .

1.2.2 The Lie Groups

The *Lie groups* occupy the most important place among continuous groups.

Marius Sophus Lie
Nordfjordeid (Norway) 17.12.1842 – Oslo 18.02.1899
Vorlesungen über continuierliche Gruppen (1893)

Lie groups occupy this special place for two reasons : first, they represent a sufficiently wide class of groups, including the most important continuous groups encountered in geometry, mathematical analysis and physics ; second, every Lie group satisfies a whole series of strict requirements which makes it possible to apply to its study the methods of the theory of differential equations. We can define a Lie group as follows.

Definition. Let G some continuous group. Consider any neighborhood V of the unit element of this group. We assume that by means of n real parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ we can define every element of the neighborhood V in such a way that:

1. there is a continuous 1 to 1 correspondence between all the different elements $g \in V$ and all the different $n-$ples of the real parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$;

2. suppose that g_1, g_2 and $g = g_1 g_2$ lie in the neighborhood V and that

$$g_1 = g_1(\alpha_1', \alpha_2', \ldots, \alpha_n') \qquad g_2 = g_2(\alpha_1'', \alpha_2'', \ldots, \alpha_n'')$$
$$g = g\,(\alpha_1, \alpha_2\,, \ldots, \alpha_n)$$

where

$$\alpha_a = \alpha_a(\alpha_1', \alpha_2', \ldots, \alpha_n'\,;\ \alpha_1'', \alpha_2''\,, \ldots, \alpha_n'') \qquad a = 1, 2, \ldots, n$$

[2] We recall that any real number x can always be uniquely decomposed into the sum of its integer $[\,x\,]$ and fractional $\{x\}$ parts, *i.e.* $x = [\,x\,] + \{x\}$, $\forall\, x \in \mathbb{R}$.

then the functions α_a $(a = 1, 2, \dots, n)$ are *analytic functions* of the parameters α'_b , α''_c $(b, c = 1, 2, \dots, n)$ of the factors.

Then the continuous group G is called a Lie group of dimensions n. We shall always choose the real parameters α_a $(a = 1, 2, \dots, n)$ in such a way that their zero values correspond to the unit element.

1.2.3 An Example : the Rotations Group

Any rotation in the three dimensional space can always be described by an oriented unit vector $\hat{\mathbf{n}}$ with origin in the center of rotation and directed along the axis of rotation and by the angle of rotation $\alpha \in [0, 2\pi[$. Therefore we can denote rotations by $g(\hat{\mathbf{n}}, \alpha)$. The angle is measured in the counterclockwise sense with respect to the positive direction of $\hat{\mathbf{n}}$. By elementary geometry it can be shown that any *active* rotation $g(\hat{\mathbf{n}}, \alpha)$ transforms the position vector $\mathbf{r}$ into the vector

$$\mathbf{r}' = \mathbf{r}\,\cos\alpha + \hat{\mathbf{n}}\,(\mathbf{r}\cdot\hat{\mathbf{n}})(1-\cos\alpha) + \hat{\mathbf{n}}\times\mathbf{r}\,\sin\alpha$$

Every rotation is defined by three parameters. If we introduce the vector $\boldsymbol{\alpha} \equiv \alpha\,\hat{\mathbf{n}}$ $(0 \le \alpha < 2\pi)$, then we can take the projections of the vector $\boldsymbol{\alpha}$ on the coordinate axes as the three numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ that label the rotation: namely,

$$\begin{aligned} &\alpha_1 = \alpha\sin\vartheta\cos\phi \qquad \alpha_2 = \alpha\sin\vartheta\sin\phi \qquad \alpha_3 = \alpha\cos\vartheta \\ &\qquad 0 \le \alpha < 2\pi \qquad\quad 0 \le \vartheta \le \pi \qquad\quad 0 \le \phi \le 2\pi \end{aligned} \tag{1.4}$$

These angular parameters are called the canonical coordinates of the rotation group and are evidently restricted to lie inside the 2-sphere

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 < (2\pi)^2$$

Thus, the rotation group is a three dimensional Lie group.

Any (abstract) rotation g can be realized by means of a linear operator $R(g)$ that transforms the position vector $\mathbf{r}$ into the new position vector $\mathbf{r}' = R(g)\,\mathbf{r}$. This corresponds to the *active* point of view, in which the reference frame is kept fixed while the position vectors are moved. Of course, one can equivalently consider the *passive* point of view, in which the position vectors are kept fixed while the reference frame is changed. As a result, the linear operators corresponding to the two points of view are inverse one of each other. Historically, the Euler angles φ, θ and ψ were firstly employed as parameters for describing rotations. According to Euler's point of view, any displacement of a solid body around a fixed point can be represented as a product of three orthogonal matrices: the rotation matrix $R_3(\varphi)$ about the OZ axis, the rotation matrix $R_1(\theta)$ about the OX' axis, which is called the *nodal line*, and the rotation matrix $R_3(\psi)$ about the OZ' axis, *i.e.*

$$R(g) = R(\varphi, \theta, \psi) = R_3(\psi)\, R_1(\theta)\, R_3(\varphi)$$

We have

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_3(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The angle φ is named the precession angle, θ is the nutation angle, while ψ is the proper rotation angle. From the parametric form in terms of the Eulerian angles it immediately follows that every rotation matrix is uniquely identified by a tern

$$R(g) = R(\varphi, \theta, \psi) \qquad 0 \le \varphi < 2\pi \;;\; 0 \le \theta \le \pi \;;\; 0 \le \psi < 2\pi \tag{1.5}$$

$$R(g) = R(\varphi, \theta, \psi) =$$
$$\begin{pmatrix} \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi & -\sin\varphi\cos\psi - \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ \sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi & -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi & -\cos\psi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \end{pmatrix}$$

and that

$$R(g^{-1}) = R^{-1}(g) = R_3(-\varphi)\, R_1(-\theta)\, R_3(-\psi) = R^{\top}(g)$$

with $\det[\, R(\varphi, \theta, \psi)\,] = 1$. It follows thereby that an isomorphism exists between the abstract rotation group and the group of the orthogonal 3×3 matrices with unit determinant: this matrix group is called the 3-dimensional special orthogonal group and is denoted by $SO(3)$. The matrix group $SO(3)$ is thereby said to be the defining representation of the rotation group around a fixed point in the three dimensional Euclidean space.

The discrete transformation that carries every position vector $\mathbf{r}$ into the vector $-\mathbf{r}$ is called inversion or parity transform. Parity $\mathbb{P}$ commutes with all rotations $\mathbb{P}g = g\mathbb{P}$, $\forall g \in SO(3)$; moreover $\mathbb{P}^2 = \mathbb{I}$, $\det \mathbb{P} = -1$. If we add to the elements of the rotation group all possible products $\mathbb{P}g\,,\ g \in SO(3)$ we still obtain a larger group, as it can be readily checked.

This matrix group is called the 3-dimensional full orthogonal group and is denoted by $O(3)$. The group $O(3)$ splits into two connected components: namely, the proper rotation group $O(3)^+ = SO(3)$, which is the subgroup connected with the unit element, and the improper component $O(3)^-$ which is connected with the parity transform. Evidently the matrices belonging to $O(3)^\pm$ have determinant equal to ± 1 respectively.

The full orthogonal group $O(3)$ is the group of transformations that leave invariant the line element

$$\mathrm{d}\mathbf{r}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}x^i \mathrm{d}x^i \qquad i = 1,2,3$$

in the 3-dimensional space. In a quite analogous way one introduces, for odd N, the $N-$dimensional full orthogonal groups $O(N)$ and the proper orthogonal Lie groups $SO(N)$ of dimensions $n = \frac{1}{2}\,N(N-1)$.

1.2.4 The Infinitesimal Operators

In the following we shall consider only those representations T of a Lie group G, the linear operators of which are analytic functions of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ in such a way that

$$T(g) = T(\alpha_1, \alpha_2, \ldots, \alpha_n) \equiv T(\alpha) \,:V \,\to \mathfrak{A}(V) \qquad (\,V \subset G\,)$$

are operator valued analytic functions of their arguments. The derivative of the operator $T(g)$ with respect to the parameter α_a taken for $g = \mathrm{e}$, *i.e.* at the values $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, is called the infinitesimal operator or generator I_a of the representation $T(g)$ corresponding to the parameter α_a. Thus, any representation $T(g)$ has n generators, *i.e.* , the number of the infinitesimal operators is equal to the dimension of the Lie group.

As an example, consider the real orthogonal matrices referred to the canonical coordinates (1.4) of the rotation group. Since those matrices $g \in SO(3)$ are linear operators acting on the vectors $\mathbf{r} \in L = \mathbb{R}^3$, the latter realize an irreducible representation of the rotation group with the same number of dimensions as the group itself, which is called the *vector* or the *adjoint* representation $\boldsymbol{\tau}_A(g)$ of the rotation group. According to the passive point of view, a rotation of coordinates $\boldsymbol{\alpha} = (\alpha_1, 0, 0)$ corresponds to the rotation of the reference frame around the positive direction of the OX axis through an angle α_1. Then the operator $\boldsymbol{\tau}_A(\alpha_1, 0, 0)$ is represented by the matrix

$$\boldsymbol{\tau}_A(\alpha_1, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}$$

Hence

$$\frac{\partial}{\partial\alpha_1}\,\boldsymbol{\tau}_A(\alpha_1,0,0) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \\ 0 & -\cos\alpha_1 & -\sin\alpha_1 \end{array}\right)$$

and consequently

$$I_1 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right)$$

In a quite analogous way we obtain

$$I_2 = \left(\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right) \qquad I_3 = \left(\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The following commutation relations among the infinitesimal operators of the rotation group $SO(3)$ hold true: namely,

$$I_a I_b - I_b I_a = -\,\varepsilon_{abc}\, I_c \qquad (a,b,c = 1,2,3) \tag{1.6}$$

where ε_{abc} is the completely anti-symmetric Levi-Civita symbol normalized to $\varepsilon_{123} = 1$. The crucial role played by the generators in the theory of the Lie groups is unraveled by the following three theorems.

- **Theorem I** . Let T_1 and T_2 any two representations of a Lie group G acting on the same vector space L and suppose that they have the same set of generators. Then T_1 and T_2 are the same representation.

- **Theorem II** . The infinitesimal operators I_a $(a = 1,2,\ldots,n)$ that correspond to any representation $T(g)$ of a Lie group G do satisfy the following commutation relations

  $$I_a I_b - I_b I_a \equiv [\, I_a\, I_b\,] = C_{abc}\, I_c \qquad a,b,c = 1,2,\ldots,n \tag{1.7}$$

  where the constant coefficients $C_{abc} = -\,C_{bac}$ are real numbers which do not depend upon the choice of the representation $T(g)$. The constant coefficients C_{abc} of a Lie algebra fulfill the Jacobi's identity

  $$C_{abe}C_{ecd} + C_{bce}C_{ead} + C_{cae}C_{ebd} = 0 \tag{1.8}$$

  because of the identity

  $$[\,[\, I_a\, I_b\,]\, I_c\,] + [\,[\, I_b\, I_c\,]\, I_a\,] + [\,[\, I_c\, I_a\,]\, I_b\,] = 0$$

  which can be readily checked by direct inspection.

- **Theorem III** . Suppose that a set of linear operators $A_1, A_2, \ldots, A_n$ acting on a certain vector space L is given and which fulfill the same commutation relations

  $$[\, A_a\, A_b \,] = C_{abc}\, A_c \qquad a, b, c = 1, 2, \ldots, n$$

  as the infinitesimal operators of the group G. Then the operators $A_a (a = 1, 2, \ldots, n)$ are the generators of a certain representation $T(g)$ of the group G in the vector space L.

The details of the proofs of the above fundamental theorems in the theory of Lie groups can be found in the excellent monographs [17]. Theorems I, II and III are very important because they reduce the problem of finding all the representations of a Lie group G to that of classifying all the possible sets of linear operators which satisfy the commutation relations (1.7).

1.2.5 Lie Algebra

The infinitesimal operators I_a $(a = 1, 2, \ldots, n)$ do generate a linear space and the commutators define a product law within this linear space. Then we have an algebra which is called the Lie algebra $\mathcal{G}$ of the Lie group G with $\dim \mathcal{G} = \dim G = n$. The constant coefficients $C_{abc} = -\, C_{bac}$ are named the structure constants of the Lie algebra $\mathcal{G}$.

Suppose that the representation $T(g)$ of the Lie group G acts on the linear space L and let A any non-singular linear operator upon L. Then the linear operators $A\, I_a\, A^{-1} \equiv J_a$ $(a = 1, 2, \ldots, n)$ do realize an equivalent representation for $\mathcal{G}$, *i.e.* the infinitesimal operators J_a $(a = 1, 2, \ldots, n)$ correspond to a new basis in $\mathcal{G}$ because

$$\begin{aligned}
[\, J_a\, J_b \,] &= [\, A\, I_a\, A^{-1}\, A\, I_b\, A^{-1} \,] = A\, [\, I_a\, I_b \,]\, A^{-1} \\
&= A\, C_{abc}\, I_c\, A^{-1} = C_{abc}\, A\, I_c\, A^{-1} \\
&= C_{abc}\, J_c \qquad a, b, c = 1, 2, \ldots, n
\end{aligned}$$

As a corollary, it turns out that, for any representation, the collections of infinitesimal operators

$$\left\{ I_a(g) \equiv T(g)\, I_a\, T^{-1}(g) \,|\, a = 1, 2, \ldots, n \right\} \qquad \forall\, g \in G \tag{1.9}$$

do indeed realize different and equivalent choices of basis in the Lie algebra. We will say that the operators $T(g)$ of a given representation of the Lie group G generate all the inner automorphisms in the given representation of the Lie algebra $\mathcal{G}$.

It is possible to regard the structure constants as the matrix elements of the $n-$dimensional representation of the generators: namely,

$$\| I_a \|_{bc} = C_{acb} \qquad (a, b, c = 1, 2, \ldots, n) \tag{1.10}$$

in such a way that we can rewrite the Jacobi identity (1.8) as the matrix identity

$$- || I_c ||_{de} || I_a ||_{eb} + || I_a ||_{de} || I_c ||_{eb} - || I_c ||_{ea} || I_e ||_{db} = 0 \tag{1.11}$$

or after relabeling of the indexes

$$|| [I_a I_b] ||_{de} = C_{abc} || I_c ||_{de}$$

Thus, for each Lie algebra $\mathcal{G}$ and Lie group G of dimensions n, there is a representation, called the adjoint representation, which has the very same dimensions n as the Lie group itself. Evidently, an Abelian Lie group has vanishing structure constants, so that its adjoint representation is trivial, *i.e.* it consists of solely the unit element. As shown above the rotation group has three generators I_a $(a = 1, 2, 3)$ and structure constants C_{abc} equal to $-\varepsilon_{abc}$. Consequently, the adjoint representation of the rotation group is a three dimensional one with generators I_a given by

$$|| I_a ||_{bc} = \varepsilon_{abc} \qquad (a, b, c = 1, 2, 3)$$

In any neighborhood of the identity operator $T(\mathrm{e}) = T(0, 0, \ldots, 0) \equiv \mathbf{1}$ we can always find a set of parameters such that

$$T(\alpha) = \exp\{I_a \alpha_a\} = \mathbf{1} + \sum_{a=1}^{n} I_a \alpha_a + O(\alpha^2) \tag{1.12}$$

1.2.6 The Exponential Representation

In order to find the representations of the group G it is sufficient to classify the representations of its Lie algebra $\mathcal{G}$. More precisely, one can prove the following theorem – see [17] for the proof.

Theorem. Let I_a $(a = 1, 2, \ldots, n)$ a given basis of the Lie algebra $\mathcal{G}$ of a Lie group G. Then, for any neighborhood $U \subset G$ of the unit element $\mathrm{e} \in G$ there exists a set of canonical coordinates α_a $(a = 1, 2, \ldots, n)$ in U and a positive number $\delta > 0$ such that

$$T(g) \equiv T(\alpha_1, \alpha_2, \ldots, \alpha_n) = \exp\{I_a \alpha_a\} \qquad g \in U \subset G, \ |\alpha_a| < \delta$$

where the exponential is understood through the Baker–Campbell–Hausdorff formula

$$T(\alpha)\,T(\beta) = T(\gamma) \qquad T(\alpha) = \exp\{I_a\,\alpha_a\} \quad T(\beta) = \exp\{I_b\,\beta_b\}$$

$$\begin{aligned} T(\gamma) = \exp\{I_\ell\,\gamma_\ell\} \;&\equiv\; \exp\big\{I_a\,\alpha_a + I_b\,\beta_b + \tfrac{1}{2}\,\alpha_a\beta_b\,[\,I_a\,I_b\,] \\ &+\; \tfrac{1}{12}\,(\alpha_a\alpha_b\beta_c + \beta_a\beta_b\alpha_c)\,[\,I_a\,[\,I_b\,I_c\,]\,] + \cdots\big\} \end{aligned}$$

in which the dots stand for higher order, iterated commutators among the generators. Then we obtain the series expansion

$$\gamma_\ell = \alpha_\ell + \beta_\ell + \tfrac{1}{2}\,\alpha_a\beta_b\,C_{ab\ell} + \tfrac{1}{12}\,(\alpha_a\alpha_b\beta_c + \beta_a\beta_b\alpha_c)\,C_{ad\ell}\,C_{bcd} + \cdots$$

which manifestly shows that all the structure constants must be real for any Lie group. To elucidate these notions let me discuss two examples.

1. Consider the translations group along the real line

   $$x \;\to\; x' = x + a \qquad\qquad (\,\forall\, x \in \mathbb{R}\,, \quad a \in \mathbb{R}\,)$$

   The translations group on the real line is a one dimensional Abelian Lie group. Let us find the representation of this group acting on the infinite dimensional functional space of all the real analytic functions $f : \mathbb{R} \to \mathbb{R}$ with $f \in C_\infty(\mathbb{R})$. To do this, we have to recall the Taylor-McLaurin formula

   $$f(x+a) = \sum_{k=0}^{\infty} \frac{1}{k\,!}\,a^k\,f^{\,(k)}(x)$$

   Now, if we define

   $$T\,f(x) \equiv \frac{\mathrm{d}f}{\mathrm{d}x} \qquad\qquad T^k\,f(x) \equiv \frac{\mathrm{d}^{\,k}f}{\mathrm{d}x^{\,k}} = f^{\,(k)}(x)$$

   then we can write

   $$\begin{aligned} f(x+a) \;&=\; \exp\{\,a\cdot T\,\}\,f(x) \\ &=\; \left(\sum_{k=0}^{\infty} \frac{1}{k\,!}\,a^k\,T^k\right) f(x) = \sum_{k=0}^{\infty} \frac{1}{k\,!}\,a^k\,f^{\,(k)}(x) \end{aligned}$$

   in such a manner that we can actually identify the generator of the infinite dimensional representation of the translations group with the derivative operator acting upon the functional space of all the real analytic functions.

2. As a second example, consider the rotations group on the plane around a fixed point, which is an Abelian and compact Lie group $SO(2,\mathbb{R})$ *i.e.* the two dimensional proper orthogonal group with unit determinant. A generic group element $g \in SO(2)$ is provided by the 2×2 orthogonal matrix

$$g(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \qquad (\, 0 \leq \varphi \leq 2\pi \,)$$

which corresponds to a passive planar rotation around the origin. If we introduce the generator or transfer matrix

$$\mathbb{T} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with $\mathbb{T}^2 = -\,\mathbf{I}$, then we readily find

$$\begin{aligned} g(\varphi) \;&\equiv\; \exp\{\,\mathbb{T}\,\varphi\,\} = \sum_{k=0}^{\infty} \frac{1}{k\,!}\,\varphi^{k}\,\mathbb{T}^{k} \\ &=\; \sum_{k=0}^{\infty} (-1)^{k} \left[\frac{\varphi^{2k}}{(2k)\,!}\,\mathbf{I} \;+\; \frac{\varphi^{2k+1}}{(2k+1)\,!}\,\mathbb{T} \right] \\ &=\; \mathbf{I}\,\cos\varphi + \mathbb{T}\,\sin\varphi \qquad \forall\,\varphi \in [\,0\,,\,2\pi\,] \end{aligned}$$

The exponential representation of the Lie group elements implies some remarkable features. The infinitesimal operators of a unitary representation are anti-Hermitean

$$T(g^{-1}) = T^{\,\dagger}(g) \quad \Leftrightarrow \quad I_a = -\,I_a^{\dagger} \quad (a = 1, 2, \ldots, n)$$

All the structure constants of an Abelian Lie group are equal to zero. For a compact Lie group G with $\dim G = n$ it is possible to show that all the sets of canonical coordinates span a bounded subset of $\mathbb{R}^n$. Moreover any element of any representation of a compact group can always be expressed in the exponential form. In a non-compact Lie group G with $\dim G = n$ the canonical coordinates run over an unbounded subset of $\mathbb{R}^n$ and, in general, not all the group elements can be always expressed in the exponential form.

Any two Lie groups G_1 and G_2 with the same structure constants are locally homeomorphic, in the sense that it is always possible to find two neighborhoods of the unit elements $U_1 \subset G_1$ and $U_2 \subset G_2$ such that there is an analytic isomorphism $f\ : U_1 \leftrightarrow U_2$ between the elements of the two groups $\forall\, g_1 \in U_1\,,\ \forall\, g_2 \in U_2\,$. Of course, this does not imply that there is a 1 to

1 analytic map over the whole parameter space, *i.e.* the two groups are not necessarily globally homeomorphic. As an example, consider the group $\mathbb{R}$ of all the real numbers and the unitary group

$$U(1) \equiv \{z \in \mathbb{C} \,|\, z\,\bar{z} = 1\}$$

of the complex unimodular numbers. These two groups are Abelian groups. In a neighborhood of the unit element we can readily set up an homeomorphic map, *e.g.* the exponential map

$$z = \exp\{i\alpha\} \qquad \bar{z}z = 1 \qquad -\pi < \alpha \le \pi$$

so that the angle α is the canonical coordinate. However, there is no such mapping in the global sense, because the unit circle in the complex plane is equivalent to a real line of which all the elements modulo 2π are considered as identical. In other words, we can associate any real number

$$x = 2k\pi + \alpha \qquad (\,k \in \mathbb{Z}\,,\ -\pi < \alpha \le \pi\,)$$

to some point of the unit circle in the complex plane with a given canonical coordinate α and a given integer winding number $k = [\,x/2\pi\,]$. If this is the case, the complex unit circle is not simply connected, because all the closed paths on a circle have a non-vanishing integer number of winding and can not be continuously deformed to a point, at variance with regard to the simply connected real line.

1.2.7 The Special Unitary Groups

Consider the group of 2×2 complex unitary matrices with unit determinant denoted as $SU(2)$, the Special Unitary 2-dimensional matrices

$$g = \begin{pmatrix} \bar{v} & -\bar{u} \\ u & v \end{pmatrix} \qquad \bar{u}u + \bar{v}v = 1$$

which evidently realize a Lie group of dimensions $n = 3$. Notice that we can always set

$$u = -x_2 + ix_1 \qquad v = x_4 + ix_3 \qquad \sum_{i=1}^{4} x_i^2 = 1$$

whence it manifestly follows that $SU(2)$ is topologically homeomorphic to the three dimensional hyper-sphere S_3 of unit radius plunged in $\mathbb{R}^4$. A convenient parametric form of S_3 is provided by the spherical angular coordinates in $\mathbb{R}^4$

$$\begin{cases} x_1 = \cos\theta_1 \\ x_2 = \sin\theta_1 \cos\theta_2 \\ x_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ x_4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \end{cases} \tag{1.13}$$

with $0 \le \theta_1 \le \pi$, $0 \le \theta_2 \le \pi$, $0 \le \theta_3 \le 2\pi$, in such a matter that one can establish the correspondence

$$u = -\sin\theta_1\cos\theta_2 + i\cos\theta_1 \qquad\qquad v = \sin\theta_1\sin\theta_2(\sin\theta_3 + i\cos\theta_3)$$

and thereby

$$g = \begin{pmatrix} \sin\theta_1\sin\theta_2(\sin\theta_3 - i\cos\theta_3) & \sin\theta_1\cos\theta_2 + i\cos\theta_1 \\ -\sin\theta_1\cos\theta_2 + i\cos\theta_1 & \sin\theta_1\sin\theta_2(\sin\theta_3 + i\cos\theta_3) \end{pmatrix}$$

Owing to the exponential representation, as the $SU(2)$ matrices are unitary, the three generators are anti-Hermitean 2×2 matrices, which can be written in terms of the Pauli matrices

$$\tau_a \equiv \tfrac{1}{2}\, i\sigma_a \qquad (a = 1, 2, 3) \tag{1.14}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.15}$$

and therefore

$$[\,\tau_a\,\tau_b\,] = -\,\varepsilon_{abc}\,\tau_c \qquad (a, b, c = 1, 2, 3) \tag{1.16}$$

It follows that $SU(2)$ has the very same Lie algebra (1.6) of $SO(3)$. Then, according to **Theorem III** of section 1.2.4, the special unitary 2×2 matrices do realize a complex 2-dimensional representation of the (abstract) rotation group. Using the canonical coordinates (1.4), in a neighborhood of the unit element we can write the $SU(2)$ elements in the exponential representation. From the very well known identities

$$\sigma_a\,\sigma_b + \sigma_b\,\sigma_a = 2\,\delta_{ab} \qquad (\sigma_a\,\alpha_a)^2 = |\,\boldsymbol{\alpha}\,|^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$

where the 2×2 identity matrix is understood, we obtain

$$\begin{aligned} g(\alpha) &= \exp\{\tau_a\,\alpha_a\} \equiv \sum_{k=0}^{\infty} \frac{1}{k!}\,(\tau_a\,\alpha_a)^k \\ &= \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k)!}\left(\frac{i}{2}\,|\,\boldsymbol{\alpha}\,|\right)^{2k} + \frac{1}{(2k+1)!}\left(\frac{i}{2}\,|\,\boldsymbol{\alpha}\,|\right)^{2k+1} \frac{\sigma_a\,\alpha_a}{|\,\boldsymbol{\alpha}\,|} \right\} \\ &= \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{(2k)!}\left(\frac{1}{2}\,|\,\boldsymbol{\alpha}\,|\right)^{2k} + \frac{i(-1)^k}{(2k+1)!}\left(\frac{1}{2}\,|\,\boldsymbol{\alpha}\,|\right)^{2k+1} \frac{\sigma_a\,\alpha_a}{|\,\boldsymbol{\alpha}\,|} \right\} \\ &= \cos\left(\tfrac{1}{2}\,|\,\boldsymbol{\alpha}\,|\right) + i\sigma_a\,\frac{\alpha_a}{|\,\boldsymbol{\alpha}\,|}\,\sin\left(\tfrac{1}{2}\,|\,\boldsymbol{\alpha}\,|\right) = x_4 + i\sigma_a\,x_a \end{aligned} \tag{1.17}$$

in which $\sum_{\imath=1}^{4} x_\imath^2 = 1$ while

$$\left\{ \begin{array}{c} x_1 = \cos\theta_1 = (\alpha_1/\alpha)\sin\frac{1}{2}\alpha \\ x_2 = \sin\theta_1\cos\theta_2 = (\alpha_2/\alpha)\sin\frac{1}{2}\alpha \\ x_3 = \sin\theta_1\sin\theta_2\cos\theta_3 = (\alpha_3/\alpha)\sin\frac{1}{2}\alpha \\ x_4 = \sin\theta_1\sin\theta_2\sin\theta_3 = \cos\frac{1}{2}\alpha \end{array} \right. \qquad \alpha = |\,\boldsymbol{\alpha}\,|$$

Notice that, by direct inspection,

$$g^\dagger(\alpha) = g(-\alpha) = g^{-1}(\alpha)$$

so that $g(\alpha)$ are unitary matrices. Furthermore we can write

$$g(\alpha) = \begin{pmatrix} x_4 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_4 - ix_3 \end{pmatrix}$$

with $\det g(\alpha) = 1$. From the explicit formula (1.17) it follows that the whole set of special unitary 2×2 matrices is spanned iff the canonical coordinates $\boldsymbol{\alpha}$ are restricted to lie inside a sphere of radius 2π, *viz.,*

$$|\,\boldsymbol{\alpha}\,|^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 < (2\pi)^2$$

The $SU(2)$ group of matrices constitutes the lowest dimensional nontrivial faithful irreducible unitary representation of the rotation group, which is thereby named its fundamental representation and denoted by $\boldsymbol{\tau}_F(g)$. Since it acts upon complex 2-components column vectors, *i.e.* the Pauli spinors of the non-relativistic quantum mechanics that describe particles of spin $\frac{1}{2}$, it is also named as the *spinorial representation* of the rotation group and further denoted by $\boldsymbol{\tau}_{\frac{1}{2}}(g) \equiv \boldsymbol{\tau}_F(g)$.

A comparison between the fundamental $\boldsymbol{\tau}_F(g) = SU(2)$ and the adjoint $\boldsymbol{\tau}_A(g) = SO(3)$ representations of the rotation group is very instructive. Since these two representations of the rotation group share the same Lie algebra they are locally homeomorphic. It is possible to obtain all the finite elements of $SO(3)$ in the exponential representation. As a matter of fact, for the infinitesimal operators (1.6) of the adjoint representation we have (the identity matrix is understood)

$$(I_a\,\alpha_a)^2 = -\,|\,\boldsymbol{\alpha}\,|^2 + \Xi(\alpha) \qquad ||\,\Xi(\alpha)\,||_{ac} \equiv \alpha_a\alpha_c$$

$$(I_a\,\alpha_a)^3 \;=\; -\,|\,\boldsymbol{\alpha}\,|^2\, I_a\,\alpha_a$$

and consequently

$$[\,\Xi(\alpha)\,]^2 = |\,\boldsymbol{\alpha}\,|^2\,\Xi(\alpha) \qquad\qquad \alpha_a\, I_a\,\Xi(\alpha) = 0 = \Xi(\alpha)\, I_a\,\alpha_a$$

$$(I_a\,\alpha_a)^{2k} = (-\,|\,\boldsymbol{\alpha}\,|^2)^k\Big(\mathbf{1} - |\,\boldsymbol{\alpha}\,|^{-2}\,\Xi(\alpha)\Big)$$

$$(I_a\,\alpha_a)^{2k+1} = (-\,|\,\boldsymbol{\alpha}\,|^2)^k\,I_a\,\alpha_a$$

in such a way that

$$\begin{aligned}
\boldsymbol{\tau}_A(\alpha) &\equiv \exp\{I_a\,\alpha_a\} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\,|\,\boldsymbol{\alpha}\,|^{2k} \\
&+ \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\,|\,\boldsymbol{\alpha}\,|^{2k}\,I_a\,\alpha_a \\
&+ \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+2)!}\,|\,\boldsymbol{\alpha}\,|^{2k}\,\Xi(\alpha) \\
&= \cos|\,\boldsymbol{\alpha}\,| + \frac{\sin|\,\boldsymbol{\alpha}\,|}{|\,\boldsymbol{\alpha}\,|}\,I_a\,\alpha_a + \frac{\Xi(\alpha)}{|\,\boldsymbol{\alpha}\,|^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\,|\,\boldsymbol{\alpha}\,|^{2k} \\
&= \cos|\,\boldsymbol{\alpha}\,| + \frac{\sin|\,\boldsymbol{\alpha}\,|}{|\,\boldsymbol{\alpha}\,|}\,I_a\,\alpha_a + \frac{\cos|\,\boldsymbol{\alpha}\,| - 1}{|\,\boldsymbol{\alpha}\,|^2}\,\Xi(\alpha) \qquad (1.18)
\end{aligned}$$

It turns out that the manifold of the canonical coordinates can be divided into two parts: an inside shell $0 \le |\,\boldsymbol{\alpha}\,| < \pi$ and an outer region $\pi \le |\,\boldsymbol{\alpha}\,| < 2\pi$. To each point in the inside shell we can assign a point in the outer region by means of the correspondence

$$\boldsymbol{\alpha}' = (-\,\boldsymbol{\alpha})\,\frac{2\pi - |\,\boldsymbol{\alpha}\,|}{|\,\boldsymbol{\alpha}\,|} \qquad 0 \le |\,\boldsymbol{\alpha}\,| < \pi\,, \quad \pi \le |\,\boldsymbol{\alpha}'\,| < 2\pi \qquad (1.19)$$

The $SU(2)$ elements corresponding to $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are then related by

$$\boldsymbol{\tau}_F(\boldsymbol{\alpha}') = -\,\boldsymbol{\tau}_F(\boldsymbol{\alpha})$$

All points on the boundary of the parameter space, a 2-sphere of radius 2π, correspond to the very same element *i.e.* $\boldsymbol{\tau}_F(\,|\,\boldsymbol{\alpha}\,| = 2\pi\,) = -\,\mathbf{1}$. Conversely, it turns out that for any pair of canonical coordinates (α, α') connected by the relation (1.19) we have

$$\boldsymbol{\tau}_A(\boldsymbol{\alpha}) = \boldsymbol{\tau}_A(\boldsymbol{\alpha}')$$

As a consequence the adjoint representation (1.18) is an irreducible and unitary representation of the rotation group that is not a faithful one on the whole domain of the canonical coordinates of the rotation group

$$D = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)\,|\,\alpha_1^2 + \alpha_2^2 + \alpha_3^2 < (2\pi)^2\}$$

the adjoint $SO(3)$ representation being an homomorphism, because the map

$$\boldsymbol{\tau}_A(\alpha) \;:\; D \;\rightarrow\; SO(3)$$

is not 1:1, contrary to the fundamental $SU(2)$ representation which is faithful. This implies in turn that, for $0 \leq |\,\boldsymbol{\alpha}\,| < 2\pi$, there are paths which are closed in the $SO(3)$ adjoint representation and which cannot be deformed in a point. Conversely, any closed path in the $SU(2)$ fundamental representation can always be deformed into a point. This means that the spinor representation of the rotation group is simply connected, though the vector representation is not. Owing to this $SU(2)$ is said to be the universal covering of $SO(3)$.

It is worthwhile to remark that, on the one hand, had we used the Euler angles (1.5) to label the manifold of the proper rotation group $SO(3)$, then the irreducible, orthogonal adjoint representation $\boldsymbol{\tau}_A(\varphi,\theta,\psi)$ turns out to be manifestly faithful. On the other hand, the generic element of the irreducible and unitary fundamental spinor representation $\boldsymbol{\tau}_F(\varphi,\theta,\psi)$ can be expressed in terms of the Euler angles as [17]

$$\boldsymbol{\tau}_{\frac{1}{2}}(\varphi,\theta,\psi) = \\ \left(\begin{array}{cc} \exp\{i(\psi+\varphi)/2\}\cos\theta/2 & i\exp\{i(\psi-\varphi)/2\}\sin\theta/2 \\ i\exp\{i(\varphi-\psi)/2\}\sin\theta/2 & \exp\{-i(\psi+\varphi)/2\}\cos\theta/2 \end{array}\right)$$

in such a manner that, for instance,

$$\boldsymbol{\tau}_{\frac{1}{2}}(0,\theta,\psi) \;=\; -\,\boldsymbol{\tau}_{\frac{1}{2}}(2\pi,\theta,\psi)$$

Hence, to any rotation $g(\varphi,\theta,\psi)$ there correspond two opposite matrices $\pm\,\boldsymbol{\tau}_{\frac{1}{2}}(\varphi,\theta,\psi)$ so that the adjoint vector representation $\boldsymbol{\tau}_A(\varphi,\theta,\psi)$ appears to be a real, faithful, orthogonal, irreducible single-valued representation of the rotation group, whereas the fundamental spinor representation $\boldsymbol{\tau}_F(\varphi,\theta,\psi)$ turns out to be a unitary irreducible double-valued representation of the rotation group. Hence, the rotation group $SO(3)$ in the three dimensional space is not simply connected, its universal enveloping group being $SU(2)$.

As it is well known from the theory of the angular momentum in quantum mechanics, it turns out that all the irreducible, unitary, finite dimensional representations of the proper rotation group are labeled by their weight, *i.e.* a positive semi-definite integer or half-integer number: namely,

$$\{\boldsymbol{\tau}_j \mid\; j = n/2 \;(n = 0,1,2,\ldots)\} \qquad \dim\boldsymbol{\tau}_j = 2j+1$$

From the main decomposition Theorem 2. of Section 1.1.2, it appears quite clear that the matrix elements of the unitary matrices $\boldsymbol{\tau}_j(\boldsymbol{\alpha})$ can be written in the form

$$\|\; \boldsymbol{\tau}_j(\boldsymbol{\alpha})\;\|_{m'm} \equiv\; \boldsymbol{\tau}^{(j)}_{m'm}(\boldsymbol{\alpha}) = \langle\, jm' \mid \exp\{-(i/\hbar)\,\widehat{\mathbf{n}}\cdot\mathbf{J}\,\alpha\} \mid jm\,\rangle$$

where $m = -j, -j+1, \ldots, j-1, j$ while

$$\mathbf{J}^2 \mid jm\rangle \;=\; \hbar^2\, j(j+1) \mid jm\rangle \qquad J_z \mid jm\rangle \;=\; \hbar\, m \mid jm\rangle \tag{1.20}$$

$$[J_x\,,\,J_y] = i\hbar\, J_z \qquad [\,J_y\,,\,J_z\,] = i\hbar\, J_x \qquad [\,J_z\,,\,J_x\,] = i\hbar\, J_y \tag{1.21}$$

In spite of its apparent simplicity, the actual detailed dependence of the matrix elements $\boldsymbol{\tau}^{(j)}_{m'm}(\boldsymbol{\alpha})$ upon the quantum numbers j, m and $\widehat{\mathbf{n}}, \alpha$ (the canonical coordinates) is rather complicated, because of the Baker-Campbell-Haussdorf formula. A much easier explicit expression can be obtained in terms of the Euler angles: namely,

$$\begin{aligned}
\boldsymbol{\tau}^{(j)}_{km}(\varphi,\theta,\psi) &= P_{km,j}(\theta)\, \exp\{ik\psi + im\varphi\} && (1.22)\\
P_{km,j}(\theta) &= \frac{(-1)^{j-k}\, i^{\,m-k}}{2^j(j-k)!}\sqrt{\frac{(j-k)!(j+m)!}{(j+k)!(j-m)!}} \\
&\times\; (1-\cos\theta)^{-(m-k)/2}(1+\cos\theta)^{-(m+k)/2} \\
&\times\; \frac{\mathrm{d}^{j-m}}{\mathrm{d}\mu^{\,j-m}}\left[(1-\mu)^{j-k}(1+\mu)^{j+k}\right]_{\mu=\cos\theta} && (1.23)
\end{aligned}$$

The functions $\boldsymbol{\tau}^{(j)}_{km}(\varphi,\theta,\psi)$ are called the generalized spherical harmonics of the $j-$th order – see G.Ya. Lyubarskii, *The Application of Group Theory in Physics*, Pergamon Press (Oxford) 1960, p. 204.

From the composition law of the angular momenta, it turns out that the product $\boldsymbol{\tau}_j \times \boldsymbol{\tau}_k$ of two irreducible unitary representations of the three dimensional rotation group of weights j and k contains just once each of the irreducible unitary representations

$$\boldsymbol{\tau}_i \qquad (i = |j-k|, |j-k|+1, \ldots, j+k-1, j+k)$$

Thus the following formula is valid

$$\boldsymbol{\tau}_j \times \boldsymbol{\tau}_k = \bigoplus_{i=|j-k|}^{j+k} \boldsymbol{\tau}_i \qquad \left(\forall\, j,k = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\right)$$

Other important examples of Lie groups are:

1. the full orthogonal groups $O(N)$ of the real orthogonal square matrices of rank N and their special or proper subgroups $SO(N)$ with unit determinant, the dimensions of which are $n = \frac{1}{2}\,N(N-1)\,$;

2. the groups $U(N)$ of the complex unitary square matrices of rank N, the dimensions of which are $n = N^2$; in fact the generic $U(N)$ matrix depends upon $2N^2$ real parameters, but the request of unitarity entails N^2 real conditions so that the dimension of the Lie group $U(N)$ is $n = N^2$.

3. the groups $SU(N)$ of the $N \times N$ special unitary complex matrices, *i.e.* with unit determinant, of dimensions $n = N^2 - 1$.

   Notice that $U(N)$ is homeomorphic to the product $SU(N) \times U(1)$.

4. The special linear groups $SL(N, \mathbb{R})$ of the $N \times N$ real matrices of unit determinant, the dimensions of which are $n = N^2 - 1$;

5. the special linear groups $SL(N, \mathbb{C})$ of the $N \times N$ complex matrices of unit determinant, the dimensions of which are $n = 2N^2 - 2$;

   The six dimensional special linear group $SL(2, \mathbb{C})$ will be of particular relevance, as it turns out to be the universal covering spinorial group of the Lorentz group, *i.e.* it will play the same role in respect to the Lorentz group as the unitary group $SU(2)$ did regard to the three dimensional rotation group.

1.3 The Non-Homogeneous Lorentz Group

1.3.1 The Lorentz Group

Consider a space-time point specified in two inertial coordinate systems S and S', where S' moves with a constant relative velocity $\mathbf{v}$ with respect to S. In S the space-time point is labeled by (x, y, z, t) and in S' by (x', y', z', t'). The transformation that relates the two inertial coordinate frames is called a Lorentz transformation: according to the postulates of the Special Theory of Relativity, it has the characteristic property ($c = 299\,792\,458$ m s^{-1} is the exact value of the velocity of light in vacuum)

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

where we are assuming that the origins of both inertial frame coordinate systems do coincide for $t = t' = 0$ – for the moment we do not consider translations under which the relative distances remain invariant; if these are included one finds non-homogeneous Lorentz transformations, also called Poincaré transformations. We shall use the following standard notations and conventions [15] – sum over repeated indexes is understood

$$x^{\mu} \equiv (x^0 = ct, x, y, z) = (x^0, \mathbf{x}) = (x^0, x^k) \qquad \mu = 0, 1, 2, 3 \quad k = 1, 2, 3$$

so that

$$x^{\mu}\, x_{\mu} \equiv x^2 = g_{\mu\nu}\, x^{\mu} x^{\nu} = g_{\mu\nu}\, x'^{\mu} x'^{\nu} \equiv x'^2$$

with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Owing to space-time homogeneity and isotropy the Lorentz transformations are linear

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

which implies

$$g_{\rho\sigma} = g_{\mu\nu}\, \Lambda^{\mu}{}_{\rho}\, \Lambda^{\nu}{}_{\sigma} \tag{1.24}$$

or even in matrix notations

$$x^{\mu}\, x_{\mu} = x^{\top} \cdot g\, x = x'^{\mu}\, x'_{\mu} \qquad x' = \Lambda\, x \qquad g = \Lambda^{\top} g\, \Lambda$$

where $^\top$ denotes transposed matrix. Eq. (1.24) does indeed define the Lorentz group L as a group of rank four square matrices acting upon the Minkowski space column four-vectors[3]. As a matter of fact we have

1. composition law : $\quad \Lambda\,,\ \Lambda' \in L \quad \Rightarrow \quad \Lambda \cdot \Lambda' = \Lambda'' \in L$
   matrix associative product : $\quad (\Lambda''\,)^{\,\mu}{}_{\rho} = \Lambda^{\,\mu}{}_{\nu}\,(\Lambda')^{\,\nu}{}_{\rho}$

2. identity matrix : $\quad \exists\,!\,\mathbf{I}$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. inverse matrix : $\quad$ from the defining relation (1.24) we have

$$\det g = \det\left(\Lambda^\top g\,\Lambda\right) \quad \Rightarrow \quad \det\Lambda = \pm 1$$

   and thereby $\exists\,!\ \Lambda^{-1} = g\cdot\Lambda^\top\cdot g \qquad \forall\Lambda\in L$

which means that L is a group of matrices that will be called the homogeneous full Lorentz group. From the relation

$$1 = g_{00} = g_{\mu\nu}\,\Lambda^{\,\mu}{}_{0}\,\Lambda^{\,\nu}{}_{0} = \left[\,\Lambda^{0}{}_{0}\,\right]^{\,2} - \Lambda^{\,k}{}_{0}\,\Lambda^{\,k}{}_{0} \quad \Rightarrow \quad \left|\,\Lambda^{0}{}_{0}\,\right| \geq 1$$

it follows that the homogeneous full Lorentz group splits into four subsets called connected components

- proper orthochronus $\quad L_{+}^{\uparrow} = \{\Lambda\in L\,|\,\det\Lambda = 1\ \cap\ \Lambda^{0}{}_{0} \geq 1\}$
- improper orthochronus $\quad L_{-}^{\uparrow} = \{\Lambda\in L\,|\,\det\Lambda = -1\ \cap\ \Lambda^{0}{}_{0} \geq 1\}$
- proper nonorthochronus $\quad L_{+}^{\downarrow} = \{\Lambda\in L\,|\,\det\Lambda = 1\ \cap\ \Lambda^{0}{}_{0} \leq -1\}$
- improper nonorthochronus $\quad L_{-}^{\downarrow} = \{\Lambda\in L\,|\,\det\Lambda = -1\ \cap\ \Lambda^{0}{}_{0} \leq -1\}$

Among the four connected components of the homogeneous full Lorentz group there is only $L_{+}^{\uparrow}$ which is a subgroup, *i.e.* the component connected with the identity element, which is also called the restricted subgroup of the homogeneous full Lorentz group. Other common notations are as follows. The homogeneous full Lorentz group is also denoted by $O(1,3)$, the proper orthochronus component $L_{+}^{\uparrow}$ by $SO(1,3)^{+}$ or $O(1,3)_{+}^{\uparrow}$ and is also called the

[3]This means in turn that contravariant indexes are row indexes while covariant ones are column indexes.

restricted homogeneous Lorentz group. Furthermore, the three remaining connected components are also respectively denoted by

$$O(1,3)_+^- = L_+^\downarrow \qquad O(1,3)_-^+ = L_-^\uparrow \qquad O(1,3)_-^- = L_-^\downarrow$$

Examples :

1. special Lorentz transformation, or even boost, in the OX direction with velocity $v > 0$ towards the positive OX axis

$$\Lambda(\eta) = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cosh\eta = (1-\beta^2)^{-1/2} \qquad \sinh\eta = \beta\,(1-\beta^2)^{-1/2} \qquad \beta = \frac{v}{c}$$

2. spatial rotation around the OZ axis

$$\Lambda(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   boosts and rotations belong to the special homogeneous Lorentz group.

3. parity transformation with respect to the OY axis

$$\Lambda_P^y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   full spatial reflection or full parity transformation

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.25}$$

   spatial reflections or parity transforms belong to $L_-^\uparrow = O(1,3)_-^+$

4. time inversion or time reflection T transformation

$$\Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in L_-^\downarrow = O(1,3)_-^-$$

5. full inversion or PT transformation

$$\Lambda_{PT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \ L_{+}^{\downarrow} = O(1,3)_{+}^{-}$$

Any Lorentz transformation can always been decomposed as the product of transformations of the four types belonging to the above described connected components. Since there are three independent rotations as well as three independent boosts, of for each spatial direction, the special homogeneous Lorentz transformations are described in terms of six parameters.

The homogeneous Lorentz group is a six dimensional Lie group, so that each element can be labeled by six real parameters. For example, one can choose the three Euler angles and the three ratios between the components of the relative velocity and the light velocity: namely,

$$\Lambda \in O(1,3) \ \to \ \Lambda = \Lambda(\varphi, \theta, \psi, \beta_1, \beta_2, \beta_3) \qquad (\, \beta_k \ \equiv \ v_k / c \,)$$
$$0 \leq \varphi \leq 2\pi \, , \ 0 \leq \theta < \pi \, , \ 0 \leq \psi \leq 2\pi \, , \ -1 < \beta_k < 1 \quad (\, k = 1, 2, 3 \,)$$

The most suitable parametric form is in terms of the canonical coordinates

$$(\boldsymbol{\alpha} \, , \boldsymbol{\eta} \,) = (\alpha_1, \alpha_2, \alpha_3 \, ; \, \eta_1, \eta_2, \eta_3) \tag{1.26}$$

where the angles α_k $(k = 1, 2, 3)$ with $\boldsymbol{\alpha}^2 < (2\pi)^2$ are related to the spatial rotations around the orthogonal axes of the chosen inertial frame, whilst the hyperbolic arguments $\eta_k \equiv \text{Arsh}\left(\beta_k \, (1 - \beta_k^2)^{-1/2}\right)$ $(k = 1, 2, 3)$ with $\boldsymbol{\eta} \in \mathbb{R}^3$ are related to boosts along the spatial directions. More specifically

$$\Lambda(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}$$

represents a rotation of the inertial reference frame in the counterclockwise sense of an angle α_1 around the OX axis whereas *e.g.*

$$\Lambda(\eta_3) = \begin{pmatrix} \cosh\eta_3 & 0 & 0 & -\sinh\eta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta_3 & 0 & 0 & \cosh\eta_3 \end{pmatrix}$$

corresponds to a boost with a rapidity parameter

$$\eta_3 = \mathrm{Arsh}\left(\beta_3\,(1-\beta_3^2)^{-1/2}\right) \tag{1.27}$$

associated to the OZ direction. Notice that the inverse transformations can be immediately obtained after sending $\alpha_k \;\mapsto\; -\alpha_k$ and $\eta_k \;\mapsto\; -\eta_k$. Since the domain of the canonical coordinates is the unbounded subset of $\mathbb{R}^6$

$$D \equiv \{(\boldsymbol{\alpha}\,,\boldsymbol{\eta}\,)\,|\,\boldsymbol{\alpha}^2 < (2\pi)^2\,,\ \boldsymbol{\eta}\,\in\,\mathbb{R}^3\}$$

it follows that the Lorentz group is non-compact. It is convenient to write the elements of the special homogeneous Lorentz group in the exponential form and to introduce the infinitesimal generators according to the standard convention [17]

$$\Lambda(\alpha_1) = \exp\{\alpha_1 I_1\} \qquad \Lambda(\eta_3) = \exp\{\eta_3 J_3\}$$

where

$$\begin{aligned} I_1 &\equiv \frac{\mathrm{d}\Lambda}{\mathrm{d}\alpha_1}\,(\alpha_1 = 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ J_3 &\equiv \frac{\mathrm{d}\Lambda}{\mathrm{d}\eta_3}\,(\eta_3 = 0) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

et cetera . It is very important to gather that the infinitesimal generators of the space rotations are anti-Hermitean $I_k^\dagger = -I_k$ $(k = 1,2,3)$ whereas the infinitesimal generators of the special Lorentz transformations turn out to be Hermitean $J_k = J_k^\dagger$ $(k = 1,2,3)$. One can also check by direct inspection that the infinitesimal generators do fulfill the following commutation relations

$$[\,I_j\,I_k\,] = -\,\varepsilon_{jkl}\,I_l \qquad [\,I_j\,J_k\,] = -\,\varepsilon_{jkl}\,J_l \qquad [\,J_j\,J_k\,] = \varepsilon_{jkl}\,I_l \tag{1.28}$$

$$(\,j,k,l = 1,2,3)$$

The above commutation relations uniquely specify the Lie algebra of the homogeneous Lorentz group.

Together with the infinitesimal generators I_k, J_k $(k = 1,2,3)$ it is very convenient to use the matrices

$$A_k \equiv \tfrac{1}{2}\,(I_k + i\,J_k) \qquad B_k \equiv \tfrac{1}{2}\,(I_k - i\,J_k) \qquad k = 1,2,3$$

It is worthwhile to remark that all the above matrices are anti-Hermitean

$$A_j^\dagger = -\,A_j \qquad B_k^\dagger = -\,B_k \qquad (\,j,k = 1,2,3)$$

The commutation relations for these matrices have an especially simple form:

$$[\,A_j\,A_k\,] = -\,\varepsilon_{jkl}\,A_l \qquad [\,B_j\,B_k\,] = -\,\varepsilon_{jkl}\,B_l \qquad [\,A_j\,B_k\,] = 0$$

$$(\,j,k,l = 1,2,3)$$

which follow from the commutation relations of the infinitesimal generators. We stress that the commutation relations for the operators A_k $(k = 1,2,3)$ are the same as those for the generators of the three dimensional rotation group $SO(3)$ and of its universal covering group $SU(2)$. This is also true for the operators B_k $(k = 1,2,3)$.

The infinite dimensional reducible and unitary representations of the total angular momentum Lie algebra are very well known. Actually, one can render simultaneously diagonal the positive semi-definite operators

$$A_j^\dagger\,A_j = A_j\,A_j^\dagger = -\,A_j\,A_j = -\,\mathbf{A}^2$$
$$B_k^\dagger\,B_k = B_k\,B_k^\dagger = -\,B_k\,B_k = -\,\mathbf{B}^2$$

the spectral resolutions of which read

$$-\,\mathbf{A}^2 = \sum_m m\,(m+1)\,\widehat{P}_m \qquad -\,\mathbf{B}^2 = \sum_n n\,(n+1)\,\widehat{P}_n$$

$$\mathrm{tr}\;\widehat{P}_m = 2m+1 \qquad \mathrm{tr}\;\widehat{P}_n = 2n+1$$
$$m\,,n \;=\; 0\,,\tfrac{1}{2}\,,1\,,\tfrac{3}{2}\,,2\,,\tfrac{5}{2}\,,3\,,\ldots\ldots$$

Those operators are called the Casimir operators of the Lorentz group, the latter ones being defined by the important property of commuting with all the infinitesimal generators of the group: namely

$$[\,\mathbf{A}^2 I_\jmath\,] = [\,\mathbf{A}^2 J_\jmath\,] = [\,\mathbf{B}^2 I_\jmath\,] = [\,\mathbf{B}^2 J_\jmath\,] = 0 \qquad (\,\jmath = 1,2,3\,)$$

It follows therefrom that

- all the irreducible finite dimensional representations of the Lorentz group are labeled by a pair (m,n) of positive semi-definite integer or half-integer numbers

- the irreducible finite dimensional representations $\boldsymbol{\tau}_{m\,n}$ contain

$$\left(\mathrm{tr}\ \widehat{P}_m\right) \cdot \left(\mathrm{tr}\ \widehat{P}_n\right) = (2m+1)\,(2n+1)$$

  linearly independent states characterized by the eigenvalues of *e.g.* A_3 and B_3 respectively

- we can identify $S_k = I_k = A_k + B_k$ $(k = 1, 2, 3)$ with the components of the spin angular momentum operator, so that we can say that $\boldsymbol{\tau}_{m\,n}$ is an irreducible representation of the Lorentz group of spin angular momentum $s = m + n$

- the lowest dimensional irreducible representations of the Lorentz group are: the 1-dimensional scalar representation $\boldsymbol{\tau}_{0\,0}$ and the pair of 2-dimensional Weyl representations of the left spinors $\boldsymbol{\tau}_{\frac{1}{2}\,0}$ and right spinors $\boldsymbol{\tau}_{0\,\frac{1}{2}}$ (the handedness being conventional)

  Hermann Klaus Hugo Weyl
  `one of the most influential personalities for Mathematics in the first half of the XXth century`
  Elmshorn, Germany, 9.11.1885 – Zurich, CH, 8.12.1955
  Gruppentheorie und Quantenmechanik (1928)

- under a parity transform $\Lambda_P = \Lambda_P^{-1}$ we find

$$I_k \ \mapsto \Lambda_P^{-1}\, I_k\, \Lambda_P = I_k \qquad J_k \ \mapsto \Lambda_P^{-1}\, J_k\, \Lambda_P = -\, J_k \tag{1.29}$$

$$A_k \ \mapsto \Lambda_P^{-1}\, A_k\, \Lambda_P = B_k \qquad B_k \ \mapsto \Lambda_P^{-1}\, B_k\, \Lambda_P = A_k$$

  meaning that the two non-equivalent irreducible Weyl's representations interchange under parity. Hence, to set up a spinor representation of the full Lorentz group out of the two Weyl's representations, one has to consider the direct sum

$$\boldsymbol{\tau}_D = \boldsymbol{\tau}_{\frac{1}{2}\,0} \oplus \boldsymbol{\tau}_{0\,\frac{1}{2}}$$

  which is a reducible four dimensional representation of the restricted Lorentz group and is called the Dirac representation

- all the other irreducible representations of higher dimensions can be obtained from the well known Clebsch-Gordan-Racah multiplication and decomposition rule

$$\boldsymbol{\tau}_{m\,n} \times \boldsymbol{\tau}_{p\,q} = \bigoplus_{r\,,\,s}\ \boldsymbol{\tau}_{r\,s}$$

$$|\,m-p\,| \le r \le m+p \qquad |\,n-q\,| \le s \le n+q$$

In particular, the spin 1 representation does coincide with the above introduced four vector representation, which we actually used to define the homogeneous Lorentz group

$$\boldsymbol{\tau}_{\frac{1}{2}\,0} \times \boldsymbol{\tau}_{0\,\frac{1}{2}} = \boldsymbol{\tau}_{\frac{1}{2}\,\frac{1}{2}}$$

The anti-symmetric Maxwell field strength $F_{\mu\nu}$ transforms according to the reducible parity symmetrical representation $\boldsymbol{\tau}_{1\,0} \oplus \boldsymbol{\tau}_{0\,1}$.

- All the irreducible finite dimensional representations of the Lorentz group are non-unitary. As a matter of fact we have *e.g.*

$$\boldsymbol{\tau}_{m\,n}(\beta_3) = \exp\{\,\beta_3 J_3\,\} = \exp\{\,-\,i\beta_3(A_3 - B_3)\,\}$$

and thereby

$$\boldsymbol{\tau}^{\dagger}_{m\,n}(\beta_3) = \boldsymbol{\tau}_{m\,n}(\beta_3) \ \neq\ \boldsymbol{\tau}^{-1}_{m\,n}(\beta_3) = \boldsymbol{\tau}_{m\,n}(-\beta_3)$$

All the unitary irreducible representations of the Lorentz group are infinite dimensional and have been classified by

I.M. Gel'fand and M.A. Naimark
Unitary representations of the Lorentz group
Izv. Akad. Nauk. SSSR, matem. **11**, 411, 1947.

`Exercise`. A second rank tensor $t_{\mu\nu}$ with $\mu,\nu = 0,1,2,3$ transforms according to the reducible representation $T = \boldsymbol{\tau}_{\frac{1}{2}\,\frac{1}{2}} \times \boldsymbol{\tau}_{\frac{1}{2}\,\frac{1}{2}}$ of the Lorentz group $O(1,3)$.

1. Decompose the representation T into the sum of irreducible representations of $O(1,3)$
2. Specify the dimensions of T and of each irreducible representations appearing in the above decomposition
3. Among the sixteen matrix elements $t_{\mu\nu}$ find the components belonging to each irreducible representations appearing in the decomposition.

Solution. From the general decomposition rule

$$\boldsymbol{\tau}_{m\,n} \times \boldsymbol{\tau}_{p\,q} = \bigoplus_{r\,,\,s}\ \boldsymbol{\tau}_{r\,s}$$

$$|\,m-p\,| \le r \le m+p \qquad |\,n-q\,| \le s \le n+q$$

we obtain for $m = n = \frac{1}{2}$

$$T = \boldsymbol{\tau}_{0\,0} \bigoplus \boldsymbol{\tau}_{1\,0} \bigoplus \boldsymbol{\tau}_{0\,1} \bigoplus \boldsymbol{\tau}_{1\,1}$$

so that the dimensions correctly match because

$$16 = 1 + 3 + 3 + 9$$

The one dimensional irreducible representation $\boldsymbol{\tau}_{0\,0}$ corresponds to the trace of the second rank tensor $t_{\mu\nu}$ *i.e.*

$$t = g^{\mu\nu}\, t_{\mu\nu}$$

which is a Lorentz scalar. The six components of the anti-symmetric part of the second rank tensor $t_{\mu\nu}$ that is

$$A_{\mu\nu} = \frac{1}{2}\,(t_{\mu\nu} - t_{\nu\mu})$$

transform according to the parity invariant representation $\boldsymbol{\tau}_{1\,0} \bigoplus \boldsymbol{\tau}_{0\,1}$, while the nine components of the traceless and symmetric part of the second rank tensor $t_{\mu\nu}$ *viz.*,

$$S_{\mu\nu} = \frac{1}{2}\,(t_{\mu\nu} + t_{\nu\mu}) - \frac{1}{4}\,g^{\mu\nu}\,t$$

transform according to the irreducible representation $\boldsymbol{\tau}_{1\,1}$. Therefore we can always write the most general decomposition

$$t_{\mu\nu} = A_{\mu\nu} + S_{\mu\nu} + \frac{1}{4}\,g_{\mu\nu}\,t \qquad\qquad (\,t = g^{\mu\nu}\,t_{\mu\nu}\,)$$

`Exercise`. Find the explicit form of the structure constants of the Lorentz group.

Solution. The commutation relations (1.28) can be written in explicit form as

$$[\,I_1\,I_2\,] = -\,I_3 \quad [\,I_1\,I_3\,] = I_2 \quad [\,I_1\,J_1\,] = 0 \quad [\,I_1\,J_2\,] = -\,J_3 \quad [\,I_1\,J_3\,] = J_2$$

$$[\,I_2\,I_3\,] = -\,I_1 \quad [\,I_2\,J_1\,] = J_3 \quad [\,I_2\,J_2\,] = 0 \quad [\,I_2\,J_3\,] = -\,J_1$$

$$[\,I_3\,J_1\,] = -\,J_2 \quad [\,I_3\,J_2\,] = J_1 \quad [\,I_3\,J_3\,] = 0$$

$$[\,J_1\,J_2\,] = I_3 \quad [\,J_1\,J_3\,] = -\,I_2$$

$$[\,J_2\,J_3\,] = I_1$$

Now it is convenient to slightly change the notation and denote the boost generators by

$$J_1 = I_4 \qquad J_2 = I_5 \qquad J_3 = I_6$$

in such a manner that all the non-vanishing structure constants of the Lorentz group

$$C_{abc} = -\,C_{bac} \qquad (\,a, b, c, \ldots = 1, 2, \ldots, 6\,)$$

can be read off the above commutation relations: namely,

$$\begin{array}{rrrr}
C_{123} = -1 & C_{132} = 1 & C_{156} = -1 & C_{165} = 1 \\
 & C_{231} = -1 & C_{246} = 1 & C_{264} = -1 \\
 & & C_{345} = -1 & C_{354} = 1 \\
 & & C_{453} = 1 & C_{462} = -1 \\
 & & & C_{561} = 1
\end{array}$$

whence it manifestly appears that the structure constants are completely anti-symmetric, as it will be proved later on.

1.3.2 Semi-Simple Groups

Lie groups and their corresponding Lie algebras can be divided into three main categories depending upon the presence or absence of some invariant subgroups and invariant sub-algebras.

By its very definition, an invariant subgroup $H \subseteq G$ satisfies the following requirement: for any elements $g \in G$ and $h \in H$ there always exists an element $h' \in H$ such that

$$gh = h'g$$

Of course, any Lie group G has two trivial invariant subgroups, G itself and the unit element.

Concerning the infinitesimal operators $J_a \in \mathcal{G}$ $(a = 1, 2, \ldots n)$ and $T_b \in \mathcal{H}$ $(b = 1, 2, \ldots m \leq n)$ of the corresponding Lie algebra and sub-algebra, we shall necessarily find

$$[\, J_a \, T_b \,] = C_{abc} \, T_c \qquad (a = 1, 2, \ldots n \,, \; b, c = 1, 2, \ldots m \leq n) \tag{1.30}$$

- Groups that do not possess any nontrivial invariant subgroup are called simple.

- A weaker requirement is that the group G had no nontrivial invariant *Abelian* subgroups : in such a circumstance G is said to be semi-simple.

  In this case it can be proved that any semi-simple Lie group G is *locally isomorphic* to the so called direct product or *cartesian product* of some mutually commuting simple non-Abelian groups $G_1, G_2, \ldots, G_s$. This means that $\forall \, g_\alpha \in G_\alpha \,,\; g_\beta \in G_\beta$ we have

$$G \doteq G_1 \times G_2 \times \cdots \times G_s$$

$$g_\alpha \, g_\beta = g_\beta \, g_\alpha \qquad \alpha \neq \beta \qquad (\alpha, \beta = 1, 2, \ldots, s)$$

$$\dim(G) = \sum_{\alpha=1}^{s} \dim(G_\alpha)$$

  where the symbol $\doteq$ means that the analytic isomorphism is true in some suitable neighborhoods of the unit elements of the involved Lie groups.

  For the corresponding Lie algebras we have that $\forall \, I_\alpha \in \mathcal{G}_\alpha \,,\; I_\beta \in \mathcal{G}_\beta$

$$\mathcal{G} = \bigoplus_{\alpha=1}^{s} \mathcal{G}_\alpha \qquad [\, I_\alpha \, I_\beta \,] = 0 \qquad \alpha \neq \beta \qquad (\alpha, \beta = 1, 2, \ldots, s)$$

  An example of a semi-simple group is $SO(4) \doteq SO(3) \times SO(3)$.

- Groups that do contain some invariant Abelian nontrivial subgroups are said to be *non-semi-simple*.

  Such groups do not always factorize into the direct product of an Abelian invariant subgroup and a semi-simple group.

The planar Euclidean group. As an example, let us consider the two dimensional Euclidean group or non-homogeneous orthogonal group $IO(2)$, which is the group of the rotations and translations in the plane with no reflections with respect to an axis of the plane. We shall denote a translation by the symbol $T(\mathbf{a})$, where $\mathbf{a} = (a_x, a_y)$ is a general displacement of all points in the OXY plane. Evidently

$$T(\mathbf{a})\,T(\mathbf{b}) = T(\mathbf{a}+\mathbf{b})$$

It is elementary to proof that every element g of the group $IO(2)$ can be represented in the form of a product of a rotation around an arbitrary point O of the plane and a certain translation:

$$g = T(\mathbf{a})\,R_O \qquad\qquad g \in IO(2)$$

It is also straightforward to prove the following identities

$$g\,T(\mathbf{a})\,g^{-1} = T(g\,\mathbf{a}) \qquad\qquad g \neq T(\mathbf{b}) \tag{1.31}$$

$$T(\mathbf{a})\,R_O\,T(-\,\mathbf{a}) = R_{O+\mathbf{a}} \tag{1.32}$$

where $g \in IO(2)$, $g\,\mathbf{a}$ is the point in the plane obtained from $\mathbf{a}$ by the displacement $g \in IO(2)$, R_O any rotation around the point O and $O+\mathbf{a}$ the point to which O moves owing to the translation of $\mathbf{a}$. Now, if we choose

$$g(\alpha) = \exp\{\alpha\,I_z\} \qquad 0 \le \alpha \le 2\pi \qquad T(\mathbf{a}) = \exp\{\mathbf{a}\cdot\mathbf{P}\}$$

where

$$I_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad\qquad g(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

from eq. (1.31) and for very small α, a_x, a_y we readily obtain the Lie algebra among the generators of $IO(2)$, *i.e.*

$$[\,P_x\,P_y\,] = 0 \qquad [\,I_z\,P_x\,] = P_y \qquad [\,I_z\,P_y\,] = -\,P_x$$

which precisely corresponds to the condition (1.30). Translations constitute an invariant Abelian subgroup of $IO(2)$ which is consequently non-semi-simple.

A very important quantity is the Cartan–Killing metric of a Lie group G, which is defined to be

$$g_{ab} \;\equiv\; C_{acd}\,C_{bdc} \qquad (a,b,c,d = 1,2,\ldots,n) \tag{1.33}$$

where C_{abc} are the structure constants of the Lie algebra $\mathcal{G}$. According to the correspondence (1.10) we can also write

$$\mathrm{tr}\,(A_a\,A_b) \;=\; g_{ab} \qquad (a,b = 1,2,\ldots,n)$$

where $n \times n$ square matrices A_a $(a = 1, 2, \ldots, n)$ denote the generators in the adjoint representation. For instance, in the case of $SU(2)$ we find

$$g_{ab} = (-\,\varepsilon_{acd})(-\,\varepsilon_{bdc}) = -\,\varepsilon_{acd}\,\varepsilon_{bcd} \;=\; -\,2\,\delta_{ab}$$

It is important to remark that the Cartan-Killing metric of a Lie group G is a group invariant. As a matter of fact, for any inner automorphism of the adjoint representation, see the definition (1.9), we find

$$\begin{aligned}
\mathrm{tr}\left(A_a(g)\,A_b(g)\right) &= \mathrm{tr}\left(T_A(g)\,A_a\,T_A^{-1}(g)\,T_A(g)\,A_b\,T_A^{-1}(g)\right)\\
&= \mathrm{tr}\left(T_A(g)\,A_a\,A_b\,T_A^{-1}(g)\right) = \mathrm{tr}\left(A_a\,A_b\,T_A^{-1}(g)\,T_A(g)\right)\\
&= \mathrm{tr}\,(A_a\,A_b) \;=\; g_{ab}\\
&\forall \quad g \in G \qquad\quad (a, b = 1, 2, \ldots, n) \qquad\qquad (1.34)
\end{aligned}$$

where $T_A(g)$ $(g \in G)$ are the linear operators of the adjoint representation and I have made use of the cycle property of the trace operation, that is

$$\mathrm{tr}\,(A_1 A_2 \cdots A_n) = \mathrm{tr}\,(A_2 \cdots A_n A_1) = \mathrm{tr}\,(A_n A_1 A_2 \cdots A_{n-1}) = \cdots$$

In particular, for an infinitesimal group transformation

$$T_A(g) = \mathbf{I}_A + \alpha_c\,A_c \qquad\qquad |\,\alpha_c\,| \ll 1$$

where $\mathbf{I}_A$ is the identity operator in the adjoint representation, the above equality entail

$$\begin{aligned}
0 &= \delta\,g_{ab} = \alpha_c\,\mathrm{tr}\left([\,A_c\,A_a\,]\,A_b\right) + \mathrm{tr}\left(A_a[\,A_c\,A_b\,]\right)\alpha_c\\
&= \alpha_c\,C_{cad}\,\mathrm{tr}\,(A_d\,A_b) + C_{cbd}\,\mathrm{tr}\,(A_a\,A_d)\,\alpha_c\\
&= (C_{cad}\,g_{bd} + C_{cbd}\,g_{ad})\,\alpha_c = (C_{cab} + C_{cba})\,\alpha_c \qquad\qquad (1.35)
\end{aligned}$$

whence

$$C_{cab} \;=\; -\,C_{cba} \qquad\qquad (1.36)$$

which means anti-symmetry of the structure constants also respect the last two indexes. As a consequence the structure constants of any Lie group turn out to be completely anti-symmetric with respect to the exchange of all three indexes: namely, $C_{abc} = -\,C_{bac} = C_{bca} = -\,C_{cba}$.

The Cartan-Killing metric is non-singular, *i.e.* $\det\|\,g\,\| \neq 0$, if and only if the group is semi-simple. Here $\|\,g\,\|$ indicates the $n \times n$ matrix of elements g_{ab} . For example we have

$$g = (-2)\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} \qquad \text{for } SU(2) \qquad\qquad (1.37)$$

$$g = (-2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{for } IO(2)$$

The Cartan-Killing metric is a real symmetric matrix: therefore, it can be set diagonal by means of an orthogonal transformation that reshuffles the infinitesimal operators.

1. If there are null eigenvalues in the diagonal form of the metric, then the Lie group is non-semi-simple.
2. If the Cartan-Killing metric is negative definite, then the Lie group is compact.
3. If the Cartan-Killing metric has both positive and negative eigenvalues, as in the case of the homogeneous Lorentz group or the group $SL(2,\mathbb{R})$, then the Lie group is non-compact.

Since the Cartan-Killing metric is a non-singular $n \times n$ matrix for semi-simple Lie groups it is possible to define its inverse matrix g^{ab}, which can be used to raise and lower the group indexes $a, b, c, \ldots = 1, 2, \ldots, n$. For example, the Lie product of two generators in the representation R, or commutator in the Quantum Physics terminology, will be written in different though equivalent forms like

$$[\, I_a^R \, I_b^R \,] = C_{abc} \, I_c^R = C_{abc} \, g^{cd} \, I_d^R = C_{ab}{}^{c} \, I_c^R$$

because contractions over lower group indexes have to be performed by means of the inverse Cartan-Killing metric tensors.

For any semi-simple Lie group G it is always possible to select a special element for any representation of its semi-simple Lie algebra $\mathcal{G}$, that is

$$C_R \equiv g_{ab} \, I_R^a \, I_R^b = g^{ab} \, I_a^R \, I_b^R \qquad I_R^a \equiv g^{ab} I_b^R \qquad (a, b = 1, 2, \ldots, n) \quad (1.38)$$

the subscript R labeling some particular representation of the infinitesimal operators. The quadratic operator C_R is called the Casimir operator of the representation R and turn out to be group invariant: namely,

$$T_R(g) \, C_R \, T_R^{-1}(g) = C_R \qquad \forall \, g \in G$$

or else

$$[\, T_R(g) \, C_R \,] = 0 \qquad \forall \, g \in G$$

`Proof` : first we notice that

$$\begin{aligned} [\, I_c^R \, C_R \,] &= g^{ab} \, [\, I_c^R \, (I_a^R \, I_b^R) \,] \\ &= g^{ab} \, [\, I_c^R \, I_a^R \,] \, I_b^R + g^{ab} \, I_a^R \, [\, I_c^R \, I_b^R \,] \\ &= g^{ab} \, C_{cad} \, g^{de} \, I_e^R \, I_b^R + g^{ab} \, I_a^R \, C_{cbd} \, g^{de} \, I_e^R \\ &= g^{ab} \, C_{cad} \, g^{de} \, I_e^R \, I_b^R + g^{ab} \, C_{cbd} \, g^{de} \, I_a^R \, I_e^R \end{aligned}$$

Reshuffling indexes in the second addendum of the last line we obtain

$$[\, I_c^R\, C_R\,] \;=\; C_{cad}\left(g^{\,ab}\, g^{\,ed} + g^{\,ea}\, g^{\,db}\right) I_e^R\, I_b^R \tag{1.39}$$

and owing to complete anti-symmetry of the structure constants with respect to all indexes we eventually find

$$[\, I_c^R\, C_R\,] = 0$$

By iterating the above procedure it is immediate to verify that

$$[\,(\,\alpha_a\, I_a^R\,)^k\, C_R\,] = 0 \qquad \forall\, k \in \mathbb{N} \qquad \alpha_a \in \mathbb{R} \qquad (\,a = 1, 2, \ldots, n\,)$$

Hence, from the exponential representation

$$\begin{aligned} T_R\,(g)\, C_R &= \exp\left\{\alpha_a\, I_a^R\right\}\, C_R = \sum_{k=0}^{\infty} \frac{1}{k!}\, \left(\,\alpha_a\, I_a^R\,\right)^k\, C_R \\ &= C_R \sum_{k=0}^{\infty} \frac{1}{k!}\, \left(\,\alpha_a\, I_a^R\,\right)^k \\ &= C_R \exp\left\{\alpha_a\, I_a^R\right\} = C_R\, T_R\,(g) \qquad \forall\, g \in G \end{aligned}$$

which completes the proof. □

The general structure of the Casimir's operators is strongly restricted by

- Schur's lemma : if a linear operator commutes with *every* element of any irreducible representation $\boldsymbol{\tau}(g)$ of a group, then it is proportional to the identity operator.

  For a Lie group this lemma may also be rephrased as follows.

- If a linear operator commutes with *all* the generators in an irreducible representation of a Lie algebra $\mathcal{G}$, then it must be proportional to the identity operator.

See [17] for the proof. According to Schur's lemma we come to the conclusion that in the **irreducible representation** labeled by R we have

$$C_R = g_{\,ab}\, I_R^{\,a}\, I_R^{\,b} = d_R\, \mathbf{I}_R$$

where d_R is a number, depending upon the irreducible representation, which is called the Dynkin's index.

- For the adjoint representation we have

  $$\operatorname{tr} C_A = g_{ab} \operatorname{tr}\,(A^a\, A^b) = d_A \operatorname{tr} \mathbf{I}_A = n\, d_A = g^{\,ab}\, g_{\,ab} = n$$

  so that

  $$g_{ab} = C_{adc}\, C_{bcd} = \varkappa\, \delta_{\,ab} \qquad g^{\,ab} = \varkappa^{-1}\, \delta^{\,ab} \qquad d_A = 1 \tag{1.40}$$

  where $\varkappa$ is a suitable normalization constant.

- In the case of the special unitary group $SU(2)$, *i.e.* in the case of the fundamental representation of the rotation group, from the very definition (1.14) and the Cartan-Killing metric (1.37), the Casimir's operator (1.38) reads – remember that $g_{ab} = (-2)\,\delta_{ab}$

$$\operatorname{tr} C_F = 2\,d_F = -\,\tfrac{1}{2}\,\delta^{\,ab}\operatorname{tr}(\tau_a\,\tau_b) = -\,\tfrac{1}{2}\,\delta^{\,ab}\,\left(-\,\tfrac{1}{4}\right)\,2\,\delta_{\,ab} = 3$$

  so that $d_F = \frac{3}{8}$ and $d_A = 1$ for the rotation group.

- For any irreducible representation of the rotation group it is possible to show that

$$C_R = \tfrac{1}{2}\,J\,(J+1)\,\mathbf{I}_R$$

  where J is the weight of the irreducible representation, which is related to the eigenvalue of the total angular momentum by $\mathbf{J}^2 = \hbar^2 J(J+1)$. Hence

$$d_J \;=\; \tfrac{1}{2}\,J\,(J+1) \qquad \left(J = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \ldots\right)$$

  are the Dynkin's indexes for the irreducible unitary representations of the rotation group.

- For a non-semi-simple Lie group G, just like the Poincaré group for example, owing to the singular nature of the Cartan-Killing metric tensor, the Casimir operators in any irreducible representation R are just defined by the very requirement of being invariant under the whole group of linear transformations $T_R\,(g)$ $(g \in G)$: namely,

$$C_R = T_R\,(g)\,C_R\,T_R^{\;-1}(g) \qquad\quad \forall\, g \in G$$

1.3.3 The Poincaré Group

The quite general symmetry group of all the relativistic classical and quantum field theories obeying the principles of the Special Relativity is the restricted non-homogeneous Lorentz group, also named the Poincaré group. We can write the passive transformations of the space-time events under the Poincaré group in the form

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu}\,(x^{\,\nu} + a^{\nu}) \qquad\qquad x^{\,\nu} = \Lambda_{\mu}^{\ \nu} x'^{\mu} - a^{\,\nu}$$

where a^{μ} is an arbitrary constant four vector. Hence the Poincaré group is a non-semi-simple ten parameters Lie group, the canonical coordinates of which, in accordance with (1.26), can be identified with

$$(\boldsymbol{\alpha}\,,\boldsymbol{\eta}\,,a\,) = \Big(\alpha_1,\alpha_2,\alpha_3\,;\,\eta_1,\eta_2,\eta_3\,;\,a^0,a^1,a^2,a^3\Big)$$

The space-time translations $T(a)$ do constitute an Abelian four parameters subgroup and fulfill

$$T(a)\,T(b) = T(a+b) = T(b)\,T(a)$$

However, space-time translations do not commute with the Lorentz group elements. Consider in fact two Poincaré transformations with parameters (Λ, a) and (Λ', a') so that

$$x^{\mu} \quad\mapsto\quad \Lambda^{\mu}_{\ \nu}\,(x^{\,\nu} + a^{\nu}) \quad\mapsto\quad \Lambda'^{\mu}_{\ \rho}\left[\Lambda'^{\rho}_{\ \nu}\Big(x^{\,\nu} + a^{\nu}\Big) + a'^{\rho}\right]$$

$$a^{\mu} \quad\mapsto\quad \Lambda'^{\mu}_{\ \rho}\left(\Lambda^{\rho}_{\ \nu}\,a^{\,\nu} + a'^{\rho}\right)$$

whence we see that the translation parameters get changed under a Lorentz transformation. Owing to this feature, called the soldering between the Lorentz transformations and the space-time translations, the Poincaré group is said to be a semi-direct product of the Lorentz group and the space-time translation Abelian group.

The generators of the space-time translations are the components of the tetra-gradient operator. As a matter of fact, for any analytic real function $f\,:\,\mathcal{M} \to \mathbb{R}$ we have

$$T(a)\,f(x) = \exp\left\{a^{\mu}\partial_{\mu}\right\}f(x) = f(x+a)$$

$$[\,\partial_{\mu}\,,\,\partial_{\nu}\,] \equiv \partial_{\mu}\,\partial_{\nu} - \partial_{\nu}\,\partial_{\mu} = 0$$

Thus, the infinitesimal operators of the space-time translations turn out to be differential operators acting of the infinite dimensional space $C_{\infty}(\mathcal{M})$ of the analytic functions on the Minkowski space.

It is necessary to obtain an infinite dimensional representation of the generators of the Lorentz group acting on the very same functional space. Introduce the six anti-Hermitean differential operators

$$\ell_{\mu\nu} \equiv g_{\mu\rho}\, x^{\rho}\, \partial_{\nu} - g_{\nu\rho}\, x^{\rho}\, \partial_{\mu} \equiv x_{\mu}\, \partial_{\nu} - x_{\nu}\, \partial_{\mu} \tag{1.41}$$

$$\ell_{\mu\nu} = -\,\ell_{\nu\mu} = -\,\ell^{\dagger}_{\mu\nu}$$

One can immediately gather that the three differential operators from the spatial components, *viz.,*

$$i\hbar\ell^{\jmath k} = x^{\jmath}\,(-\,i\hbar\,\partial_k) - x^k\,(-\,i\hbar\,\partial_{\jmath}) = x^{\jmath}\,p^k - x^k\,p^{\jmath}$$

are nothing but the components of the orbital angular momentum self-adjoint operators of non-relativistic Quantum Mechanics $\mathbf{L} = \mathbf{r} \times \mathbf{p}$: namely,

$$i\hbar\,\ell_{23} = L_x \qquad i\hbar\,\ell_{31} = L_y \qquad i\hbar\,\ell_{12} = L_z$$

which fulfill the well known $SU(2)$ Lie algebra

$$[\,L_x\,,\,L_y\,] = i\hbar\,L_z \qquad [\,L_y\,,\,L_z\,] = i\hbar\,L_x \qquad [\,L_z\,,\,L_x\,] = i\hbar\,L_y$$

so that for example

$$[\,L_x\,,\,L_y\,] = i\hbar\,L_z \quad \Leftrightarrow \quad [\,\ell_{23}\,,\,\ell_{31}\,] = \ell_{12} \qquad \textit{et cetera} \tag{1.42}$$

By direct inspection it is straightforward to verify the commutation relations

$$[\,\ell_{\mu\nu}\,,\,\ell_{\rho\sigma}\,] = -\,g_{\mu\rho}\,\ell_{\nu\sigma} + \,g_{\mu\sigma}\,\ell_{\nu\rho} - \,g_{\nu\sigma}\,\ell_{\mu\rho} + \,g_{\nu\rho}\,\ell_{\mu\sigma} \tag{1.43}$$

Notice that the latter are fixed by the requirement of anti-symmetry with respect to the two pairs of indexes $\mu\nu$ and $\rho\sigma$, together with the overall sign which is provided by the commutator (1.42). Thus, if we make the correspondences

$$I_j \;\leftrightarrow\; -\,\tfrac{1}{2}\,\varepsilon_{jkl}\,\ell_{kl} \qquad J_k \;\leftrightarrow\; \ell_{0k} \qquad (\,j,k,l = 1,2,3, \quad \varepsilon_{123} = 1\,)$$

it can be checked that the above commutation relations among the differential operators $\ell_{\mu\nu}$ do realize a representation of the Lie algebra (1.28) of the Lorentz group, which is infinite dimensional as the generators are differential operators acting upon the infinite dimensional functional space $C_{\infty}(\mathcal{M})$ of the analytic functions on the Minkowski space. Furthermore, it is important to remark that in the above infinite dimensional representation of the Lorentz group all six generators are anti-Hermitean, at variance with the infinitesimal operators of all the finite dimensional irreducible representations, in which only the three generators of the rotation subgroup are anti-Hermitean.

Moreover we find

$$[\,\ell_{\mu\nu}\,,\,\partial_\rho\,] = -\,g_{\mu\rho}\,\partial_\nu + g_{\nu\rho}\,\partial_\mu$$

hence we eventually come to the infinite dimensional representation of the Lie algebra of the Poincaré group

$$\begin{aligned}
&[\,\partial_\mu\,,\,\partial_\nu\,] = 0 \qquad\qquad [\,\ell_{\mu\nu}\,,\,\partial_\rho\,] = -\,g_{\mu\rho}\,\partial_\nu + g_{\nu\rho}\,\partial_\mu \\
&[\,\ell_{\mu\nu}\,,\,\ell_{\rho\sigma}\,] = -\,g_{\mu\rho}\,\ell_{\nu\sigma} + \,g_{\mu\sigma}\,\ell_{\nu\rho} - \,g_{\nu\sigma}\,\ell_{\mu\rho} + \,g_{\nu\rho}\,\ell_{\mu\sigma}
\end{aligned} \tag{1.44}$$

However, in view of the applications to Quantum Mechanics and Quantum Field Theory, it is convenient and customary to introduce a set of Hermitean generators of the Poincaré group: namely,

$$P_\mu \equiv i\hbar\,\partial_\mu = (i\hbar\,\partial_0\,,\,i\hbar\,\boldsymbol{\nabla}) \tag{1.45}$$

$$L_{\mu\nu} \equiv i\hbar\,\ell_{\mu\nu} = x_{\,\mu}\,P_\nu - x_{\,\nu}\,P_\mu \tag{1.46}$$

and the corresponding Lie algebra

$$[\,P_\mu\,,\,P_\nu\,] = 0 \qquad [\,L_{\mu\nu}\,,\,P_\rho\,] = -\,i\hbar\,g_{\mu\rho}\,P_\nu + i\hbar\,g_{\nu\rho}\,P_\mu$$

$$[\,L_{\mu\nu}\,,\,L_{\rho\sigma}\,] = -\,i\hbar\,g_{\mu\rho}\,L_{\nu\sigma} + \,i\hbar\,g_{\mu\sigma}\,L_{\nu\rho} - \,i\hbar\,g_{\nu\sigma}\,L_{\mu\rho} + \,i\hbar\,g_{\nu\rho}\,L_{\mu\sigma} \tag{1.47}$$

One can easily recognize that the six Hermitean differential operators $L_{\mu\nu}$ do actually constitute the relativistic generalization of the orbital angular momentum operator of non-relativistic Quantum Mechanics. We find indeed, as already noticed,

$$\mathbf{L} = (L_{23}, L_{31}, L_{12}) = \mathbf{r}\,\times\,\frac{\hbar}{i}\,\boldsymbol{\nabla} \tag{1.48}$$

On the other hand, it turns out that the most general infinite dimensional representation of the Poincaré Lie algebra is given by

$$\begin{aligned}
&[\,P_\mu\,,\,P_\nu\,] = 0 \qquad [\,M_{\mu\nu}\,,\,P_\rho\,] = -\,i\hbar\,g_{\mu\rho}\,P_\nu + i\hbar\,g_{\nu\rho}\,P_\mu \\
&[\,M_{\mu\nu}\,,\,M_{\rho\sigma}\,] = -\,i\hbar\,g_{\mu\rho}\,M_{\nu\sigma} + \,i\hbar\,g_{\mu\sigma}\,M_{\nu\rho} - \,i\hbar\,g_{\nu\sigma}\,M_{\mu\rho} + \,i\hbar\,g_{\nu\rho}\,M_{\mu\sigma}
\end{aligned}$$

where

$$M_{\mu\nu} \equiv L_{\mu\nu} + S_{\mu\nu} \tag{1.49}$$

in which the relativistic spin angular momentum operator $S_{\mu\nu}$ must satisfy

$$[\, P_\mu \, , \, S_{\nu\rho} \,] = 0 = [\, S_{\mu\nu} \, , \, L_{\rho\sigma} \,] \tag{1.50}$$

$$[\, S_{\mu\nu} \, , \, S_{\rho\sigma} \,] = -\, i\hbar\, g_{\mu\rho}\, S_{\nu\sigma} + i\hbar\, g_{\mu\sigma}\, S_{\nu\rho} - i\hbar\, g_{\nu\sigma}\, S_{\mu\rho} + i\hbar\, g_{\nu\rho}\, S_{\mu\sigma} \tag{1.51}$$

Also the above construction is nothing but the relativistic generalization of the most general representation for the Lie algebra of the three dimensional Euclidean group $IO(3)$ in the non-relativistic Quantum Mechanics

$$\mathbf{p} = \frac{\hbar}{i}\,\boldsymbol{\nabla} \qquad \mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{r} \times \mathbf{p} + \mathbf{S}$$

where $\mathbf{p}, \mathbf{L}, \mathbf{S}$ are the self-adjoint operators of the momentum, orbital angular momentum and spin angular momentum respectively of a point-like particle with spin. Notice that from the correspondences (1.46) and (1.48) we get in turn

$$\mathbf{S} = (S_{23}, S_{31}, S_{12}) \tag{1.52}$$

Since the translation infinitesimal operators P_μ do evidently constitute an Abelian invariant sub-algebra, the restricted Poincaré group turns out to be a non-semi-simple and non-compact Lie group, the diagonal form of the Cartan–Killing symmetric square matrix of rank ten leading to three positive (boosts), three negative (rotations) and four null (translations) eigenvalues.

In order to single out the Casimir operators of the Poincaré group, it is useful to introduce the Pauli-Lubanski operator

$$W^{\,\mu} \equiv -\,\tfrac{1}{2}\,\varepsilon^{\,\mu\nu\rho\sigma}\, P_\nu\, M_{\rho\sigma} = -\,\tfrac{1}{2}\,\varepsilon^{\,\mu\nu\rho\sigma}\, P_\nu\, S_{\rho\sigma} \tag{1.53}$$

$$W^{\,\mu} = \begin{cases} \mathbf{P}\cdot\mathbf{S} & \text{for } \mu = 0 \\ P_0\,\mathbf{S} - \mathbf{P}\times\boldsymbol{\Upsilon} & \text{for } \mu = 1,2,3 \end{cases} \tag{1.54}$$

$$\mathbf{S} = (S_{23}, S_{31}, S_{12}) \qquad\qquad \boldsymbol{\Upsilon} = (S_{01}, S_{02}, S_{03}) \tag{1.55}$$

and its dual counterpart, *i.e.* the *Shirkov* operator

$$V^{\,\mu} \equiv S^{\,\mu\nu}\, P_\nu = \begin{cases} \mathbf{P}\cdot\boldsymbol{\Upsilon} & \text{for } \mu = 0 \\ P_0\,\boldsymbol{\Upsilon} - \mathbf{P}\times\mathbf{S} & \text{for } \mu = 1,2,3 \end{cases} \tag{1.56}$$

which satisfy the relationships

$$W_\mu\, P^{\,\mu} = 0 = V_\mu\, P^{\,\mu}$$

From the identity

$$
\begin{aligned}
-\,\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu}^{\;\;\lambda\rho\sigma} &= g^{\nu\lambda}g^{\alpha\rho}g^{\beta\sigma}+g^{\nu\rho}g^{\alpha\sigma}g^{\beta\lambda}+g^{\nu\sigma}g^{\alpha\lambda}g^{\beta\rho} \\
&- g^{\nu\rho}g^{\alpha\lambda}g^{\beta\sigma}-g^{\nu\lambda}g^{\alpha\sigma}g^{\beta\rho}-g^{\nu\sigma}g^{\alpha\rho}g^{\beta\lambda} \qquad (1.57)
\end{aligned}
$$

with $\varepsilon^{0123}=1$ a straightforward calculation yields

$$
\begin{aligned}
W^{2} &= \tfrac{1}{4}\,\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu}^{\;\;\lambda\rho\sigma}\,P_{\nu}\,P_{\lambda}\,S_{\alpha\beta}\,S_{\rho\sigma} \\
&= \tfrac{1}{2}\,P^{2}\,S^{2}-V^{2} \qquad (1.58)
\end{aligned}
$$

where we have set

$$\tfrac{1}{2}\,S^{2}=\tfrac{1}{2}\,S_{\rho\sigma}\,S^{\rho\sigma}=\mathbf{S}^{2}-\boldsymbol{\Upsilon}^{2}$$

A tedious computation allows to check that the following commutators hold true, *viz.*,

$$[\,W_{\mu}\,,\,P_{\nu}\,]=0 \qquad [\,M_{\mu\nu}\,,\,W_{\rho}\,]=-\,i\hbar\,g_{\mu\rho}\,W_{\nu}+i\hbar\,g_{\nu\rho}\,W_{\mu}$$

Thus, one can readily recognize the two Casimir operators of the Poincaré group to be the scalar and pseudo-scalar differential operators

$$C_{m}=P^{2}=P^{\mu}P_{\mu} \qquad C_{s}=W^{2}=W^{\mu}W_{\mu} \qquad (1.59)$$

which are respectively said the mass and the spin operators. Now, according to Wigner's theorem [27], any symmetry transformation in Quantum Mechanics must be realized only by means of unitary or anti-unitary operators

Eugene Paul Wigner (Budapest 17.11.1902 – Princeton 1.1.1995) *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektrum*, Fredrick Vieweg und Sohn, Braunschweig, Deutschland, 1931, pp. 251–254, *Group Theory and its Application to the Quantum Theory of Atomic Spectra*, Academic Press Inc., New York, 1959, pp. 233–236.

The representation theory for the Poincaré group has been worked out by Bargmann and Wigner (1948) [3]. The result is that ALL THE UNITARY IRREDUCIBLE INFINITE DIMENSIONAL REPRESENTATIONS of the Poincaré group have been classified and fall into four classes.

1. The eigenvalues of the Casimir operator

$$C_{m}=m^{2}c^{2}$$

are real and positive, while the eigenvalues of the Casimir operator

$$C_{s}=-\,(m\hbar c)^{2}\,s(s+1)$$

where s is the spin, do assume discrete values $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

Such a kind of unitary irreducible representations $\boldsymbol{\tau}_{m\,,\,s}$ are labeled by the rest mass $m > 0$ and the spin s . The states belonging to this kind of unitary irreducible representations are distinguished, for instance, by *e.g.* the component of the spin along the OZ axis

$$s_z = -s, -s+1, \ldots, s-1, s\ ; \qquad\qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

and by the continuous eigenvalues of the spatial momentum $\mathbf{p}$, so that $p_0^2 = \mathbf{p}^2 + m^2$: namely,

$$|\, m, s\,;\, \mathbf{p}, s_z\,\rangle \qquad m > 0 \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$\mathbf{p} \in \mathbb{R}^3 \qquad s_z = -s, -s+1, \ldots, s-1, s$$

Physically, these states will describe some elementary particle of rest mass m , spin s , momentum $\mathbf{p}$ and spin projection s_z along the OZ axis. Massive particles of spin s are described by wave fields which correspond to $2s + 1$ wave functions on the Minkowski space.

2. The eigenvalues of both the Casimir operators vanish, *i.e.* $P^{\,2} = 0$ and $W^{\,2} = 0$, and since $P^{\,\mu}\, W_\mu = 0$ it follows that W_μ and P_μ are light-like and proportional

$$W_\mu = \pm\, \hbar s\, P_\mu$$

the constant of proportionality being called the helicity, which is equal to $\pm\hbar s$, where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ is again the spin of the mass-less representation.

The states belonging to this kind of unitary irreducible representations $\boldsymbol{\tau}_{0\,,\,s}$ are distinguished by *e.g.* two possible values of the the helicity $\pm\,\hbar s$ and by the continuous eigenvalues of the momentum $\mathbf{p}$, so that $p_0^2 = \mathbf{p}^2$

$$|\, \mathbf{p}, \pm\, s\,\rangle \qquad \mathbf{p} \in \mathbb{R}^3 \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

Thus, this kind of unitary irreducible representations of the Poincaré group will correspond to the mass-less particles that, for $s \neq 0$, are described by two independent real (or one complex) wave functions on the Minkowski space. Examples of particles falling in this category are photons with spin 1 and two helicity or polarization states, the spin $\frac{1}{2}$ mass-less left-handed neutrinos ($\mathrm{h} = -\,\frac{1}{2}$) and mass-less right-handed anti-neutrinos ($\mathrm{h} = +\,\frac{1}{2}$) and maybe the graviton with spin 2 and two polarization states.

3. The eigenvalue of the Casimir mass operator is zero, $P^2 = 0$, while that one of the Casimir spin operator is continuous, *i.e.* $W^2 = -w^2$ where $w > 0$. Thus, this kind of unitary irreducible representations $\boldsymbol{\tau}_{0,w}$ of the Poincaré group would correspond to elementary particles of zero rest mass, with an infinite and continuous number of polarization. These objects have never been detected in Nature.

4. There are also tachyon-like unitary irreducible representations which $P^\mu P_\mu < 0$ which are not physical as they drive to a-causality.

There are further irreducible representations of the Poincaré group but they are neither unitary nor anti-unitary. As already remarked, Wigner's theorem [27] generally states that all symmetry transformations – just like Poincaré transformations – in Quantum Mechanics can be consistently realized *solely* by means of some unitary or anti-unitary operators.

- All the unitary representations of the Poincaré group turn out to be infinite dimensional, corresponding to particle states with unbounded momenta $\mathbf{p} \in \mathbb{R}^3$.

- Elementary particles correspond to the irreducible representations, the reducible representations being naturally associated to the composite objects. For example, the six massive quarks with spin $s = \frac{1}{2}$ and with fractional electric charge $q = \frac{2}{3}$ or $q = -\frac{1}{3}$ are considered as truly elementary particles, although not directly detectable because of the dynamical confinement mechanism provided by Quantum-Chromo-Dynamics (QCD), while hadrons, like the nucleons and π mesons, are understood as composite objects.

- As already seen, all finite dimensional irreducible representations of the Lorentz group are non-unitary.

- The relativistic quantum fields on the Minkowski space do constitute the only available way to actually implement the unitary irreducible representations of the Poincaré group, that will be thereof associated to the elementary particles obeying the general principles of Quantum Mechanics and of the Special Theory of Relativity.

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Chapter 2

The Action Functional

In this chapter I will review the main dynamical properties and structural features of the classical relativistic wave fields, as they follow from the definite transformation laws under the Poincaré group. All those fundamental aspects will be crucial in order to set up the quantum field theory. Moreover, the general requirements to be fulfilled by the classical Action, together with the meaning and the role of the continuous symmetry transformations in the classical relativistic field theory will be considered in some detail.

2.1 The Classical Relativistic Wave Fields

As we shall see further on, in order to build up the infinite dimensional Hilbert spaces that carry on all the irreducible unitary representations of the Poincaré group, which describe the quantum states of the elementary particles we detect in Nature, we will apply the canonical quantum procedure to some classical mechanical systems with an infinite number of degrees of freedom. These mechanical systems consist in a collection of real or complex functions defined on the Minkowski space $\mathcal{M}$

$$u_A(x) \,:\, \mathcal{M} \longrightarrow \left\{ \begin{array}{l} \mathbb{R} \\ \mathbb{C} \end{array} \right. \qquad (A = 1, 2, \ldots, N)$$

with a well defined transformation law under the action of the Poincaré group $IO(1,3)$. We shall call these systems the classical relativistic wave fields. More specifically, if we denote the elements of the Poincaré group by $g = (\Lambda, a) \in IO(1,3)$, where $\Lambda \in L$ is an element of the Lorentz group while a^μ specify a translation in the four dimensional Minkowski space $\mathcal{M}$, we have

$$u_A(x) \stackrel{g}{\longmapsto} u'_A(x') \;\equiv\; u'_A(\Lambda(x+a))$$

$$
\begin{aligned}
u'_A(x') &= [\,T(\Lambda)\,]_{AB}\, u_B(x) \\
&= [\,T(\Lambda)\,]_{AB}\, u_B\Big(\Lambda^{-1}x' - \mathrm{a})\Big)
\end{aligned}
\tag{2.1}
$$

$$
(\Lambda, a) \in IO(1,3) \qquad (A, B = 1, 2, \ldots, N)
$$

where $T(\Lambda)$ are the operators of a representation of the Lorentz group of finite dimensions N . This means that, if the collection of the wave field functions at the point $P \in \mathcal{M}$ of coordinates x^{μ} is given by $u_A(x)$ $(A = 1, 2, \ldots, N)$ in a certain inertial frame S , then in the new inertial frame S' , related to S by the Poincaré transformation $(\Lambda, a) \in IO(1,3)$, the space-time coordinates of P will be changed to $x' = \Lambda(x{+}a)$ and contextually the wave field functions will be reshuffled as $u'_A(x')$ $(A = 1, 2, \ldots, N)$ because the functional relationships will be in general frame dependent.

2.1.1 Field Variations

We can always represent the collection of the classical relativistic wave field functions as an $N-$component column vector

$$
u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_{N-1}(x) \\ u_N(x) \end{pmatrix} \qquad x \in \mathcal{M}
$$

Then we can suitably introduce some finite quantities which are said to be the total variation, the local variation and the differential of the classical relativistic wave field $u(x)$ according to

$$
u'(x') - u(x) \qquad \text{total variation} \tag{2.2}
$$

$$
u'(x') - u(x') \qquad \text{local variation} \tag{2.3}
$$

$$
u(x') - u(x) \qquad \text{differential variation} \tag{2.4}
$$

so that we can write

$$
u'(x') - u(x) = [\,u'(x') - u(x')\,] + [\,u(x') - u(x)\,]
$$

that is the total variation is equal to the sum of the local and of the differential variations for any classical relativistic wave field.

In order to suitably label the Lorentz matrices, it is very convenient to make use of the Minkowski notation. According to the general theorem of section 1.2.6, any rank four square matrix $\Lambda^{\mu}{}_{\nu}$ close to the identity can always be expressed in the exponential form (1.12) that reads

$$\Lambda = \exp\{\alpha_k\, I_k + \eta_\ell\, J_\ell\} = \mathbb{I} + \alpha_k\, I_k + \eta_\ell\, J_\ell + \cdots$$

where $I_k = -\, I_k^{\top}$ ($k = 1, 2, 3$) are the 4×4 real anti-symmetric generators of the rotation subgroup, while $J_\ell = J_\ell^{\top}$ ($\ell = 1, 2, 3$) are the 4×4 symmetric real generators of the special Lorentz transformations or boosts. We first rename the six canonical coordinates $(\boldsymbol{\alpha}, \boldsymbol{\eta})$ in terms of an anti-symmetric tensor parametric form according to

$$\alpha_1 = -\,\omega^{23} \qquad \alpha_2 = -\,\omega^{31} \qquad \alpha_3 = -\,\omega^{12}$$
$$\eta_1 = \omega^{01} \qquad \eta_2 = \omega^{02} \qquad \eta_3 = \omega^{03}$$

with

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0 \qquad\qquad (\, \mu, \nu = 0, 1, 2, 3\,)$$

Next we rename the generators by introducing the spin anti-symmetric tensor

$$I_j = \frac{i}{2\hbar}\, \varepsilon_{jkl}\, S_{kl} \qquad\qquad J_k = -\,\frac{i}{\hbar}\, S_{0k} \qquad\qquad S_{\rho\sigma} + S_{\sigma\rho} = 0 \tag{2.5}$$

where ε_{jkl} is the Levi-Civita symbol, totally anti-symmetric in all of its three spatial indexes and normalized as $\varepsilon_{123} \;=\; +1 \;=\; -\,\varepsilon^{123}$. Notice that we have the induced Hermitean properties

$$S_{\imath\jmath} = S_{\imath\jmath}^{\dagger} \qquad\qquad S_{0k} = \,-\, S_{0k}^{\dagger} \qquad\qquad (\, \imath, \jmath, k = 1, 2, 3\,)$$

Thus we can eventually write

$$\alpha_k\, I_k + \eta_\ell\, J_\ell \;=\; -\,(i/2\hbar)\, \omega^{\rho\sigma}\, S_{\rho\sigma}$$

in such a manner that the exponential representation of any Lorentz matrix close to the identity can always be expressed in the so called manifestly covariant form

$$\Lambda(\omega) = \exp\{-\,(i/2\hbar)\, \omega^{\rho\sigma}\, S_{\rho\sigma}\}$$

Consider now a first order infinitesimal (passive) Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu\rho}\, g_{\rho\nu} \qquad |\, \varepsilon^{\mu}{}_{\nu}\, | \ll 1 \tag{2.6}$$

in which we have neglected all the terms of higher order in the very small quantities $\varepsilon^{\mu\rho}$. On the one hand, from the defining relation (1.24) of the Lorentz matrices we obtain

$$0 = g_{\mu\rho}\, \varepsilon^{\rho}{}_{\nu} + g_{\nu\rho}\, \varepsilon^{\rho}{}_{\mu} \quad \Leftrightarrow \quad \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0$$

that is, the infinitesimal parameters $\varepsilon_{\mu\nu}$ constitute an anti-symmetric matrix with six independent entries that can be evidently identified as

$$\varepsilon_{\mu\nu} = g_{\mu\rho}\, g_{\nu\sigma}\, \delta\,\omega^{\rho\sigma}$$

in such a manner that we can also write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \frac{i}{2\hbar}\,\delta\,\omega^{\rho\sigma}\,(S_{\rho\sigma})^{\mu}{}_{\nu} \tag{2.7}$$

It is worthwhile to realize that, up to the first order, the infinitesimal rapidity parameters can be identified with the infinitesimal velocities since we find

$$\delta\eta_k = \varepsilon_{k0} = \delta\omega^{0k} = \delta\mathrm{Arsh}\left(\beta_k/\sqrt{1-\beta_k^2}\right) = \delta\beta_k$$

For example, an infinitesimal rotation around the OZ axis corresponds to

$$\begin{aligned} \alpha_1 = \alpha_2 = 0\,, \quad \alpha_3 = -\,\delta\,\omega^{12} = -\,\varepsilon_{12} \\ \Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\,\omega^{12}\,(\delta^{\mu}{}_{1}\, g_{2\nu} - \delta^{\mu}{}_{2}\, g_{1\nu}) \\ \Lambda(\alpha_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha_3 & 0 \\ 0 & -\,\alpha_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{2.8}$$

Moreover, an infinitesimal boost along the OX axis is described by

$$\begin{aligned} \beta_1 = \delta\,\omega^{01}\,, \qquad \beta_2 = \beta_3 = 0 \\ \Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\,\omega^{01}\,(\delta^{\mu}{}_{0}\, g_{1\nu} - \delta^{\mu}{}_{1}\, g_{0\nu}) \\ \Lambda(\beta_1) = \begin{pmatrix} 1 & -\beta_1 & 0 & 0 \\ -\beta_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{2.9}$$

A comparison between the expressions (2.6) and (2.7) yields

$$\begin{aligned} \varepsilon^{\mu\lambda}\, g_{\lambda\nu} &= -\,\varepsilon^{\rho\sigma}\,\delta^{\mu}{}_{\sigma}\, g_{\rho\nu} \\ &= \tfrac{1}{2}\,\varepsilon^{\rho\sigma}\,(\delta^{\mu}{}_{\rho}\, g_{\sigma\nu} - \delta^{\mu}{}_{\sigma}\, g_{\rho\nu}) \\ &= (-\,i/2\hbar)\,\delta\,\omega^{\rho\sigma}\,(S_{\rho\sigma})^{\mu}{}_{\nu} \end{aligned} \tag{2.10}$$

and thereby

$$(S_{\rho\sigma})^{\mu}{}_{\nu} = i\hbar\,\left(\delta^{\mu}{}_{\rho}\, g_{\sigma\nu} - \delta^{\mu}{}_{\sigma}\, g_{\rho\nu}\right) \tag{2.11}$$

This means that the six matrices $S_{\rho\sigma}$ do actually realize the relativistic spin angular momentum tensor of the irreducible vector representation $\boldsymbol{\tau}_{\frac{1}{2}\frac{1}{2}}$ of the Lorentz group. By the way, it is worthwhile to stress once more that the components S_{jk} are Hermitean matrices and generate rotations, while the components S_{0k} are anti-Hermitean matrices and generate boosts.

Turning back to the three kinds of infinitesimal variation we can write

$$x'^{\mu} \approx x^{\mu} + \delta x^{\mu} = x^{\mu} + \delta\omega^{\mu\nu} g_{\nu\rho} x^{\rho} + \delta\omega^{\mu} \tag{2.12}$$

$$\delta\omega^{\mu\nu} + \delta\omega^{\nu\mu} = 0 \qquad |\,\delta\omega^{\mu\nu}\,| \ll 1 \qquad |\,\delta\omega^{\mu}\,| \ll |\,x^{\mu}\,|$$

$$\begin{aligned}
\Delta u(x) &\equiv u'(x+\delta x) - u(x) \\
&\equiv [\,u'(x+\delta x) - u(x+\delta x)\,] + [\,u(x+\delta x) - u(x)\,] \\
&\equiv \delta u(x+\delta x) + \mathrm{d}u(x) \\
&= \delta u(x) + \delta x^{\mu}\,\partial_{\mu}\,\delta u(x) + \cdots + \mathrm{d}u(x) \\
&= \delta u(x) + \delta x^{\mu}\partial_{\mu} u(x) + O\,(\delta x \cdot \partial\,\delta u)
\end{aligned} \tag{2.13}$$

so that we can safely write the suggestive symbolic relation

$$\Delta \;=\; \delta \;+\; \mathrm{d} \;=\; \delta \;+\; \delta x^{\mu}\,\partial_{\mu} \tag{2.14}$$

among the first order infinitesimal variations together with

$$\Delta u(x) \approx \delta u(x) + \mathrm{d}u(x) = \delta u(x) + \delta x \cdot \partial\, u(x) \tag{2.15}$$

It is important to remark that, by definition, the local variations do commute with the tetra-gradient differential operator

$$\partial_{\mu}\,\delta u(x) = \delta\,\partial_{\mu}\,u(x) \quad \Leftrightarrow \quad [\,\delta\,,\,\partial_{\mu}\,] = 0 \tag{2.16}$$

Notice that the infinitesimal form of the Poincaré transformations for the space-time coordinates can be written in terms of the Hermitean generators

$$\delta x^{\mu} = \frac{i}{\hbar}\left[\,\tfrac{1}{2}\,\delta\omega^{\rho\sigma}\,L_{\rho\sigma}\,x^{\mu} - \,\delta\omega^{\rho}\,P_{\rho}\,x^{\mu}\,\right]$$

where – see the definition (1.46)

$$P_{\mu} = i\hbar\,\partial_{\mu} \qquad L_{\mu\nu} \equiv x_{\mu}\,P_{\nu} - x_{\nu}\,P_{\mu}$$

Here below we shall analyse the most relevant cases.

2.1.2 The Scalar and Vector Fields

1. **Scalar field**: the simplest case is that of a single invariant real function

$$\phi\,:\,\mathcal{M}\longrightarrow\mathbb{R}\qquad \phi^{\,\prime}\,(x^{\,\prime}\,)=\phi\,(x) \tag{2.17}$$

so that

$$\Delta\,\phi\,(x)=0\quad\Leftrightarrow\quad\delta\,\phi\,(x)=-\,\mathrm{d}\,\phi\,(x)=-\,\delta\,x\cdot\partial\,\phi\,(x)$$

From the infinitesimal change (2.12) we get the local variation

$$\begin{aligned}\delta\,\phi\,(x)\;&=\;-\,\delta\,\omega^{\,\mu\nu}\,g_{\,\nu\rho}\,x^{\,\rho}\,\partial_{\mu}\,\phi(x)-\,\delta\,\omega^{\,\mu}\,\partial_{\mu}\,\phi(x)\\&=\;-\,\frac{i}{\hbar}\left[\,\tfrac{1}{2}\,\delta\,\omega^{\,\mu\nu}\,L_{\,\mu\nu}-\,\delta\,\omega^{\,\mu}\,P_{\mu}\,\right]\phi\,(x)\end{aligned} \tag{2.18}$$

whence it follows that for a scalar field we find by definition

$$M_{\,\mu\nu}\;\phi\,(x)\equiv L_{\,\mu\nu}\;\phi\,(x)\quad\Leftrightarrow\quad S_{\,\mu\nu}\;\phi\,(x)\equiv 0$$

i.e. the scalar field carries relativistic orbital angular momentum but not relativistic spin angular momentum.

It is worthwhile to consider also the pseudo-scalar field, which are odd with respect to improper orthochronus Lorentz transformations, *i.e.*

$$\widetilde{\phi}^{\,\prime}\,(\Lambda x+a)=(\det\Lambda)\,\widetilde{\phi}\,(x)=-\,\widetilde{\phi}\,(x)\qquad\forall\,\Lambda\in L_{-}^{\uparrow} \tag{2.19}$$

Complex scalar (pseudo-scalar) field are complex invariant functions with scalar (pseudo-scalar) real and imaginary parts.

2. **Vector field**: a relativistic contravariant vector wave field is defined by the transformation law under the Poicaré group that reads

$$V^{\,\prime\,\mu}(x^{\,\prime}\,)=V^{\,\prime\,\mu}(\Lambda x+a)\equiv\Lambda^{\mu}{}_{\nu}\,V^{\,\nu}(x) \tag{2.20}$$

The above transformation law can be obviously generalized to the aim of defining the arbitrary relativistic tensor wave fields of any rank with many contravariant and covariant Minkowski indexes

$$\begin{aligned}T^{\,\prime\,\alpha\beta\ldots}_{\mu\nu\ldots}\,(x^{\,\prime}\,)\;&=\;T^{\,\prime\,\alpha\beta\ldots}_{\mu\nu\ldots}\,(\Lambda x+a)\\&\equiv\;\Lambda^{\,\rho}_{\,\mu}\,\Lambda^{\,\sigma}_{\,\nu}\,\ldots\,\Lambda^{\,\alpha}_{\,\lambda}\,\Lambda^{\,\beta}_{\,\kappa}\,\ldots\,T^{\,\lambda\kappa\ldots}_{\rho\sigma\ldots}\,(x)\end{aligned} \tag{2.21}$$

It follows therefrom that the components of a vector field transform according to the irreducible vector representation $\boldsymbol{\tau}_{\frac{1}{2}\frac{1}{2}}$ of the Lorentz group. For infinitesimal Lorentz transformations we can write

$$\Lambda^{\,\mu}{}_{\nu}=\delta^{\,\mu}{}_{\nu}+\delta\omega^{\,\mu\rho}\,g_{\,\rho\nu}\qquad\delta\omega^{\,\mu\rho}+\delta\omega^{\,\rho\mu}=0$$

and consequently

$$\begin{aligned}\Delta V_\mu(x) &= \delta\omega^{\rho\sigma} g_{\mu\rho} V_\sigma(x) \\ &\equiv -\frac{i}{2\hbar}\,\delta\omega^{\rho\sigma}\,(S_{\rho\sigma})_\mu^{\ \nu}\,V_\nu(x) \qquad (2.22)\end{aligned}$$

the generators of the total variation of the covariant vector field under a Poincaré transformation being the relativistic spin angular momentum matrices (2.5), the action of which actually reads

$$\begin{aligned}(S_{\rho\sigma})_\mu^{\ \nu} V_\nu(x) &= i\hbar\, g_{\mu\rho} V_\sigma(x) - i\hbar\, g_{\sigma\mu} V_\rho(x) \\ &\equiv S_{\rho\sigma} * V_\mu(x) \qquad (2.23)\end{aligned}$$

in which

$$(S_{\rho\sigma})_\mu^{\ \nu} = i\hbar\left[\, g_{\mu\rho}\,\delta_\sigma^{\ \nu} - g_{\mu\sigma}\,\delta_\rho^{\ \nu}\,\right] \qquad (2.24)$$

The above expression can be readily checked taking the four gradient of a real scalar field. As a matter of fact we obtain

$$\partial_\mu \phi(x) \equiv V_\mu(x) \qquad (2.25)$$

so that from eq. (2.18) we find the infinitesimal transformation law

$$\begin{aligned}\Delta V_\mu(x) &= \Delta\partial_\mu\phi(x) \\ &= [\,\Delta\,,\,\partial_\mu\,]\,\phi(x) + \partial_\mu\Delta\,\phi(x) \\ &= [\,\Delta\,,\,\partial_\mu\,]\,\phi(x) \qquad (2.26)\end{aligned}$$

Now we have

$$\begin{aligned}[\,\Delta\,,\,\partial_\mu\,] &= [\,\delta\,,\,\partial_\mu\,] + [\,\delta x\cdot\partial\,,\,\partial_\mu\,] \\ &= \delta\omega^{\lambda\nu} g_{\nu\rho}\,[\,x^\rho\,,\,\partial_\mu\,]\,\partial_\lambda \\ &= -\,\delta\omega^{\lambda\nu} g_{\nu\rho}\,\delta^{\ \rho}_\mu\,\partial_\lambda \\ &= \delta\omega^{\lambda\nu} g_{\lambda\mu}\,\partial_\nu \qquad (2.27)\end{aligned}$$

and thereby

$$\Delta\partial_\mu\phi(x) = \delta\omega^{\rho\sigma} g_{\mu\rho}\,\partial_\sigma\phi(x)$$

whence eq. (2.22) immediately follows.

From the symbolic relation (2.14) we obtain the expression for the local variation of a relativistic covariant vector wave field

$$\begin{aligned}\delta V_\rho(x) &= \Delta V_\rho(x) - \delta x^\mu\,\partial_\mu V_\rho(x) \\ &= -\,\delta\omega^{\mu\nu}\left[\frac{i}{2\hbar}\,S_{\mu\nu} * V_\rho(x) + x_\nu\,\partial_\mu V_\rho(x)\right] - \delta\omega^\mu\,\partial_\mu V_\rho(x) \\ &= -\,\frac{i}{\hbar}\left[\,\tfrac{1}{2}\,\delta\omega^{\mu\nu} M_{\mu\nu} * V_\rho(x) - \delta\omega^\mu P_\mu V_\rho(x)\,\right] \qquad (2.28)\end{aligned}$$

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} = x_{\mu} P_{\nu} - x_{\nu} P_{\mu} + S_{\mu\nu}$$

Well known examples of vector and tensor wave fields are the vector potential and the field strength of the electromagnetic field

$$\begin{aligned}
&A'_{\mu}(x') = \Lambda_{\mu}^{\ \nu} A_{\nu}(x) \\
&\Delta A_{\mu}(x) = \delta\omega^{\lambda\nu} g_{\mu\lambda} A_{\nu}(x) \\
&F'_{\mu\nu}(x') = \Lambda_{\mu}^{\ \rho} \Lambda_{\nu}^{\ \sigma} F_{\rho\sigma}(x) \\
&\Delta F_{\mu\nu}(x) = \delta\omega^{\lambda\rho} \left[g_{\mu\lambda} F_{\rho\nu}(x) + g_{\nu\lambda} F_{\mu\rho}(x) \right]
\end{aligned} \tag{2.29}$$

A tensor wave field of any rank with many contravariant and covariant indexes will exploit a local variation in accordance with the obvious generalization of the infinitesimal change (2.28). In particular, the action of the relativistic spin matrix on a tensor wave field will be the (algebraic) sum of expressions like (2.5), one for each index. For instance, the action of the spin matrix on the electromagnetic field strength is given by

$$\begin{aligned}
-\frac{i}{\hbar} S_{\rho\sigma} * F_{\mu\nu}(x) &= g_{\rho\mu} F_{\sigma\nu}(x) - g_{\sigma\mu} F_{\rho\nu}(x) \\
&+ g_{\rho\nu} F_{\mu\sigma}(x) - g_{\sigma\nu} F_{\mu\rho}(x)
\end{aligned} \tag{2.30}$$

If we consider a full parity transformation (1.25) $\Lambda_P \in L_{-}^{\uparrow}$ we have

$$A'_{\mu}(\Lambda_P x) = (\Lambda_P)_{\mu}^{\ \nu} A_{\nu}(x) \tag{2.31}$$

that yields

$$A'_0(x_0, -\mathbf{x}) = A_0(x_0, \mathbf{x}) \qquad \mathbf{A}'(x_0, -\mathbf{x}) = -\mathbf{A}(x_0, \mathbf{x}) \tag{2.32}$$

Conversely, a covariant pseudo-vector wave field will be defined by the transformation law

$$\widetilde{V}'_{\mu}(\Lambda x) = (\det \Lambda) \Lambda_{\mu}^{\ \nu} \widetilde{V}_{\nu}(x) \qquad \forall \Lambda \in L_{-}^{\uparrow} \tag{2.33}$$

so that under parity

$$\widetilde{V}'_0(x_0, -\mathbf{x}) = -\widetilde{V}_0(x_0, \mathbf{x}) \qquad \widetilde{\mathbf{V}}'(x_0, -\mathbf{x}) = \widetilde{\mathbf{V}}(x_0, \mathbf{x}) \tag{2.34}$$

2.1.3 The Spinor Fields

The two irreducible fundamental representations $\boldsymbol{\tau}_{\frac{1}{2}\,0}$ and $\boldsymbol{\tau}_{0\,\frac{1}{2}}$ of the homogeneous Lorentz group can be realized by means of $SL(2,\mathbb{C})$, *i.e.* the group of complex 2×2 matrices of unit determinant. The $SL(2,\mathbb{C})$ matrices belonging to $\boldsymbol{\tau}_{\frac{1}{2}\,0}$ act upon the so called Weyl's 2-component left spinors, whilst the $SL(2,\mathbb{C})$ matrices belonging to $\boldsymbol{\tau}_{0\,\frac{1}{2}}$ act upon the so called Weyl's 2-component right spinors.

In any neighborhood of the unit element, the $SL(2,\mathbb{C})$ matrices can always be presented in the exponential form

$$\Lambda_L \equiv \exp\left\{\tfrac{1}{2}\, i\,\sigma_k\,(\alpha_k - i\eta_k)\right\} \tag{2.35}$$

$$\Lambda_R \equiv \exp\left\{\tfrac{1}{2}\, i\,\sigma_k\,(\alpha_k + i\eta_k)\right\} \tag{2.36}$$

in which

$$\alpha_k\,,\ \beta_k = \frac{v_k}{c}\,,\ \eta_k = \mathrm{Arsh}\left(\beta_k(1-\beta_k^2)^{-1/2}\right) \qquad (\,k=1,2,3\,)$$

are respectively the angular canonical coordinates, the relative velocity components (1.26) and rapidity (1.27) parameters of the Lorentz group, whereas σ_k $(k=1,2,3)$ are the Pauli matrices. Notice that

$$\sigma_2\,\sigma_k\,\sigma_2 = -\,\sigma_k^* = -\,\sigma_k^{\top} \qquad (k=1,2,3) \tag{2.37}$$

$$\begin{aligned} &\sigma_j\,\sigma_k = \delta_{jk} + i\varepsilon_{\,jkl}\,\sigma_l \qquad && \{\sigma_j\,\sigma_k\} = 2\delta_{jk} \\ &[\,\sigma_j\,\sigma_k\,] = 2i\varepsilon_{\,jkl}\,\sigma_l && (\,j,k,l=1,2,3\,) \end{aligned} \tag{2.38}$$

It is clear that for $\beta_k = \eta_k = 0$ $(k=1,2,3)$, *i.e.* for rotations, we have that $\Lambda_L(\alpha) = \Lambda_R(\alpha) \in SU(2)$ whereas for $\alpha_k = 0$ $(k=1,2,3)$, *i.e.* for boosts, the matrices $\Lambda_{L,R}(\eta)$ are Hermitean and non-unitary.

Let us therefore introduce the relativistic two-component Weyl's spinor wave fields

$$\psi_L(x) \equiv \begin{pmatrix} \psi_{L1}(x) \\ \psi_{L2}(x) \end{pmatrix} \qquad \psi_R(x) \equiv \begin{pmatrix} \psi_{R1}(x) \\ \psi_{R2}(x) \end{pmatrix} \tag{2.39}$$

which, by definition, transform according to

$$\psi_L'(x') = \Lambda_L\,\psi_L(x) \qquad \psi_R'(x') = \Lambda_R\,\psi_R(x) \tag{2.40}$$

The infinitesimal form of the above transformation laws give rise to the total variations

$$
\begin{aligned}
\Delta\,\psi_L\,(x) &= \tfrac{1}{2}\,i\,(\sigma_j\,\delta\alpha_{\,j} - i\,\sigma_k\,\delta\beta_k)\,\psi_L\,(x) \\
&= -\,\tfrac{1}{2}\,i\,\left(\sigma_j\,\tfrac{1}{2}\,\varepsilon_{jkl}\,\delta\,\omega^{\,kl} + i\sigma_k\,\delta\,\omega^{\,0k}\right)\psi_L\,(x) \\
&\equiv -\,\frac{i}{2\hbar}\,\delta\,\omega^{\,\rho\sigma}\,(S_{\rho\sigma})_L\,\psi_L\,(x)
\end{aligned} \tag{2.41}
$$

and quite analogously

$$
\begin{aligned}
\Delta\,\psi_R\,(x) &= \tfrac{1}{2}\,i\,(\sigma_j\,\delta\alpha_{\,j} + i\sigma_k\,\delta\beta_k)\,\psi_R\,(x) \\
&= -\,\tfrac{1}{2}\,i\,\left(\sigma_j\,\tfrac{1}{2}\,\varepsilon_{jk\ell}\,\delta\,\omega^{\,k\ell} - i\sigma_k\,\delta\,\omega^{\,0k}\right)\psi_R\,(x) \\
&\equiv -\,\frac{i}{2\hbar}\,\delta\,\omega^{\,\rho\sigma}\,(S_{\rho\sigma})_R\,\psi_R\,(x)
\end{aligned} \tag{2.42}
$$

whence we identify

$$
(S_{k\ell})_L \;=\; \tfrac{1}{2}\,\hbar\varepsilon_{jk\ell}\,\sigma_j \;=\; (S_{k\ell})_R \tag{2.43}
$$

$$
(S_{0k})_L \;=\; \tfrac{1}{2}\,i\hbar\sigma_k \;=\; (S^{0k})_R \tag{2.44}
$$

From the symbolic relation (2.14) we can easily obtain the expression for the local variation of both relativistic Weyl's spinor fields.

The $SL(2,\mathbb{C})$ matrices do satisfy some important properties :

(a) $\quad \Lambda_L^{-1} = \Lambda_R^{\dagger} \qquad \Lambda_R^{-1} = \Lambda_L^{\dagger}$

(b) $\quad \sigma_2\,\Lambda_L\,\sigma_2 = \Lambda_R^{*} \qquad \sigma_2\,\Lambda_R\,\sigma_2 = \Lambda_L^{*}$

(c) $\quad \Lambda_L^{\top} = \sigma_2\,\Lambda_L^{-1}\,\sigma_2 \qquad \Lambda_R^{\top} = \sigma_2\,\Lambda_R^{-1}\,\sigma_2$

`Proof`

(a) We have

$$
\begin{aligned}
\Lambda_R^{\dagger} &= \exp\left\{-\,\tfrac{1}{2}\,i\sigma_k\,(\alpha_k - i\eta_k)\right\} \\
&= \exp\left\{\tfrac{1}{2}\,i\sigma_k\,(-\,\alpha_k + i\eta_k)\right\} \\
&= \Lambda_L(-\,\alpha, -\,\eta) = \Lambda_L^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_L^{\dagger} &= \exp\left\{-\,\tfrac{1}{2}\,i\sigma_k\,(\alpha_k + i\eta_k)\right\} \\
&= \exp\left\{\tfrac{1}{2}\,i\sigma_k\,(-\,\alpha_k - i\eta_k)\right\} \\
&= \Lambda_R(-\,\alpha, -\,\eta) = \Lambda_R^{-1}
\end{aligned}
$$

(b) From the definition of the exponential of a matrix we get

$$\begin{aligned}
\sigma_2\,\Lambda_L\,\sigma_2 &= \sigma_2\,\exp\left\{\frac{i}{2}\,\sigma_k\,(\alpha_k - i\eta_k)\right\}\,\sigma_2 \\
&= \sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n \frac{1}{n!}\,\sigma_2\,\sigma_{k_1}\,\sigma_{k_2}\,\cdots\,\sigma_{k_n}\,\sigma_2 \\
&\times\ (\alpha_{k_1} - i\eta_{k_1})\,(\alpha_{k_2} - i\eta_{k_2})\,\cdots\,(\alpha_{k_n} - i\eta_{k_n}) \\
&= \sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n \frac{1}{n!}\,\sigma^*_{k_1}\,\sigma^*_{k_2}\,\cdots\,\sigma^*_{k_n} \\
&\times\ (-\,\alpha_{k_1} + i\eta_{k_1})\,(-\,\alpha_{k_2} + i\eta_{k_2})\,\cdots\,(-\,\alpha_{k_n} + i\eta_{k_n}) \\
&= \exp\left\{\frac{i}{2}\,\sigma^*_k\,(-\,\alpha_k + i\eta_k)\right\} \\
&= \left(\exp\left\{\frac{i}{2}\,\sigma_k\,(\alpha_k + i\eta_k)\right\}\right)^* = \Lambda^*_R
\end{aligned}$$

and in the very same way we prove that $\sigma_2\,\Lambda_R\,\sigma_2 = \Lambda^*_L$.

(c) Finally we get

$$\begin{aligned}
\sigma_2\,\Lambda_L^{-1}\,\sigma_2 &= \sigma_2\,\Lambda_R^{\dagger}\,\sigma_2 \\
&= \sigma_2\,\exp\left\{-\,\tfrac{1}{2}\,i\sigma_k\,(\alpha_k - i\eta_k)\right\}\,\sigma_2 \\
&= \left(\exp\left\{\tfrac{1}{2}\,i\sigma_k\,(\alpha_k - i\eta_k)\right\}\right)^{\top} = \Lambda^{\top}_L
\end{aligned}$$

and repeating step-by-step we prove that $\sigma_2\,\Lambda_R^{-1}\,\sigma_2 = \Lambda^{\top}_R$ □

The above listed relations turn out to be rather useful to single out Lorentz invariant combinations out of the Weyl's spinors. Consider for instance

$$\left(\sigma_2\,\psi^*_L\right)' = \sigma_2\,(\Lambda_L\psi_L)^* = \sigma_2\Lambda^*_L\sigma_2\,\sigma_2\psi^*_L = \Lambda_R\,\sigma_2\psi^*_L$$

which means that $\sigma_2\,\psi^*_L \in \boldsymbol{\tau}_{0\,\frac{1}{2}}$ and correspondingly $\sigma_2\,\psi^*_R \in \boldsymbol{\tau}_{\frac{1}{2}\,0}$.
We show now that the anti-symmetric combination of the two Weyl representations of the Lorentz group transforms according to the scalar representation. As a matter of fact, on the one hand we find

$$\chi^{\top}_L\,i\sigma_2\,\psi_L = \chi_{L\,1}\psi_{L\,2} - \chi_{L\,2}\psi_{L\,1} \tag{2.45}$$

On the other hand we have

$$(\chi^{\top}_L\,\sigma_2\,\psi_L)' = \chi^{\top}_L\Lambda^{\top}_L\,\sigma_2\,\Lambda_L\psi_L = \chi^{\top}_L\,\sigma_2\,\psi_L \in \boldsymbol{\tau}_{0\,0} \tag{2.46}$$

From the anti-symmetry of the above invariant combination, it follows that $\psi_L^\top\,\sigma_2\,\psi_L \equiv 0$ for ordinary $c\,-$number valued Weyl's left spinors. If we now take $\chi_L \equiv \sigma_2\,\psi_R^*$ then we obtain the complex invariants

$$\mathfrak{I} = \psi_R^{\top *}\sigma_2^\top\sigma_2\,\psi_L = -\,\psi_R^\dagger\,\psi_L \qquad \mathfrak{I}^* = -\,\psi_L^\dagger\,\psi_R \tag{2.47}$$

We can build up a four vector out of a single Weyl left spinor fields. To this purpose, let me recall the transformation laws of a contravariant four vector under passive infinitesimal boosts and rotations respectively

$$\left.\begin{array}{c} \delta V^{\,0} = \varepsilon^{0}{}_{k}\,V^k = -\,\varepsilon^{0k}\,V^k = -\,\delta\,\beta_k\,V^k \\ \delta V^{j} = \varepsilon^{j}{}_{0}\,V^{\,0} = \varepsilon^{j0}\,V^{\,0} = -\,\delta\,\beta_j\,V^{\,0} \end{array}\right\} \qquad (\,\mathsf{boosts}\,)$$

$$\left.\begin{array}{c} \delta V^{\,0} = 0 \\ \delta V^{j} = \varepsilon^{j}{}_{k}\,V^k = -\,\varepsilon^{jk}\,V^k = -\,\varepsilon^{jk\ell}\,V^k\,\delta\,\alpha_\ell \end{array}\right\} \qquad (\,\mathsf{rotations}\,)$$

Consider in fact the left combination

$$\psi_L^\dagger(x)\,\sigma^{\,\mu}\,\psi_L(x) \qquad \sigma^{\,\mu}\;\equiv\;(\mathbf{1},-\,\sigma_k) \qquad (\,k=1,2,3\,)$$

Under a passive infinitesimal Lorentz transformation the total variation (2.41) yields

$$\begin{aligned} & \Delta\Big(\psi_L^\dagger(x)\,\sigma^{\,\mu}\,\psi_L(x)\Big) \;= \\ - & \;\tfrac{1}{2}\,\psi_L^\dagger(x)\,i\sigma_k\,\sigma^{\,\mu}\,\psi_L(x)\,(\delta\,\alpha_{\,k}+i\delta\,\beta_{\,k}) \\ + & \;\tfrac{1}{2}\,\psi_L^\dagger(x)\,i\sigma^{\,\mu}\,\sigma_k\,\psi_L(x)\,(\delta\,\alpha_{\,k}-i\delta\,\beta_{\,k}) \\ = & \;\tfrac{1}{2}\,\psi_L^\dagger(x)\,[\,\sigma^{\,\mu}\,,\,\sigma_k\,]\,\psi_L(x)\,i\delta\,\alpha_{\,k}+\tfrac{1}{2}\,\psi_L^\dagger(x)\,\{\sigma_k\,,\,\sigma^{\,\mu}\}\,\psi_L(x)\,\delta\,\beta_{\,k} \\ = & \;\left\{\begin{array}{ll} \psi_L^\dagger(x)\,\sigma_k\,\psi_L(x)\,\delta\,\beta_{\,k} = -\,\psi_L^\dagger(x)\,\sigma^k\,\psi_L(x)\,\delta\,\beta_{\,k} & (\,\mu=0\,) \\ -\,\psi_L^\dagger(x)\,\psi_L(x)\,\delta\,\beta_{\,j}+\varepsilon_{\,jk\ell}\,\psi_L^\dagger(x)\,\sigma_\ell\,\psi_L(x)\,\delta\alpha_k & (\,\mu=j\,) \end{array}\right. \\ = & \;-\,\frac{i}{2\hbar}\,\delta\,\omega^{\,\rho\sigma}\,(S_{\,\rho\sigma})^{\,\mu}{}_{\nu}\,\psi_L^{\,\dagger}\,(x)\,\sigma^{\,\nu}\,\psi_L(x) \end{aligned} \tag{2.48}$$

which means that the left-handed combination

$$V_L^{\mu}\,(x) \equiv \psi_L^\dagger(x)\,\sigma^{\,\mu}\,\psi_L(x) = V_L^{\mu\,\dagger}(x) \qquad \sigma^{\,\mu} = (\mathbf{1},-\,\sigma_k)$$

transforms under the restricted Lorentz group like a contravariant real vector field, that is

$$V_L^{\prime\,\mu}\,(x^{\,\prime}) = \Lambda^{\mu}{}_{\nu}\,\psi_L^\dagger(x)\,\sigma^{\,\nu}\,\psi_L(x) = \Lambda^{\mu}{}_{\nu}\,V_L^{\,\nu}\,(x) \tag{2.49}$$

In the very same way one can see that the right combination

$$V_R^{\mu}(x) \equiv \psi_R^{\dagger}(x)\,\bar{\sigma}^{\mu}\,\psi_R(x) = V_R^{\mu\,\dagger}(x) \qquad \bar{\sigma}^{\mu} = (\mathbf{1}, \sigma_k)$$

transform under the Lorentz group like a contravariant real vector field

$$V_R'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}\,\psi_R^{\dagger}(x)\,\bar{\sigma}^{\nu}\,\psi_R(x) = \Lambda^{\mu}{}_{\nu}\,V_R^{\nu}(x) \tag{2.50}$$

In conclusion we see that the following relationships hold true: namely,

$$\Lambda_L^{\dagger}\,\sigma^{\mu}\,\Lambda_L = \Lambda^{\mu}{}_{\nu}\,\sigma^{\nu} \qquad \Lambda_R^{\dagger}\,\bar{\sigma}^{\mu}\,\Lambda_R = \Lambda^{\mu}{}_{\nu}\,\bar{\sigma}^{\nu} \tag{2.51}$$

It becomes now easy to build up the Lorentz invariant real kinetic terms

$$\begin{aligned} \mathfrak{T}_L &= \tfrac{1}{2}\,\psi_L^{\dagger}(x)\,\sigma^{\mu}\,i\partial_{\mu}\psi_L(x) - \tfrac{1}{2}[\,i\partial_{\mu}\psi_L^{\dagger}(x)\,]\sigma^{\mu}\,\psi_L(x) \\ &\equiv \tfrac{1}{2}\,\psi_L^{\dagger}(x)\,i\sigma^{\mu}\,\overset{\leftrightarrow}{\partial}_{\mu}\psi_L(x) \qquad (2.52) \\ \mathfrak{T}_R &= \tfrac{1}{2}\,\psi_R^{\dagger}(x)\,\bar{\sigma}^{\mu}\,i\partial_{\mu}\psi_R(x) - \tfrac{1}{2}[\,i\partial_{\mu}\psi_R^{\dagger}(x)\,]\bar{\sigma}^{\mu}\,\psi_R(x) \\ &\equiv \tfrac{1}{2}\,\psi_R^{\dagger}(x)\,i\bar{\sigma}^{\mu}\,\overset{\leftrightarrow}{\partial}_{\mu}\psi_R(x) \qquad (2.53) \end{aligned}$$

When the full Lorentz group is a concern, we have already seen (1.29) that it is necessary to consider the direct sum of the two nonequivalent irreducible Weyl's representation. In so doing, we are led to the so called four components bispinor or Dirac relativistic spinor wave field

$$\psi(x) \equiv \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \begin{pmatrix} \psi_{L\,1}(x) \\ \psi_{L\,2}(x) \\ \psi_{R\,1}(x) \\ \psi_{R\,2}(x) \end{pmatrix} \tag{2.54}$$

The full parity transformation, also named space inversion,

$$\mathbb{P}\,:\;\psi_{L\,,\,R} \leftrightarrow \psi_{R\,,\,L}$$

is then represented by the 4×4 matrix

$$\gamma^0 \equiv \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

so that

$$
\begin{aligned}
\psi'(x') &= \psi'(x_0, -\mathbf{x}) = (\mathbb{P}\,\psi)(x) \\
&= \gamma^0\,\psi(x) = \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix}
\end{aligned}
\tag{2.55}
$$

that means

$$
\psi'_L(x_0, -\mathbf{x}) = \psi_R(x_0, \mathbf{x}) \qquad\qquad \psi'_R(x_0, -\mathbf{x}) = \psi_L(x_0, \mathbf{x})
$$

The left and right Weyl's components can be singled out by means of the two projectors

$$
P_L \equiv \tfrac{1}{2}\,(\mathbb{I} - \gamma_5\,) \qquad P_R \equiv \tfrac{1}{2}\,(\mathbb{I} + \gamma_5\,)
$$

where

$$
\mathbb{I} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.56}
$$

$$
\gamma_5 = \gamma^5 \equiv \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.57}
$$

Starting from the Lorentz invariant complex bilinears in the Weyl spinors (2.47), one can easily build up the Lorentz and parity invariant real bilinear in the Dirac spinors, *viz.,*

$$
\mathfrak{I} + \mathfrak{I}^* = -\,\psi_R^\dagger\,\psi_L - \psi_L^\dagger\,\psi_R = -\,\psi^\dagger\gamma^0\psi \equiv -\,\bar{\psi}\psi \tag{2.58}
$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$ is said to be the adjoint spinor. In the very same way we can construct the parity even and Lorentz invariant Dirac kinetic term

$$
\mathfrak{T}_D = \tfrac{1}{2}\,\bar{\psi}(x)\,\gamma^\mu\, i\overset{\leftrightarrow}{\partial}_\mu\,\psi(x) = \mathfrak{T}_L + \mathfrak{T}_R \tag{2.59}
$$

which is real, where we have eventually introduced the set of matrices

$$
\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.60}
$$

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.61}$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{2.62}$$

$$\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.63}$$

$$\gamma_5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.64}$$

The above set of five 4×4 matrices are said to be the Dirac matrices in the Weyl or chiral or even spinorial representation. The gamma matrices do satisfy the so called Clifford algebra

$$\{\gamma^{\mu}\,,\,\gamma^{\nu}\} = 2g^{\mu\nu} \qquad \{\gamma^{\mu}\,,\,\gamma_5\} = 0 \tag{2.65}$$

in which

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

Notice that we have the Hermitean conjugation properties

$$\gamma^{\mu\,\dagger} = \gamma^0\gamma^{\mu}\gamma^0 \qquad \gamma_5^{\dagger} = \gamma_5 \tag{2.66}$$

and moreover

$$\gamma^{\mu} = \begin{pmatrix} 0 & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & 0 \end{pmatrix} \qquad \bar{\sigma}^{\mu} \equiv (\mathbf{1}, \boldsymbol{\sigma}) \qquad \sigma^{\mu} \equiv (\mathbf{1}, -\,\boldsymbol{\sigma})$$

$$\gamma^0\gamma^{\mu} = \begin{pmatrix} \sigma^{\mu} & 0 \\ 0 & \bar{\sigma}^{\mu} \end{pmatrix} \equiv \alpha^{\mu}$$

We have already seen the transformation laws of the left-handed (2.41) and right-handed (2.42) Weyl's spinors under the restricted Lorentz group. If we consider an infinitesimal boost $\delta\,\omega^{\,kl} = 0$ $(k, l = 1, 2, 3)$ both transformation laws can be written in terms of a Dirac spinor as

$$\Delta\,\psi(x) = \tfrac{1}{4}\,\delta\,\omega_{\,0k}\,(\gamma^0\gamma^k - \gamma^k\gamma^0)\,\psi(x) \qquad (k = 1, 2, 3) \tag{2.67}$$

Analogously, the transformation laws under an infinitesimal rotation $\delta\omega^{0k} = 0$ $(k = 1,2,3)$ can be written together in the form

$$\Delta\,\psi(x) = \tfrac{1}{8}\,\delta\omega^{jk}(\gamma^j\gamma^k - \gamma^k\gamma^j)\,\psi(x) \qquad (j,k = 1,2,3) \tag{2.68}$$

This means that, if we introduce

$$(\,S^{\mu\nu}\,)_D \;\equiv\; \hbar\sigma^{\mu\nu} \;=\; \frac{i\hbar}{4}\,[\,\gamma^{\mu}\,,\,\gamma^{\nu}\,] \tag{2.69}$$

we eventually obtain that the transformation law for the Dirac spinors under the restricted Lorentz group becomes

$$\Delta\,\psi(x) \;=\; -\,\tfrac{1}{2}\,i\sigma^{\mu\nu}\,\psi(x)\,\delta\omega_{\mu\nu} \tag{2.70}$$

which leads us to identify, in natural units, the boost and the rotation generators for the Dirac bispinors as

$$\hbar^{-1}(\,S^{0k}\,)_D = \frac{i}{4}\,[\,\gamma^0\,,\,\gamma^k\,] = \frac{1}{2}\begin{pmatrix} -i\sigma_k & 0 \\ 0 & i\sigma_k \end{pmatrix}$$

and respectively

$$\hbar^{-1}(\,S^{jk}\,)_D = \frac{i}{4}\,[\,\gamma^j\,,\,\gamma^k\,] = \tfrac{1}{2}\,\varepsilon^{jk\ell}\begin{pmatrix} \sigma_\ell & 0 \\ 0 & \sigma_\ell \end{pmatrix} \equiv \tfrac{1}{2}\,\varepsilon^{jk\ell}\,\Sigma^\ell$$

Hence the covariant spin angular momentum tensor operator for the relativistic Dirac spinor wave field reads

$$\sigma_{\mu\nu} \;=\; \tfrac{1}{4}\,ig_{\mu\lambda}\,g_{\nu\kappa}\,[\,\gamma^{\lambda}\,,\,\gamma^{\kappa}\,] \tag{2.71}$$

the spatial components being Hermitean whilst the spatial temporal components being anti-Hermitean, that is

$$(\,\sigma^{\mu\nu}\,)^{\dagger} = \gamma^0\,\sigma^{\mu\nu}\,\gamma^0 \qquad\qquad \sigma_{\mu\nu} = \gamma^0\,\sigma^{\dagger}_{\mu\nu}\,\gamma^0 \tag{2.72}$$

By construction, the six components of the spin angular momentum tensor of the Dirac field enjoy the Lie algebra of the Lorentz group

$$[\,\sigma^{\mu\nu}\,,\,\sigma^{\lambda\kappa}\,] = -\,ig^{\mu\lambda}\,\sigma^{\nu\kappa} + ig^{\mu\kappa}\,\sigma^{\nu\lambda} -\,ig^{\nu\kappa}\,\sigma^{\mu\lambda} + ig^{\nu\lambda}\,\sigma^{\mu\kappa}$$

Actually, it is worthwhile to remark that, *mutatis mutandis*, the above construction keeps true in any $D-$dimensional space with a symmetry group $O(m,n)$ in which

$$D = m + n \qquad m \geq 0 \quad n \geq 0$$

$$x^2 = \sum_{k=1}^{m} x_k^2 - \sum_{j=1}^{n} x_{m+j}^2$$

For any finite passive transformation of the restricted Lorentz group we have

$$\psi'(x') \;=\; \Lambda_{\frac{1}{2}}(\omega)\,\psi(x) \;=\; \exp\left\{-\,\tfrac{1}{2}\,i\sigma_{\mu\nu}\,\omega^{\mu\nu}\right\}\,\psi(x) \qquad (2.73)$$

As an example, for a boost along the positive OY axis we find

$$\begin{aligned}
\Lambda_{\frac{1}{2}}(\eta) \;&=\; \exp\left\{-\,i\sigma_{02}\,\omega^{02}\right\} = \cosh\frac{\eta}{2} - \gamma^0\gamma^2\sinh\frac{\eta}{2} = \Lambda^{\dagger}_{\frac{1}{2}}(\eta) \\
&=\; \begin{pmatrix} \cosh\eta/2 & i\sinh\eta/2 & 0 & 0 \\ -i\sinh\eta/2 & \cosh\eta/2 & 0 & 0 \\ 0 & 0 & \cosh\eta/2 & -i\sinh\eta/2 \\ 0 & 0 & i\sinh\eta/2 & \cosh\eta/2 \end{pmatrix}
\end{aligned}$$

where $\sinh\eta = v_y\,(1 - v_y^2)^{-1/2}\;(v_y > 0)\,.$

From the Hermitean conjugation property (2.72) we can write

$$\begin{aligned}
\Lambda^{\dagger}_{\frac{1}{2}}(\omega) \;&=\; \exp\left\{\tfrac{1}{2}\,i\sigma^{\dagger}_{\mu\nu}\,\omega^{\mu\nu}\right\} \\
&=\; \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\left(\frac{i}{2}\right)\,\gamma^0\,\sigma_{\mu_j\rho_j}\,\gamma^0\,\omega^{\mu_j\rho_j} \\
&=\; \gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n\,(\,\sigma_{\mu\nu}\,\omega^{\mu\nu}\,)^n\,\gamma^0 \\
&=\; \gamma^0\,\exp\left\{\tfrac{1}{2}\,i\sigma_{\mu\nu}\,\omega^{\mu\nu}\right\}\,\gamma^0 \\
&=\; \gamma^0\,\Lambda^{-1}_{\frac{1}{2}}(\omega)\,\gamma^0 = \gamma^0\,\Lambda_{\frac{1}{2}}(-\,\omega)\,\gamma^0 \qquad (2.74)
\end{aligned}$$

which entails in turn the two further relations

$$\gamma^0\,\Lambda^{\dagger}_{\frac{1}{2}}(\omega)\,\gamma^0 = \Lambda^{-1}_{\frac{1}{2}}(\omega) = \Lambda_{\frac{1}{2}}(-\,\omega) \qquad (2.75)$$

$$\left(\Lambda^{-1}_{\frac{1}{2}}(\omega)\right)^{\dagger}\gamma^0\,\Lambda^{-1}_{\frac{1}{2}}(\omega) = \Lambda^{\dagger}_{\frac{1}{2}}(-\omega)\,\gamma^0\,\Lambda^{-1}_{\frac{1}{2}}(\omega) = \gamma^0 \qquad (2.76)$$

From the Lorentz invariance of the Dirac real kinetic term (2.59) it follows that the bilinear

$$\bar{\psi}(x)\,\gamma^{\mu}\,\psi(x) \;\equiv\; J^{\mu}(x)$$

transforms as a contravariant four vector and is thereby named the **Dirac vector current**. Actually the finite transformation law holds true

$$\Lambda_{\frac{1}{2}}^{-1}(\omega)\,\gamma^{\lambda}\,\Lambda_{\frac{1}{2}}(\omega) = \Lambda^{\lambda}{}_{\kappa}\,\gamma^{\kappa} \tag{2.77}$$

$$\gamma^{\lambda} = \Lambda^{\lambda}{}_{\kappa}\,\Lambda_{\frac{1}{2}}(\omega)\,\gamma^{\kappa}\,\Lambda_{\frac{1}{2}}^{-1}(\omega) \tag{2.78}$$

`Proof.` Consider any Lorentz infinitesimal transformation up to the first order approximation in the arbitrary infinitesimal parameters $\delta\omega_{\mu\nu}$. Then we can write

$$\left\{\mathbf{1} + \tfrac{1}{2}\,i\sigma^{\mu\nu}\,\delta\omega_{\mu\nu}\right\}\gamma^{\lambda}\left\{\mathbf{1} - \tfrac{1}{2}\,i\sigma^{\mu\nu}\,\delta\omega_{\mu\nu}\right\} = \left\{\delta^{\lambda}{}_{\kappa} + g^{\lambda\rho}\,\delta\omega_{\rho\kappa}\right\}\gamma^{\kappa}$$

$$\tfrac{1}{2}\,i\left[\sigma^{\mu\nu}, \gamma^{\lambda}\right]\delta\omega_{\mu\nu} = g^{\lambda\rho}\,\delta\omega_{\rho\kappa}\,\gamma^{\kappa} = \tfrac{1}{2}\,\delta\omega_{\rho\kappa}\left(g^{\lambda\rho}\,\gamma^{\kappa} - g^{\lambda\kappa}\,\gamma^{\rho}\right)$$

$$\left[\gamma^{\nu}\gamma^{\mu}, \gamma^{\lambda}\right] - \left[\gamma^{\mu}\gamma^{\nu}, \gamma^{\lambda}\right] = 2g^{\lambda\mu}\,\gamma^{\nu} - 2g^{\lambda\nu}\,\gamma^{\mu}$$

Now we can make use of the algebraic identity

$$[\,ab, c\,] = abc - cab = abc - acb + acb - cab = a\,\{\,b, c\,\} - \{\,a, c\,\}\,b$$

that immediately yields the result from the Clifford algebra. Now, owing to the arbitrariness of the six infinitesimal parameters $\delta\omega_{\mu\nu}$ the finite transformation rules (2.77-2.78) hold actually true. As a matter of fact, any finite transformation matrix can always be written as the product of the one parameter matrices for each one of the canonical coordinate $(\boldsymbol{\alpha}, \boldsymbol{\eta})$ of the Lorentz group O(1,3). □

As we have seen above, it turns out that $\sigma_2\psi_L^* \in \boldsymbol{\tau}_{0\,\frac{1}{2}}$ while $\sigma_2\psi_R^* \in \boldsymbol{\tau}_{\frac{1}{2}\,0}$. Thus, we can build up the **charge conjugated spinor** of a given relativistic Dirac wave field $\psi(x)$ as follows :

$$\psi \quad \mapsto \quad \psi^{c} = C\psi^{*} = \begin{pmatrix} \sigma_2\,\psi_R^* \\ -\,\sigma_2\,\psi_L^* \end{pmatrix} \qquad C = \gamma^2 \tag{2.79}$$

Notice that $(\psi^{c})^{c} = \gamma^2\,(\gamma^2\,\psi^*)^* = \psi$. As a consequence, starting from a solely left or right handed Weyl spinor, it is possible to build up **charge self-conjugated Majorana bispinors**

$$\begin{aligned} \chi_L &= \begin{pmatrix} \psi_L \\ -\,\sigma_2\,\psi_L^* \end{pmatrix} = \chi_L^c \\ \chi_R &= \begin{pmatrix} \sigma_2\,\psi_R^* \\ \psi_R \end{pmatrix} = \chi_R^c \end{aligned} \tag{2.80}$$

Ettore Majorana

(5 Agosto 1906, Catania, via Etnea 251 – 26 marzo 1938, ?)

Teoria simmetrica dell'elettrone e del positrone

Il Nuovo Cimento, volume **14** (1937) pp. 171-184

The classical charge self-conjugated Majorana spinors are real spinors in a four components form, so that their field degrees of freedom content is half of that one of a Dirac spinor wave field.

2.2 The Action Principle

In the previous sections we have seen how to build up Poincaré invariant expressions out of the classical relativistic wave fields corresponding to the irreducible tensor and spinor representations of the non-homogeneous Lorentz group. The requirement of Poincaré invariance will ensure that these classical field theories will obey the axioms of the Special Theory of Relativity.

The general properties that will specify the Action for the collection of the classical relativistic wave field functions $u_A(x)$ $(A = 1, 2, \ldots, N)$ will be assumed in close analogy with the paradigmatic and very well known case of the classical electromagnetic field, as it was developed since the early days of the field theory [1].

1. The Action integrand $\mathcal{L}(x)$ is called the Lagrange density or Lagrangian for short

   Giuseppe Lodovico Lagrangia (Torino 25.1.1736 – Paris 10.4.1813)

   Mécanique Analytique (1788)

   In the absence of preassigned background fields, the Lagrangian can not explicitly depend on the coordinates x^μ , so as to ensure space-time translation invariance, and must be a Lorentz invariant to ensure that the corresponding theory will obey the axioms of Special Relativity.

2. To fulfill causality, the differential equations for the field functions must be at most of the second order in time, in such a way that the related Cauchy problem has a unique solution. Classical field theories described by differential equations of order higher than the second in time will typically develop a-causal solutions, a well known example being the Abraham-Lorentz equation [2] of electrodynamics, which is a third order in time differential equation that encodes the effects of the radiation reaction and shows a-causal effects such as pre-acceleration of charged particles yet to be hit by radiation.

3. The wave equations for all the fundamental fields that describe matter and radiation are assumed to be partial differential equations and **not**

[1] Henri Poincaré, *Sur la dynamique de l'électron*, Rendiconti del Circolo Matematico di Palermo **21** (1906) 129-175; J.J. Larmor, *Aether and Matter*, Cambridge University Press (1900) Chap.**6**; Karl Schwarzschild, *Zur Elektrodynamik I. Zwei Formen des Princips der Aktion in der Elektonentheorie*, Gött. Nachr. Math.-Phys. Kl. (1903) 126-131; Max Born, *Die Träge Masse und das Relativitätstheorie*, Ann. Phys. **28** (1909) 571.

[2] M. Abraham, Ann. Phys. **10**, 105 (1903), H. A. Lorentz, Amst. Versl. **12**, 986 (1904); *cfr.* [14] pp. 578–610.

integral-differential equations, which do satisfy Lorentz covariance in accordance with the special theory of Relativity: as a consequence the Lagrangian must be a Lorentz invariant LOCAL functional of the field functions and their first partial derivatives

$$\mathcal{L}(x) = \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)$$

The classical Action

$$S = \int_{t_\imath}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \mathcal{L}(t, \mathbf{x})$$

has the very same physical dimensions of the Planck constant or the angular momentum, *i.e.*

$$[\,S\,] = [\,\hbar\,] = \mathrm{g\ cm^2\ s^{-1} = eV\ s}$$

while the Lagrangian has engineering dimensions of eV cm^{-3}. In the natural units system the Action is dimensionless and the Lagrangian in four space-time dimensions has natural dimensions of cm^{-4}.

4. According to the variational principle of classical mechanics, the Action must be real and must exhibit a local minimum in correspondence of the Euler-Lagrange equations of motion: in classical Physics complex potentials lead to absorption, *i.e.* disappearance of matter into nothing, a phenomenon that will not be considered in the sequel – it will be found *a posteriori* that a real classical Action is crucial to obtain a satisfactory quantum field theory where the total probability is conserved.

5. The astonishing phenomenological and theoretical success of the gauge theories in the construction of the present day standard model of the fundamental interactions in particle physics does indeed suggest that the Action functional will be invariant under further symmetry groups of transformations beyond the non-homogeneous Lorentz group. Such a kind of transformations does not act upon the space-time coordinates and will be thereby called internal symmetry groups of transformations. These transformations will involve new peculiar field degrees of freedom such as the electric charge, the weak charge, the color charge and maybe other charges yet to be discovered. In particular, gauge theories are described by an Action functional which is invariant under local – *i.e.* space-time point dependent – transformations among those internal degrees of freedom.

Consider the Action functional

$$S(t_i, t_f\,; [\,u\,]) = \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \mathcal{L}\Big(u(x), \partial_\mu u(x)\Big)$$

where we shall denote by

$$u(x) = \{u_A(x) \mid A = 1, 2, \ldots, N\,,\ x = (x_0, \mathbf{x})\,,\ t_i \le t \le t_f\,,\ \mathbf{x} \in \mathbb{R}^3\}$$

the collection of all classical relativistic local wave fields. The index $A = 1, 2, \ldots, N < \infty$ runs over the Lorentz group as well as all the internal symmetry group representations, so that we can suppose the local wave field component to be real valued functions.

We recall that, by virtue of the principle of the least Action, the field variations are assumed to be local and infinitesimal

$$u(x)\ \mapsto\ u'(x) = u(x) + \delta u(x) \qquad |\,\delta u(x)\,| \ll |\,u(x)\,|$$

and to vanish at the initial and final times t_i and t_f

$$\delta u(t_i\,, \mathbf{x}) = \delta u(t_f\,, \mathbf{x}) = 0 \qquad \forall\, \mathbf{x} \in \mathbb{R}^3 \tag{2.81}$$

A local variation with respect to the wave field amplitudes gives

$$\begin{aligned} \delta S &= \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \delta\,\mathcal{L}\Big(u(x), \partial_\mu u(x)\Big) \\ &= \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \left[\frac{\delta\mathcal{L}}{\delta u(x)}\,\delta u(x) + \frac{\delta\mathcal{L}}{\delta\partial_\mu u(x)}\,\delta\,\partial_\mu u(x)\right] \end{aligned}$$

The local infinitesimal variations do satisfy by definition

$$\delta\,\partial_\mu u(x) = \partial_\mu\,\delta u(x) \qquad \Rightarrow \qquad [\,\delta\,,\,\partial_\mu\,] = 0$$

so that

$$\begin{aligned} \delta S &= \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \left[\frac{\delta\mathcal{L}}{\delta u(x)}\ -\ \partial_\mu\,\frac{\delta\mathcal{L}}{\delta\partial_\mu u(x)}\right]\delta u(x) \\ &+\ \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu u(x)}\ \delta u(x)\right] \end{aligned}$$

The very last term can be rewritten, using the Gauß theorem, in the form

$$\begin{aligned} &\int \mathrm{d}\mathbf{x}\ \left[\frac{\delta\mathcal{L}}{\delta\,\partial_0 u(x)}\ \delta u(x)\right]_{t_i}^{t_f} + \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \boldsymbol{\nabla}\cdot\left[\frac{\delta\mathcal{L}}{\delta\,\boldsymbol{\nabla} u(x)}\ \delta u(x)\right] \\ &= \lim_{R\to\infty}\ R^2 \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\Omega\ \widehat{\mathbf{r}}\cdot\left[\frac{\delta\mathcal{L}}{\delta\,\boldsymbol{\nabla} u(t, R, \Omega)}\ \delta u(t, R, \Omega)\right] \end{aligned}$$

where R is the radius of a very large sphere centered at $\mathbf{x} = 0$, $\Omega = (\theta, \phi)$ is the solid angle in the three dimensional space and $\widehat{\mathbf{r}}$ is the radial unit vector, *i.e.* the exterior unit normal vector to the sphere. If we assume the asymptotic radial behavior

$$\lim_{R\to\infty} R^2 \,[\, \delta\mathcal{L}/\delta\, \partial_r u(t,R,\Omega)\,]\, \delta u(t,R,\Omega) = 0 \qquad \forall t \in [\, t_i, t_f\,] \qquad (2.82)$$

where $\partial_r \equiv \widehat{\mathbf{r}}\cdot\boldsymbol{\nabla}$ is the radial derivative, then the above boundary term indeed disappears and consequently, from the arbitrariness of the local variations $\delta u(x)$, we eventually come to the Euler–Lagrange equations of motion for the classical relativistic wave field

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu u(x)} \;=\; \frac{\delta\mathcal{L}}{\delta u(x)} \qquad (2.83)$$

2.3 The Nöther Theorem

For the construction of the constants of motion in field theory we shall use Nöther theorem :

> Amalie Emmy Nöther
> Erlangen 23.03.1882 – Brynn 14.04.1935
> *Invariante Varlationsprobleme*, Nachr. d. König. Gesellsch. d. Wiss. Göttingen, Math–phys. Klasse (1918), 235–257
> English translation M. A. Travel
> *Transport Theory and Statistical Physics* **1** (1971), 183–207 .

This theorem states that to every continuous transformation of coordinates and fields, which makes the variation of the Action equal to zero, there always corresponds a definite constant of motion, *i.e.* a combination of the field functions and their derivatives which remains conserved in time. Such a transformation of coordinates and fields will be called a continuous symmetry and will correspond to some representation of a Lie group of transformations of finite dimension. In order to prove Nöther theorem we shall consider the infinitesimal active transformation of coordinates

$$x^{\,\mu} \quad \mapsto \quad x^{\prime\,\mu} = x^{\,\mu} + \delta x^{\,\mu} \qquad \delta x^{\,\mu} = X^{\,\mu}_{\,a}\, \delta\, \omega^{\,a} \qquad (2.84)$$

with coefficients X^{μ}_{a} that may depend or not upon the space-time points and s space-time independent infinitesimal parameters

$$\delta\, \omega^{\,a} \qquad (a = 1, 2, \ldots, s)$$

which include, for example, the cases of the active infinitesimal translations $\delta\,\omega^{\,\mu}$ and homogeneous Lorentz transformations $\delta\,\omega^{\,\mu\nu}$:

$$\delta x^{\,\mu} = -\,\delta\,\omega^{\,\mu} - x_{\,\nu}\,\delta\,\omega^{\,\mu\nu} \qquad \delta\,\omega^{\,\mu\nu} = -\,\delta\,\omega^{\,\nu\mu}$$

In general we shall suppose to deal with a collection of N field functions

$$u_A(x) \qquad (A = 1, 2, \ldots, N)$$

with a well defined infinitesimal transformation law under the continuous symmetry (2.84) that may be written in the form

$$u_A(x) \quad \mapsto \quad u'_A(x') = u'_A(x + \delta x) = u_A(x) + \Delta\,u_A(x)$$

$$\Delta\,u_A(x) = u'_A(x + \delta x) - u_A(x) = Y^{\,a}_{AB}\,u_B\,(x)\;\delta\,\omega^{\,a} \tag{2.85}$$

where the very last variation is just the already introduced total variation (2.2) of the field function. The fields total variations are assumed to be space-time point independent. For example, in the case of an infinitesimal Lorentz active transformation, the definitions (2.24), (2.41), (2.42), (2.70) and (2.71) yield

$$\Delta\,u_A(x) = Y^{\,a}_{AB}\,u_B\,(x)\;\delta\,\omega^{\,a} \;=\; \frac{i}{2}\,(S_{\mu\nu})_{AB}\,u_B\,(x)\;\delta\,\omega^{\,\mu\nu} \tag{2.86}$$

The Jacobian that corresponds to the infinitesimal transformation of the coordinates is provided by

$$\delta J \;=\; \delta\det\,\|\,\partial x'/\partial x\,\| = \;\partial_\mu\,\delta x^{\mu}$$

`Proof.` As a matter of fact we have in general

$$\mathrm{d}^4 x' = \det\left\|\frac{\partial x'}{\partial x}\right\|\,\mathrm{d}^4 x$$

and for infinitesimal coordinate transformations $x' = x + \delta x$ one obtains the 4×4 matrix $\mathbb{J}$ with matrix elements

$$\mathbb{J}^{\mu}_{\nu} \equiv \partial_\nu(x^{\mu} + \delta x^{\mu}) = \delta^{\mu}_{\nu} + \partial_\nu \delta x^{\mu}$$

that is

$$\mathbb{J} = \left(\begin{array}{cccc} 1+\partial_t\delta x^0 & \partial_x\delta x^0 & \partial_y\delta x^0 & \partial_z\delta x^0 \\ \partial_t\delta x & 1+\partial_x\delta x & \partial_y\delta x & \partial_z\delta x \\ \partial_t\delta y & \partial_x\delta y & 1+\partial_y\delta y & \partial_z\delta y \\ \partial_t\delta z & \partial_x\delta z & \partial_y\delta z & 1+\partial_z\delta z \end{array}\right)$$

so that the explicit evaluation of its determinant yields

$$J \equiv \det\mathbb{J} = 1 + \partial_\mu\,\delta x^{\mu} + O\Big((\partial_\alpha\delta x^{\rho})(\partial_\beta\delta x^{\sigma})\Big)$$

in such a manner that, up to the first order in $\partial_\nu\,\delta x^\mu$, one gets $\delta J = \partial_\mu\,\delta x^\mu$, which completes the proof. □

Hence we can eventually write

$$
\begin{aligned}
\delta S \;&=\; \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\; \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)\, \partial_\lambda\, \delta x^\lambda \\
&+\; \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\; \Delta\,\mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big) \\
&=\; \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\; \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)\, \partial_\lambda\, \delta x^\lambda \\
&+\; \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\; \delta x^\lambda\, \partial_\lambda\, \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big) \\
&+\; \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\; \delta\mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)
\end{aligned}
$$

where use has been made of the relation (2.14). If the Euler-Lagrange wave field equations (2.83) are assumed to be valid then we can recast the local variation of the Lagrangian in the form

$$
\delta\,\mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big) = \partial_\mu \left[\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\, u_A(x)}\; \delta u_A(x) \right]
$$

Hence we immediately obtain

$$
\delta S \;=\; \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\; \partial_\mu \left[\mathcal{L}(x)\,\delta x^\mu + \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\, u_A(x)}\; \delta u_A(x) \right] \qquad (2.87)
$$

Alternatively, we can always recast the local variations (2.3) in terms of the total variations (2.2) so that

$$
\begin{aligned}
\delta S \;&=\; \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\; \partial_\mu \left\{ \left[\delta^\mu_\nu\, \mathcal{L}(x) \;-\; \frac{\delta\mathcal{L}}{\delta\,\partial_\mu u_A(x)}\; \partial_\nu\, u_A(x) \right] \delta x^\nu \right\} \\
&+\; \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\; \partial_\mu \left[\frac{\delta\mathcal{L}}{\delta\,\partial_\mu u_A(x)}\; \Delta u_A(x) \right] \qquad (2.88)
\end{aligned}
$$

Consider now the infinitesimal symmetry transformations depending upon constant parameters, *i.e.* space-time independent, so that

$$
\delta x^\mu \equiv \left(\frac{\partial\, x^\mu}{\partial\,\omega^a} \right)\, \delta\,\omega^a = X^\mu_a(x)\,\delta\,\omega^a \qquad (a = 1, 2, \ldots, s)
$$

$$\Delta\, u_A(x) \equiv (\,Y_a\,)_{AB}\; u_B\,(x)\;\delta\omega^{\,a} \qquad (A = 1, 2, \ldots, N)$$

then we have

$$\frac{\delta S}{\delta\,\omega^{\,a}} \;=\; \int_{t_i}^{t_f} \mathrm{d}\,t \int \mathrm{d}\mathbf{x}\;\partial_\mu J_a^\mu(x) \tag{2.89}$$

where

$$\begin{aligned} J_a^\mu(x) \;&\equiv\; -\Big[\,\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\;\partial_\nu\,u_A(x) \;-\; \mathcal{L}(x)\,\delta^{\,\mu}_{\,\nu}\,\Big]\,X_a^{\,\nu}(x) \\ &+\;\; \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\;\,(\,Y_a\,)_{AB}\;u_B\,(x) \end{aligned} \tag{2.90}$$

are the Nöther currents associated to the parameters $\omega^{\,a}$ $(a = 1, 2, \ldots, s)$ of the Lie group of global symmetry transformations. Suppose the Action functional to be invariant under this group of global transformations

$$(\delta S\,/\,\delta\,\omega^{\,a}) = 0 = \int_{t_i}^{t_f} \mathrm{d}\,t \int \mathrm{d}\mathbf{x}\;\partial_\mu J_a^\mu(x) \qquad (a = 1, 2, \ldots, s)$$

Then from Nöther and Gauß theorems we get

$$\begin{aligned} 0 \;&=\; \int_{t_i}^{t_f} \mathrm{d}\,t \int \mathrm{d}\mathbf{x}\;\partial_\mu J_a^\mu(x) \\ &=\; \int \mathrm{d}\mathbf{x}\,\Big[\,J_a^0(t_f,\mathbf{x}) - J_a^0(t_i,\mathbf{x})\,\Big] + c\int_{t_i}^{t_f} \mathrm{d}\,t \int \mathrm{d}\mathbf{x}\;\boldsymbol{\nabla}\,\cdot\,\mathbf{J}\,(x_0,\mathbf{x}) \\ &=\; \int \mathrm{d}\mathbf{x}\,\Big[\,J_a^0(t_f,\mathbf{x}) - J_a^0(t_i,\mathbf{x})\,\Big] \\ &+\;\; c\lim_{R\to\infty}\;R^{\,2}\int_{t_i}^{t_f} \mathrm{d}\,t \int \mathrm{d}\Omega\;\widehat{\mathbf{r}}\,\cdot\,\mathbf{J}_a\,(x_0, R, \Omega) \end{aligned} \tag{2.91}$$

where R is the radius of a very large sphere centered at $\mathbf{x} = 0$, $\Omega = (\theta,\phi)$ is the solid angle in the three dimensional space and $\widehat{\mathbf{r}}$ is the exterior unit normal vector to the sphere, *i.e.* the radial unit vector. Once again, if we assume the radial asymptotic behavior

$$\lim_{R\to\infty}\;R^{\,2}\;\widehat{\mathbf{r}}\,\cdot\,\mathbf{J}_a(x_0, R, \Omega) = 0 \qquad \forall x_0 \in [\,t_i\,,\,t_f\,] \tag{2.92}$$

then the above boundary term indeed disappears and we eventually come to the conservation laws

$$\frac{\delta S}{\delta\,\omega^{\,a}} = 0 \qquad\Leftrightarrow\qquad \frac{\mathrm{d}}{\mathrm{d}\,t}\,Q_a(t) \;=\; 0 \tag{2.93}$$

where the conserved Nöther charges are defined to be

$$Q_a \;\equiv\; \int \mathrm{d}\mathbf{x}\; J_a^0\,(t,\mathbf{x}) \qquad (a = 1, 2, \dots, s) \tag{2.94}$$

Notice that the Nöther current is not uniquely identified: as a matter of fact, if we redefine the tetra-currents (2.90) according to

$$\tilde{J}_a^{\,\mu}\,(x) \equiv J_a^{\,\mu}\,(x) + \partial_\nu\,\mathcal{A}_a^{\mu\nu}\,(x) \tag{2.95}$$

where $\mathcal{A}_a^{\mu\nu}\,(x)$ $(a = 1, 2, \dots, s)$ is an arbitrary set of anti-symmetric tensor fields

$$\mathcal{A}_a^{\mu\nu}\,(x) + \mathcal{A}_a^{\nu\mu}\,(x) = 0$$

then by construction

$$\partial_\mu\,\tilde{J}_a^{\,\mu}\,(x) = \partial_\mu\,J_a^{\,\mu}\,(x) = \partial_t\,J_a^0(t,\mathbf{r}) + \boldsymbol{\nabla}\cdot\mathbf{J}_a(t,\mathbf{r})$$

and the same conserved Nöther charges (2.94) are obtained. This is Nöther theorem. Let us now examine some important examples.

1. Space-time translations

$$\delta x^{\,\mu} = -\,\delta\,\omega^{\,\mu} \qquad a \equiv \rho = 0, 1, 2, 3$$
$$X_a^{\,\nu} \equiv -\,\delta_\rho^{\,\nu} \qquad \Delta\,u_A(x) \equiv 0$$

for there is no change under space-time translations for any classical relativistic wave field. In this case the corresponding Nöther's current yields the canonical energy-momentum tensor

$$J_a^{\,\mu}\,(x) \quad\mapsto\quad T^{\,\mu}{}_{\rho}\,(x)$$

$$T^{\,\mu\rho}\,(x) \;=\; \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\,\partial^\rho u_A(x) \;-\; \mathcal{L}(x)\,g^{\mu\rho} \tag{2.96}$$

the corresponding conserved charge being the total energy-momentum tetra-vector of the system

$$P_\mu = \int \mathrm{d}\mathbf{x}\; T^{\,0}{}_{\mu}\,(x) \tag{2.97}$$

Actually, it can be readily shown that the canonical energy-momentum tensor does fulfill the continuity equation $\partial_\mu T^{\mu}{}_{\nu}(x) = 0$ because

$$\begin{aligned}
\partial_\mu\,T^{\,\mu}{}_{\nu}(x) \;&=\; \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\,\partial_\nu u_A(x)\right] \;-\; \partial_\mu\,\mathcal{L}(x)\,\delta_\nu^\mu \\
&=\; \frac{\delta\mathcal{L}}{\delta\,u_A(x)}\,\partial_\nu u_A(x) + \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\,\partial_\mu\,\partial_\nu u_A(x) \\
&-\; \left[\frac{\delta\mathcal{L}}{\delta\,u_A(x)}\,\partial_\mu u_A(x) + \frac{\delta\mathcal{L}}{\delta\,\partial_\rho\,u_A(x)}\,\partial_\mu\,\partial_\rho u_A(x)\right]\delta_\nu^\mu = 0
\end{aligned}$$

where use has been suitably made of the Euler-Lagrange field equation.

2. Lorentz transformations

$$\delta\, x^{\nu} = -\,\delta\,\omega^{\,\nu\rho}\, x_{\,\rho} \qquad a \equiv \{\rho\,\sigma\} = 1, \ldots, 6$$

and from (2.84) and (2.86) with $a = [\,\rho\sigma\,]$ and $\delta\omega^{a} = -\,\delta\omega^{\,\rho\sigma}$ we obtain

$$X_a^{\,\nu} \equiv \tfrac{1}{2}\left(x_{\,\sigma}\,\delta_{\rho}^{\,\nu} - x_{\,\rho}\,\delta_{\sigma}^{\,\nu}\right) \qquad \Delta\, u_A(x) \equiv \frac{i}{2\hbar}\,\left(S_{\,\rho\sigma}\,\right)_{AB}\, u_B\,(x)\,\delta\,\omega^{\,\rho\sigma}$$

In this case the corresponding Nöther current yields the relativistic **total angular momentum density** third rank tensor

$$J_a^{\,\mu}\,(x) = J^{\,\mu}{}_{\rho\sigma}\,(x) = \tfrac{1}{2}\, M^{\,\mu}{}_{\rho\sigma}\,(x)$$

$$\begin{aligned} M^{\,\mu}{}_{\rho\sigma}\,(x) &= x_{\,\rho}\, T^{\,\mu}{}_{\sigma}\,(x) - x_{\,\sigma}\, T^{\,\mu}{}_{\rho}\,(x) \\ &+ \frac{\delta\mathcal{L}}{\delta\,\partial_{\mu}\, u_A\,(x)}\,\left(iS_{\,\rho\sigma}\,\right)_{AB}\, u_B\,(x) \\ &\stackrel{\text{def}}{=} L^{\,\mu}{}_{\rho\sigma}\,(x) + S^{\,\mu}{}_{\rho\sigma}\,(x) \end{aligned} \tag{2.98}$$

where the third rank tensors

$$L^{\,\mu}{}_{\rho\sigma}\,(x) = -\, L^{\,\mu}{}_{\sigma\rho}\,(x) = x_{\,\rho}\, T^{\,\mu}{}_{\sigma}\,(x) - x_{\,\sigma}\, T^{\,\mu}{}_{\rho}\,(x) \tag{2.99}$$

$$S^{\,\mu}{}_{\rho\sigma}\,(x) = -\, S^{\,\mu}{}_{\sigma\rho}\,(x) = \frac{\delta\mathcal{L}}{\delta\,\partial_{\mu}\, u_A\,(x)}\,\left(iS_{\,\rho\sigma}\,\right)_{AB}\, u_B\,(x) \tag{2.100}$$

are respectively the relativistic **orbital angular momentum density** and the relativistic **spin angular momentum density** of the wave field. The corresponding charge is the total angular momentum anti-symmetric tensor of the system

$$M_{\mu\nu} = \int \mathrm{d}\mathbf{x}\; M^{\,0}{}_{\mu\nu}\,(t,\mathbf{x}) \qquad M_{\,\mu\nu} + M_{\,\nu\mu} = 0 \tag{2.101}$$

Notice however that, as we shall see further on, the actual validity of the continuity equation for the total angular momentum density, *viz.,*

$$\partial_{\lambda}\, M^{\lambda}{}_{\mu\nu}(x) = 0 \tag{2.102}$$

does indeed entails

$$T_{\mu\nu}(x) - T_{\nu\mu}(x) + \partial_{\lambda}\, S^{\lambda}{}_{\mu\nu}(x) = 0 \tag{2.103}$$

which indicates that any component

$$S_{\mu\nu} = \int \mathrm{d}\mathbf{x}\; S^{0}{}_{\mu\nu}(t,\mathbf{x})$$

of the relativistic spin angular momentum is conserved if and only if the corresponding component of the canonical energy-momentum tensor is symmetric: namely,

$$\dot{S}_{\mu\nu} = 0 \quad \Leftrightarrow \quad T_{\mu\nu}(x) = T_{\nu\mu}(x)$$

3. Internal symmetries

$$X_a^{\,\nu} \equiv 0 \qquad \Delta\, u_A(x) \equiv \mathrm{Y}^{a}_{AB}\, u_B\,(x)\,\delta\,\omega^{\,a} \tag{2.104}$$

where $\mathrm{Y}^{\,a}$ $(a = 1, 2, \ldots, s)$ are the generators of the internal symmetry Lie group in some given representation. It follows thereby that the corresponding Nöther current and charge yields the internal symmetry current and charge multiplets

$$J^{a}_{\mu}(x) = \frac{\delta \mathcal{L}}{\delta\,\partial_\mu\, u_A(x)}\; \mathrm{Y}^{a}_{AB}\, u_B\,(x) \qquad (a = 1, 2, \ldots, s) \tag{2.105}$$

$$Q^{\,a} = \int \mathrm{d}\mathbf{x}\; J^{a}_{0}(x) \qquad (a = 1, 2, \ldots, s) \tag{2.106}$$

`Lorentz invariance of the conserved charges.` We have seen that Nöther theorem implies the occurrence of s tetra-vector fields (2.90), which satisfy the continuity equation and lead to the existence of s conserved charges (2.94), where s is the dimension of the Lie group of the symmetry transformation of the classical Action. It is very important to gather that those conserved charges are also Lorentz invariant quantities. Since the rotation invariance is obviously true from the very structure (2.94), let us concentrate on a boost that, without loss of generality by just taking the rotation invariance into account, we can always suppose to be along the OZ axis, *viz.,*

$$x'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\,\nu} \quad \Longleftrightarrow \quad \begin{cases} c\mathrm{t}' = c\mathrm{t}\cosh\eta - \mathrm{z}\sinh\eta \\ \mathrm{x}' = \mathrm{x} \qquad\quad \mathrm{y}' = \mathrm{y} \\ \mathrm{z}' = \mathrm{z}\cosh\eta - c\mathrm{t}\sinh\eta \end{cases}$$

where we have set $\eta \equiv \mathrm{Arsh}\Big(\beta_z/\sqrt{1-\beta_z^2}\Big)$ $(\,\beta_z = v_z/c > 0\,)$ in such a manner that we can write

$$\jmath_0^{\,a\,\prime}(x') = \jmath_0^{\,a}(x)\,\cosh\eta + \jmath_3^{\,a}(x)\,\sinh\eta \qquad (\,a = 1, 2, \ldots, s\,)$$

in which the Nöther tetra-currents (2.90) do satisfy the continuity equation owing to the validity of the Euler-Lagrange field equations. For the sake of simplicity, but without loss of generality, we shall here suppose the symmetry group to be internal, the straightforward generalization to the IO(1,3) symmetry being left to the interested reader. Hence, in order to prove the Lorentz invariance of any Nöther internal charge, it is expedient to recast it as follows. Let K and K' the laboratory and moving inertial reference frames respectively: then, in the moving inertial reference frame K', we can write

$$\begin{aligned} Q' &= \int \mathrm{d}\mathbf{x}'\, \jmath_0'(t',\mathbf{x}') = \int \mathrm{d}\mathbf{x}'\, \jmath_0'(0,\mathbf{x}') = \int \mathrm{d}^4x'\, \jmath_0'(\mathrm{t}',\mathbf{x}')\,\delta(\mathrm{t}') \\ &= \int \mathrm{d}^4x'\, \jmath_0'(\mathrm{t}',\mathbf{x}')\,\partial_0'\theta(\mathrm{t}') = g^{\mu\nu}\int \mathrm{d}^4x'\, \jmath_\mu'(x')\,\partial_\nu'\theta(\mathrm{t}') \\ &= g^{\mu\nu}\int \mathrm{d}^4x'\,\partial_\nu'\left[\,\jmath_\mu'(x')\,\theta(\mathrm{t}')\,\right] \end{aligned}$$

where the Lie algebra index $a = 1,2,\ldots,s$ has been omitted to lighten notations, while use has been made of :

$a)$ the time independence of the charge integral for any inertial observer;
$b)$ the invariant continuity equation $\partial_\mu\,\jmath^\mu(x) = 0$ for the Nöther current.

Notice that the charge spatial integrals are constant quantities, *i.e.* proper time independent, for any inertial Observer. From the transformation law

$$\jmath_\mu'(x')\,\theta(\mathrm{t}') = \begin{cases} \jmath_\mu'(x') & \text{for} \quad \mathrm{t}'>0 \\ 0 & \text{for} \quad \mathrm{t}'<0 \end{cases} = \Lambda_\mu^{\ \nu}\,\jmath_\nu(x)\,\theta(\mathrm{t}\cosh\eta - \mathrm{z}\sinh\eta)$$

after turning to the laboratory inertial reference frame K we obtain

$$Q' = \int \mathrm{d}^4x\,\partial_\mu\left[\,\jmath^\mu(x)\,\theta(\mathrm{t}\cosh\eta - \mathrm{z}\sinh\eta)\,\right]$$

Thus we can eventually write

$$Q' - Q = \int \mathrm{d}^4x\,\partial_\mu\Big(\jmath^\mu(\mathrm{t},\mathbf{x})\,[\,\theta(\mathrm{t}\cosh\eta - \mathrm{z}\sinh\eta) - \theta(\mathrm{t})\,]\Big)$$

Now, as we have already stressed, the charge integral is a constant for any inertial Observer, in such a manner that we can evaluate the above difference in terms of *e.g.* the laboratory frame Minkowski coordinates: namely,

$$\begin{aligned} Q' - Q &= \int \mathrm{d}\mathbf{x}\int_{-\infty}^{\infty}\mathrm{d}\mathrm{t}\,\partial_\mu\Big(\jmath^\mu(\mathrm{t},\mathbf{x})\,[\,\theta(\mathrm{t}\cosh\eta - \mathrm{z}\sinh\eta) - \theta(\mathrm{t})\,]\Big) \\ &= \int \mathrm{d}\mathbf{x}\,\Big\{\jmath_0(\mathrm{t},\mathbf{x})[\,\theta(\mathrm{t}\cosh\eta - \mathrm{z}\sinh\eta) - \theta(\mathrm{t})\,]\Big\}_{\mathrm{t}\to-\infty}^{\mathrm{t}\to\infty} \\ &+ \int_{-\infty}^{\infty}\mathrm{d}\mathrm{t}\int \mathrm{d}\mathbf{x}\,\nabla_k\Big\{\,\jmath^k(\mathrm{t},\mathbf{x})\,[\,\theta(\mathrm{t}\cosh\eta - \mathrm{z}\sinh\eta) - \theta(\mathrm{t})\,]\Big\} = 0 \end{aligned} \tag{2.107}$$

owing to the Gauß divergence theorem and under the customary assumption of the asymptotic spatial behavior

$$u_A(\mathrm{t},\mathbf{x}) \ \sim\ O\left(\,|\,\mathbf{x}\,|^{-1-\epsilon}\right) \qquad\qquad (\,|\,\mathbf{x}\,|\to\infty\,,\ \epsilon>0\,)$$

that proves the Lorentz invariance of any charge integral, as expected.

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2.3.1 Problems

1. Maxwell Field Equations. Write the Maxwell equations in Lorentz covariant form and derive them from an Action principle.

Solution. The celebrated Maxwell's field equations of Electromagnetism are usually grouped into two pairs: namely,

$$\begin{cases} \nabla\cdot\mathbf{B}=0 \\ \dot{\mathbf{B}}+c\nabla\times\mathbf{E}=0 \end{cases} \qquad \text{First Couple}$$

$$\begin{cases} \nabla\cdot\mathbf{E}=\rho \\ -\dot{\mathbf{E}}+c\nabla\times\mathbf{B}=\mathbf{J} \end{cases} \qquad \text{Second Couple}$$

Notice that in the C.G.S. system of Heaviside-Lorentz electromagnetic units we have the following simple dimensional correspondences: namely,

$$\alpha=\frac{e^{2}}{4\pi\hbar c}\simeq\frac{1}{137} \qquad \Longleftrightarrow \qquad [\,e\,]=\text{esu}=\sqrt{\text{eV cm}}$$

$$[\,\mathbf{E}^{2}+\mathbf{B}^{2}\,]=\text{eV cm}^{-3} \qquad\qquad [\,e\mathbf{B}\,]=[\,e\mathbf{E}\,]=\text{eV cm}^{-1}$$

whence

$$[\,\mathbf{E}\,]=[\,\mathbf{B}\,]=\text{Gauss}=\text{eV}^{\frac{1}{2}}\ \text{cm}^{-\frac{3}{2}}=\text{esu cm}^{-2}$$

$$[\,\mathbf{J}\,]=\text{esu cm}^{-2}\ \text{s}^{-1}=\text{eV}^{\frac{1}{2}}\ \text{cm}^{-\frac{3}{2}}\ \text{s}^{-1}=\text{Gauss s}^{-1}$$

As a check we have that the quantity $\Phi = hc/e = e/2\alpha$ is a unit of quantum flux or electric charge because

$$[\,\Phi\,] = [\,e\,] = \mathrm{eV}^{\frac{1}{2}}\ \mathrm{cm}^{\frac{1}{2}} = \mathrm{Gauss\ cm}^2 = \mathrm{esu}$$

The second couple of the Maxwell field equations can be readily set into a Lorentz covariant form, once we identify

$$\mathbf{E} = (F^{10}, F^{20}, F^{30}) \qquad\qquad \mathbf{B} = (F^{32}, F^{13}, F^{21})$$

As a matter of fact, we find that the second couple of the Maxwell field equations can be recast into the manifestly Poincaré covariant form: namely,

$$c\,\partial_\mu F^{\mu\nu} = J^\nu$$

that is immediately verified by direct inspection. Next, if we introduce the so called dual electromagnetic tensor

$$F_*^{\mu\nu} = \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,F_{\rho\sigma} \qquad\qquad (\,\varepsilon^{0123} = 1\,)$$

in which the electric and magnetic components are interchanged, for $F_*^{10} = B_{32}$ *et cetera*, then we can write the first couple in the Lorentz covariant form

$$\partial_\mu F_*^{\mu\nu} = 0$$

which can also readily verified by direct inspection. If the charged matter is made up by n point-like charges $q_\imath(\,\imath = 1, 2, \ldots, n\,)$ with instantaneous position vectors $\mathbf{r}_\imath(t)$, then we have the following charge and density current

$$\rho(t, \mathbf{x}) = \sum_{\imath=1}^{n} q_\imath\,\delta(\mathbf{x} - \mathbf{r}_\imath(t))$$

$$\mathbf{J}(t, \mathbf{x}) = \sum_{\imath=1}^{n} q_\imath\,\dot{\mathbf{r}}_\imath(t)\,\delta(\mathbf{x} - \mathbf{r}_\imath(t))$$

which satisfy by construction the continuity equation

$$\dot\rho + \nabla\cdot\mathbf{J} = 0$$

that can also be written with the Minkowski notations

$$\partial_\mu J^\mu(x) = 0 \qquad\qquad J^\mu = (\,c\rho, \mathbf{J}\,)$$

It should be stressed that the charge and current densities are supposed to be fixed preassigned distributions, which do not undergo any dynamics but merely act as external sources for the electromagnetic fields. Contextually, the external sources are provided in some given inertial reference frame, which is understood to be definitely chosen.

To the purpose of building up a Lagrangian for the classical radiation field, we notice that there are two Lorentz and gauge invariant quadratic expressions in the tetra-vector potential, *viz.,*

$$F_{\mu\nu}F^{\mu\nu} = -\,2\mathbf{E}^2 + 2\mathbf{B}^2 \qquad\qquad F_{\mu\nu}F_*^{\mu\nu} = -\,2\mathbf{E}\cdot\mathbf{B}$$

both being null for the electromagnetic radiation in vacuum. The second invariant is a tetra-divergence because

$$\tfrac{1}{2}\,\varepsilon^{\,\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 2\varepsilon^{\,\mu\nu\rho\sigma}\partial_\mu\left(A_\nu\partial_\rho A_\sigma\right)$$

the other terms being null for symmetry reasons. In such a circumstance, the second couple of Maxwell field equation can be derived through an Action principle from the following Lagrangian

$$\mathcal{L} = -\,\tfrac{1}{4}\,F^{\,\mu\nu}\,F_{\mu\nu} - c^{-1}A^\mu J_\mu$$

As a matter of fact we immediately find

$$\delta\mathcal{L}/\delta\partial_\mu A_\nu(x) = -\,F^{\,\mu\nu}(x) \qquad\qquad \delta\mathcal{L}/\delta A_\nu(x) = -\,c^{-1}J^\nu(x)$$

whence the second pair of Maxwell equations follows in the Euler-Lagrangian form. As a final remark it is worthwhile to stress that the radiation-matter coupling is gauge invariant, up to an irrelevant tetra-divergence, owing to the continuity equation satisfied by the tetra-current external source: namely

$$A'_\nu(x)J^\nu(x) = [\,A_\nu(x) + \partial_\nu f(x)\,]J^\nu(x) = A_\nu(x)J^\nu(x) + \partial_\nu[\,f(x)J^\nu(x)\,]$$

so that $A'_\nu J^\nu \doteq A_\nu J^\nu$, where $\doteq$ means up to an irrelevant tetra-divergence which does not affect the dynamics.

2. Energy-Momentum Tensor of the Radiation Field. Construct the symmetric energy momentum tensor and the total angular momentum tensor density for the classical radiation field in the absence of charge and current sources [3].

Solution. The classical Maxwell Lagrangian that describes the radiation field is provided by

$$\mathcal{L}(x) = -\,\tfrac{1}{4}\,F^{\,\mu\nu}(x)\,F_{\mu\nu}(x)$$

[3] See *e.g.* L.D. Landau, E.M. Lifšits, *Teoria dei campi*, Editori Riuniti/Edizioni Mir, Roma, 1976, § 33 p. 114.

where the electromagnetic field strength tensor

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

is invariant under the gauge transformations of the vector potential

$$A_\mu(x) \quad \mapsto \quad A'_\mu(x) = A_\mu(x) + \partial_\mu f(x)$$

for any arbitrary analytic function $f(x)$. Then from Nöther theorem we get the canonical energy-momentum tensor

$$\begin{aligned} T^{\mu\rho}(x) &= \frac{\delta\mathcal{L}}{\delta\,\partial_\mu A_\nu(x)}\,\partial^\rho A_\nu(x) - \mathcal{L}(x)\,g^{\mu\rho} \\ &= \tfrac{1}{4}\,F^{\lambda\nu}(x)\,F_{\lambda\nu}(x)\,g^{\mu\rho} - F^{\mu\nu}(x)\,\partial^\rho A_\nu(x) \\ &= -\,F^{\mu\nu}(x)F^\rho{}_\nu(x) + \tfrac{1}{4}\,F^{\lambda\nu}(x)\,F_{\lambda\nu}(x)\,g^{\mu\rho} \\ &- F^{\mu\nu}(x)\,\partial_\nu A^\rho(x) \end{aligned} \tag{2.108}$$

which is not symmetric with respect to μ and ρ owing to the last addendum in the right-hand-side of the above equality. Actually we find

$$\begin{aligned} T^{\mu\rho}(x) - T^{\rho\mu}(x) &= \partial_\nu\left(A^\mu(x)\,F^{\rho\nu}(x) - A^\rho(x)\,F^{\mu\nu}(x)\right) \\ &= \partial_\nu\,S^{\nu\rho\mu}(x) \end{aligned}$$

where use has been done of the Maxwell equations $\partial_\nu F^{\mu\nu}(x) = 0$ and of the spin angular momentum density rank-three tensor as obtained from equations (2.22) and (2.98)

$$S^\mu{}_{\rho\sigma} = i\,F^{\mu\nu}\,S_{\rho\sigma} * A_\nu = A_\rho\,F^\mu{}_\sigma - A_\sigma\,F^\mu{}_\rho$$

which enjoys the anti-symmetry property $S^\mu{}_{\rho\sigma} + S^\mu{}_{\sigma\rho} = 0$. In accordance with the general rule (2.95), we shall introduce the so called improved symmetric energy-momentum tensor

$$\Theta^{\mu\rho}(x) \equiv \tfrac{1}{4}\,F^{\lambda\nu}(x)\,F_{\lambda\nu}(x)\,g^{\mu\rho} - F^{\mu\nu}(x)\,F^\rho{}_\nu(x) = \Theta^{\rho\mu}(x)$$

which is manifestly symmetric by construction. Hence we can write

$$T^{\mu\nu}(x) \doteq \Theta^{\mu\nu}(x) - \tfrac{1}{2}\,\partial_\lambda\left[\,F^{\mu\lambda}(x)\,A^\nu(x) + F^{\nu\lambda}(x)\,A^\mu(x) + S^{\lambda\mu\nu}(x)\,\right]$$

where $\doteq$ means up to the use of the Maxwell equations $\partial_\nu F^{\mu\nu}(x) = 0$. From the obvious equality

$$\begin{aligned} & F^{\mu\lambda}(x)\,A^\nu(x) + F^{\nu\lambda}(x)\,A^\mu(x) + A^\lambda(x)\,F^{\mu\nu}(x) - A^\lambda(x)\,F^{\mu\nu}(x) \\ =\; & S^{\mu\nu\lambda}(x) + S^{\nu\mu\lambda}(x) \end{aligned}$$

we eventually get the Belifante-Rosenfeld relationship [4]

$$\Theta^{\mu\nu}(x) \doteq T^{\mu\nu}(x) + \tfrac{1}{2}\,\partial_\lambda\left[\,S^{\lambda\mu\nu}(x) + S^{\mu\nu\lambda}(x) + S^{\nu\mu\lambda}(x)\,\right]$$

in such a manner that Nöther theorem yields

$$\partial_\mu\,T^{\mu\rho}(x) = \partial_\mu\,\Theta^{\mu\rho}(x) = \partial_\rho\,\Theta^{\mu\rho}(x) \tag{2.109}$$

$$P_\mu = \int \mathrm{d}\mathbf{x}\;T^{0}{}_{\mu}(x) = \int \mathrm{d}\mathbf{x}\;\Theta^{0}{}_{\mu}(x) \qquad \dot{P}_\mu = 0 \tag{2.110}$$

It turns out that the improved symmetric energy-momentum tensor of the electromagnetic field is trace-less, *i.e.* $g_{\mu\rho}\,\Theta^{\mu\rho} = 0$. Let us express explicitly the well known components:

$$\begin{array}{lll} \Theta_{jk} = \frac{1}{2}\,\delta_{jk}\,(\mathbf{E}^2 + \mathbf{B}^2) - E_j\,E_k - B_j\,B_k & & (\,\text{Maxwell stress tensor}\,) \\ \Theta_{j0} = \mathrm{S}^j/c = \varepsilon_{jkl}\,E^k\,B^l & \mathbf{S} = c\,\mathbf{E}\times\mathbf{B} & (\,\text{Poynting vector}\,) \\ \Theta_{00} = \frac{1}{2}\,(\mathbf{E}^2 + \mathbf{B}^2) & & (\,\text{Energy density}\,) \end{array}$$

that yield

$$\begin{aligned} c\,\mathbf{P} &= \frac{1}{c}\int \mathrm{d}\mathbf{x}\;\mathbf{S}(t,\mathbf{x}) = \int \mathrm{d}\mathbf{x}\;\mathbf{E}(t,\mathbf{x})\times\mathbf{B}(t,\mathbf{x}) \\ W &= \int \mathrm{d}\mathbf{x}\;\Theta_{00}(t,\mathbf{x}) = \int \mathrm{d}\mathbf{x}\;\tfrac{1}{2}\left(\mathbf{E}^2(t,\mathbf{x}) + \mathbf{B}^2(t,\mathbf{x})\right) \end{aligned}$$

From equations (2.22) and (2.98) we can derive the canonical total angular momentum density of the radiation field

$$\begin{aligned} M^{\mu}{}_{\rho\sigma} &= x_\rho\,T^{\mu}{}_{\sigma} - x_\sigma\,T^{\mu}{}_{\rho} + A_\rho\,F^{\mu}{}_{\sigma} \;-\; A_\sigma\,F^{\mu}{}_{\rho} \\ &= x_\rho\,T^{\mu}{}_{\sigma} - x_\sigma\,T^{\mu}{}_{\rho} + S^{\mu}{}_{\rho\sigma} \end{aligned} \tag{2.111}$$

which satisfies the continuity equation according to Noether's theorem

$$\begin{aligned} \partial_\mu\,M^{\mu}{}_{\rho\sigma} &= T_{\rho\sigma} - T_{\sigma\rho} + \partial_\mu\left(A_\rho\,F^{\mu}{}_{\sigma} - A_\sigma\,F^{\mu}{}_{\rho}\right) \\ &= \partial_\mu\,S^{\mu}{}_{\rho\sigma} - \tfrac{1}{2}\,\partial_\nu\,S^{\nu}{}_{\rho\sigma} + \tfrac{1}{2}\,\partial_\nu\,S^{\nu}{}_{\sigma\rho} = 0 \end{aligned}$$

One can also express the canonical energy-momentum tensor in terms of the improved and symmetric energy momentum tensor. Taking the Maxwell equations into account one gets

$$\begin{aligned} M^{\mu\rho\sigma} &= x^\rho\Big(\Theta^{\mu\sigma} - F^{\mu\nu}\partial_\nu A^\sigma\Big) - x^\sigma\Big(\Theta^{\mu\rho} - F^{\mu\nu}\partial_\nu A^\rho\Big) + S^{\mu\rho\sigma} \\ &= x^\rho\,\Theta^{\mu\sigma} - x^\sigma\,\Theta^{\mu\rho} - \partial_\nu\Big(x^\rho F^{\mu\nu}A^\sigma - x^\sigma F^{\mu\nu}A^\rho\Big) \\ &= x^\rho\,\Theta^{\mu\sigma} - x^\sigma\,\Theta^{\mu\rho} - \partial_\nu\left[\,F^{\mu\nu}\left(x^\rho A^\sigma - x^\sigma A^\rho\right)\right] \end{aligned}$$

[4] Frederik J. Belifante, *On the spin angular momentum of mesons*, Physica **6** (1939) 887-898; L. Rosenfeld, *Sur le tenseur d'impulsion-énergie*, Mémoires de l'Academie Roy. Belgique **18** (1940) 1-30.

The very last term does not contribute at all to the continuity equation by virtue of the anti-symmetry with respect to the pair of indexes $\mu\,\nu$. Hence we obtain the improved total angular momentum density, a third rank tensor of a pure orbital form

$$\begin{aligned}\mathfrak{M}^{\mu}{}_{\rho\sigma} &= M^{\mu}{}_{\rho\sigma} + \partial_\nu \left[F^{\mu\nu} \left(x_\rho\, A_\sigma - x_\sigma\, A_\rho \right) \right] \\ &= x_\rho\, \Theta^{\mu}{}_{\sigma} - x_\sigma\, \Theta^{\mu}{}_{\rho}\end{aligned}$$

in such a manner that the continuity equations hold true, *viz.,*

$$\partial_\mu\, M^{\mu}{}_{\rho\sigma} = \partial_\mu\, \mathfrak{M}^{\mu}{}_{\rho\sigma} = 0$$

On the one hand, it is clear that the classical spin angular momentum second rank tensor of the radiation field is neither conserved in time nor gauge invariant since we have

$$\partial_\lambda S^{\lambda\mu\rho}(x) = T^{\rho\mu}(x) - T^{\mu\rho}(x) = F^{\mu\nu}(x)\ \partial^\rho A_\nu(x) - F^{\rho\nu}(x)\ \partial^\mu A_\nu(x) \neq 0$$

and thereby the integrated quantities

$$\begin{aligned}\Sigma_{\varrho\sigma}(t) &= \int \mathrm{d}\mathbf{x}\, \Big(A_\varrho(t,\mathbf{x})\, F^{0}{}_{\sigma}(t,\mathbf{x}) - A_\sigma(t,\mathbf{x})\, F^{0}{}_{\varrho}(t,\mathbf{x}) \Big) \\ \Sigma^{\jmath 0}(t) &= \int \mathrm{d}\mathbf{x}\ E^{\jmath}(t,\mathbf{x})\, A^{0}(t,\mathbf{x}) \\ \Sigma^{\jmath k}(t) &= \int \mathrm{d}\mathbf{x}\ \Big\{ E^{\jmath}(t,\mathbf{x})\, A^{k}(t,\mathbf{x}) - E^{k}(t,\mathbf{x})\, A^{\jmath}(t,\mathbf{x}) \Big\}\end{aligned}$$

turn out to be time and gauge dependent, in general. However, if we select the radiation gauge or Coulomb gauge $\nabla\cdot\mathbf{A} = 0$, $A_0 = 0$, in which only the transverse and physical components of the vector potential are involved, we find $\Sigma^{0\jmath} \equiv 0$, $\dot{\boldsymbol{\Sigma}} = 0$ because

$$c\,\mathbf{E} = -\,\dot{\mathbf{A}} \qquad\qquad \dot{\mathbf{E}} = c\,\nabla\times\mathbf{B} \tag{2.112}$$

$$\dot{\boldsymbol{\Sigma}} = \int \mathrm{d}\mathbf{x}\, [\,\nabla\times\mathbf{B}(t,\mathbf{x})\,] \times \mathbf{A}(t,\mathbf{x}) \doteq 0 \tag{2.113}$$

On the other hand, it turns out to be manifest that the total angular momentum tensor can always appear to be of a purely orbital form. As a consequence we get the three spatial components

$$M^{\imath\jmath} = \frac{1}{c^2} \int \mathrm{d}\mathbf{x}\, \Big\{ x^{\imath}\, \mathrm{S}^{\jmath}(t,\mathbf{x}) - x^{\jmath}\, \mathrm{S}^{\imath}(t,\mathbf{x}) \Big\}$$

of the total angular momentum spatial vector for the radiation field

$$\mathbf{M} = (M_{23}, M_{31}, M_{12}) \equiv \frac{1}{c} \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \ \mathrm{d}^3 x$$

while the spatial-temporal components

$$M^{0k} = x^0 \, c \, P^k - \int \mathrm{d}\mathbf{x} \ x^k \, \Theta_{00}(t, \mathbf{x})$$

lead to the definition of the center of the energy for the radiation, *viz.*,

$$\mathbf{X}_t \equiv \int \mathrm{d}^3 x \ \frac{\mathbf{x}}{W} \ \Theta_{00}(t, \mathbf{x})$$

that satisfies the suggestive particle light ray relationship

$$\dot{M}^{0k} = 0 \quad \Longleftrightarrow \quad \dot{\mathbf{X}}_t \equiv \partial_t \mathbf{X}_t = c^2 \, \mathbf{P}/W$$

Chapter 3

The Scalar Field

In this chapter we shall analyze the quantum theory for a real Klein-Gordon field. As a first step we obtain the most general real solution of the Klein-Gordon evolution equation. This leads to the normal modes expansions of a Klein-Gordon field. Once this expansion is available we shall investigate the Hamiltonian formulation of the dynamics and quantize the system according to the canonical Heisenberg-Dirac procedure. The general structure of the Fock space of the states and the Lorentz covariance properties of the Klein-Gordon quantum field will be discussed. Finally, the Feynman propagator and the Euclidean formulation for scalar field theory will be developed.

3.1 General Features

The simplest though highly nontrivial example of a quantum field theory involves a real scalar field $\phi : \mathcal{M} \to \mathbb{R}$. As we shall see below, the Lorentz scalar Hermitean quantum field describes neutral spin-less massive or massless particles. The most general Poincaré invariant Lagrange density that fulfill all the criteria listed in section 2.2 takes the general form

$$\mathcal{L} = \tfrac{1}{2}\,\partial_\mu \phi(x)\,\partial^\mu \phi(x) - \mathcal{V}[\,\phi(x)\,] \tag{3.1}$$

where $\mathcal{V}(\phi)$ is assumed to be a real analytic functional of its relativistic wave field argument, that is

$$\mathcal{V}[\,\phi(x)\,] = \upsilon\, m^4 + \varkappa\, m^3\,\phi(x) \pm \frac{1}{2}\, m^2\,\phi^2(x) + \frac{g}{3!}\, m\,\phi^3(x) + \frac{\lambda}{4!}\,\phi^4(x) + \cdots$$

with $m > 0$ and $\upsilon, \varkappa, g, \lambda \in \mathbb{R}$. Notice that the structure of the kinetic term is such to fix the canonical dimension of the scalar field : $[\,\phi\,] = \mathrm{cm}^{-1} = \mathrm{eV}$ in natural units $\hbar = c = 1$. Moreover, the important case of the negative quadratic mass term does generally provide the remarkable phenomenon of the spontaneous symmetry breaking, which lies at the heart of the present day Standard Model of the High Energy Particle Physics and Quantum Field Theory of the Fundamental Interactions.

The Euler-Lagrange field equations of motion read

$$\Box\,\phi(x) + \varkappa\, m^3 \pm m^2\,\phi(x) + \frac{g}{2}\,m\,\phi^2(x) + \frac{\lambda}{6}\,\phi^3(x) + \cdots = 0 \tag{3.2}$$

while the conserved energy-momentum tensor (2.96) and vector (2.97) are

$$T_{\mu\nu}\,(x) = \partial_\mu\,\phi\,(x)\,\partial_\nu\,\phi\,(x) - g_{\mu\nu}\,\mathcal{L}(x) = T_{\nu\mu}\,(x) \tag{3.3}$$

$$P_\mu = \int T^{0}{}_{\mu}\,(x)\,\mathrm{d}\mathbf{x} = \int \mathrm{d}\mathbf{x}\,\Big[\,\partial_0\,\phi(x)\partial_\mu\phi(x) - g_{0\mu}\,\mathcal{L}(x)\,\Big] \tag{3.4}$$

It is very easy to check that thanks to the Euler-Lagrange equations of motion the energy-momentum current is conserved, *i.e.*

$$\partial_\mu\,T^{\mu}{}_{\nu}\,(x) = 0$$

`Proof.` As a matter of fact we readily find

$$\begin{aligned} \partial_\mu T^{\mu}{}_{\nu}(x) &= (\Box\phi(x))\,\partial_\nu\phi(x) + (\partial^{\mu}\phi(x))\,\partial_\mu\partial_\nu\phi(x) - \partial_\nu\,\mathcal{L}(x) \\ &= (\Box\phi(x))\,\partial_\nu\phi(x) + (\partial^{\mu}\phi(x))\,\partial_\mu\partial_\nu\phi(x) - \partial_\nu\left(\tfrac{1}{2}\,\partial_\rho\,\phi\,(x)\,\partial^{\rho}\,\phi\,(x) - \mathcal{V}\,[\,\phi(x)\,]\right) \\ &= (\Box\phi(x))\,\partial_\nu\phi(x) + [\,\delta\,\mathcal{V}/\delta\phi(x)\,]\partial_\nu\phi(x) = \left(\Box\phi(x) + \frac{\delta\,\mathcal{V}}{\delta\phi(x)}\right)\partial_\nu\phi(x) = 0 \end{aligned}$$

owing to the Euler-Lagrange equations of the motion $\Box\phi + \delta\mathcal{V}/\delta\phi = 0$. □

Moreover, the stability requirement that the total energy must be bounded from below, to avoid a collapse of the mechanical system, entails that the analytic potential has to be a bounded from below functional of the real scalar field. Finally, as it will be better focused later on, the constraining criterion of power counting renormalizability for the corresponding quantum field theory will forbid the presence of coupling parameters with canonical dimensions equal to positive integer powers of length. In such a circumstance, the Lagrange density for a self-interacting, stable, renormalizable real scalar field theory reduces to

$$\mathcal{L}(x) = \tfrac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi(x)\partial_\nu\phi(x) - \mathcal{V}\,[\,\phi(x)\,] \tag{3.5}$$

$$\mathcal{V}\,[\,\phi(x)\,] = \mathcal{V}_0 + \varkappa\,m^3\,\phi\,(x) \pm \frac{1}{2}\,m^2\,\phi^2(x) + \frac{mg}{3!}\,\phi^3(x) + \frac{\lambda}{4!}\,\phi^4(x)$$

in which $\mathcal{V}_0$ is a finite classical zero point energy, while $\varkappa, g$ and λ are real numerical constants, the dimensionless positive coupling $\lambda > 0$ just endorsing stability. Hence we obtain the canonical conjugate momentum field

$$\Pi(x) \equiv \frac{\delta L}{\delta \partial_0 \phi(x)} = \frac{\delta S}{\delta \partial_0 \phi(x)} = \partial_0 \phi(x) = \dot{\phi}(x) \tag{3.6}$$

$$S \equiv \int_{t_i}^{t_f} \mathrm{d}t \; L(t) \qquad L(t) \equiv \int \mathrm{d}\mathbf{x} \, \mathcal{L}(t, \mathbf{x}) \tag{3.7}$$

To proceed further on we shall restrict ourselves to the paradigmatic case of the $\lambda\phi_4^4$ even functional potential

$$\mathcal{V}[\phi(x)] = \frac{1}{2} m^2 \phi^2(x) + \frac{\lambda}{4!} \phi^4(x) \tag{3.8}$$

which corresponds to the self-interacting theory of a neutral spin-less field in four space-time dimensions, the so called $\lambda\phi_4^4$ field theory, with a null minimum classical energy. This model is actually picked out if we impose the internal discrete symmetry $\mathbb{Z}_2$ that is the invariance of the classical Action under the transformation

$$\phi(x) \quad \longmapsto \quad \phi'(x) \; = \; -\phi(x)$$

Hence the Lagrangian must be either even or odd, but stability enforces the even choice *i.e.* the $\lambda\phi_4^4$ self-interaction. The classical Hamiltonian functional can be obtained through the functional Legendre transformation

$$H[\phi, \Pi] = \int \mathrm{d}\mathbf{x} \; \left\{ \dot{\phi}(t, \mathbf{x}) \, \Pi(t, \mathbf{x}) - \mathcal{L}[\phi(t, \mathbf{x}), \dot{\phi}(t, \mathbf{x})] \right\} \tag{3.9}$$

and reads

$$H \equiv P_0 = \int T_0^{\;0}(x^0, \mathbf{x}) \, \mathrm{d}\mathbf{x} = H_0 + H_{\mathrm{I}} \geq 0$$

$$H_0 = \int \tfrac{1}{2} \left[\Pi^2(x) - \phi(x) \boldsymbol{\nabla}^2 \phi(x) + m^2 \phi^2(x) \right] \mathrm{d}\mathbf{x} \tag{3.10}$$

$$H_{\mathrm{I}} = \int \frac{\lambda}{4!} \phi^4(x) \, \mathrm{d}\mathbf{x} \tag{3.11}$$

Furthermore, from Nöther's theorem we get the total momentum

$$\mathbf{P} \equiv -\int \mathrm{d}\mathbf{x} \; \Pi(x) \, \boldsymbol{\nabla}\phi(x) \tag{3.12}$$

and the total angular momentum, which is of the pure orbital form for a scalar field, *viz.,*

$$L_{\mu\nu} = \int \mathrm{d}\mathbf{x}\, \Big[x_\mu T_{0\nu}(t,\mathbf{x}) - x_\nu T_{0\mu}(t,\mathbf{x}) \Big] \tag{3.13}$$

We observe *en passant* that the above expressions are indeed obtained by making use of the asymptotic radial behavior

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|\, \phi(x_0,\mathbf{x})\, \mathbf{x}\cdot \boldsymbol{\nabla}\phi(x_0,\mathbf{x}) = 0$$

by virtue of which and of the Gauß theorem we can write

$$\int \mathrm{d}\mathbf{x}\ \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) = -\int \mathrm{d}\mathbf{x}\ \phi(x)\, \boldsymbol{\nabla}^2\phi(x)$$

It is also worthwhile to remark that the angular momentum density third rank tensor

$$M^{\lambda}{}_{\mu\nu}(x) = x_\mu T^{\lambda}{}_{\nu}(x) - x_\nu T^{\lambda}{}_{\mu}(x)$$

does fulfill the continuity equation thanks to the symmetry of the canonical energy-momentum tensor $T_{\rho\sigma}(x)$.

The field equations can also be rewritten in the Hamiltonian formalism. To this concern, let me remind that for any pair of functional of the real scalar field and its conjugate momentum one can readily define the functional Poisson's bracket

$$\begin{aligned} &\Big\{ F[\phi,\Pi]\,,\, G[\phi,\Pi] \Big\} \equiv \\ &\int \mathrm{d}\mathbf{x}\, \left(\frac{\delta F[\phi,\Pi]}{\delta\phi(t,\mathbf{x})}\cdot\frac{\delta G[\phi,\Pi]}{\delta\Pi(t,\mathbf{x})} - \frac{\delta F[\phi,\Pi]}{\delta\Pi(t,\mathbf{x})}\cdot\frac{\delta G[\phi,\Pi]}{\delta\phi(t,\mathbf{x})} \right) \end{aligned} \tag{3.14}$$

Hence, from the fundamental canonical Poisson's brackets

$$\begin{aligned} \{\phi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\} &= \int \mathrm{d}\mathbf{z}\, \frac{\delta\phi(t,\mathbf{x})}{\delta\phi(t,\mathbf{z})}\cdot\frac{\delta\Pi(t,\mathbf{y})}{\delta\Pi(t,\mathbf{z})} \\ &= \int \mathrm{d}\mathbf{z}\, \delta(\mathbf{x}-\mathbf{z})\, \delta(\mathbf{y}-\mathbf{z}) \\ &= \delta(\mathbf{x}-\mathbf{y}) \end{aligned}$$

$$\{\phi(t,\mathbf{x})\,,\,\phi(t,\mathbf{y})\} = 0 = \{\Pi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\}$$

one can immediately get

$$\begin{cases} \dot\phi(x) = \{\phi(t,\mathbf{x})\,,\,H\} = \delta H/\delta\Pi(x) \\ \dot\Pi(x) = \{\Pi(t,\mathbf{x})\,,\,H\} = -\,\delta H/\delta\phi(x) \end{cases}$$

and thereby

$$\dot{\phi}(x) = \Pi\,(x) \tag{3.15}$$

$$\dot{\Pi}\,(x) = \ddot{\phi}(x) = \boldsymbol{\nabla}^2\phi(x) - m^2\,\phi(x) - \frac{\lambda}{3!}\,\phi^3(x) \tag{3.16}$$

which is nothing but the Euler-Lagrange field equation

$$(\,\Box + m^2\,)\phi(x) = -\,\tfrac{1}{6}\,\lambda\,\phi^3(x) \tag{3.17}$$

`Proof.` As a matter of fact we find

$$\begin{aligned}
\dot{\phi}\,(t,\mathbf{x}) &= \{\,\phi\,(t,\mathbf{x})\,,\,H\,\} = \int \mathrm{d}\mathbf{z}\,\frac{\delta\phi(t,\mathbf{x})}{\delta\phi(t,\mathbf{z})}\cdot\frac{\delta H[\phi,\Pi]}{\delta\,\Pi(t,\mathbf{z})} = \int \mathrm{d}\mathbf{z}\,\delta(\mathbf{x}-\mathbf{z})\,\frac{\delta H[\phi,\Pi]}{\delta\,\Pi(t,\mathbf{z})} \\
&= \frac{\delta H[\phi,\Pi]}{\delta\,\Pi(t,\mathbf{x})} = \tfrac{1}{2}\int \mathrm{d}\mathbf{y}\,\frac{\delta\,\Pi^2(t,\mathbf{y})}{\delta\,\Pi(t,\mathbf{x})} = \int \mathrm{d}\mathbf{y}\,\Pi(t,\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y}) = \Pi(t,\mathbf{x}) \\
\dot{\Pi}(t,\mathbf{x}) &= \{\,\Pi\,(t,\mathbf{x})\,,\,H\,\} = -\int \mathrm{d}\mathbf{z}\,\frac{\delta\,\Pi(t,\mathbf{x})}{\delta\,\Pi(t,\mathbf{z})}\cdot\frac{\delta H[\phi,\Pi]}{\delta\,\phi(t,\mathbf{z})} = -\int \mathrm{d}\mathbf{z}\,\delta(\mathbf{x}-\mathbf{z})\,\frac{\delta H[\phi,\Pi]}{\delta\,\phi(t,\mathbf{z})} \\
&= -\,\frac{\delta H[\phi,\Pi]}{\delta\,\phi(t,\mathbf{x})} = \tfrac{1}{2}\int \mathrm{d}\mathbf{y}\,\frac{\delta}{\delta\,\phi(t,\mathbf{x})}\,[\,\phi(t,\mathbf{y})\boldsymbol{\nabla}^2\phi(t,\mathbf{y})\,] \\
&- \int \mathrm{d}\mathbf{y}\,\delta(\mathbf{x}-\mathbf{y})\left[m^2\,\phi(t,\mathbf{y}) + \frac{\lambda}{6}\,\phi^3(t,\mathbf{y})\right]
\end{aligned}$$

Now we get

$$\begin{aligned}
&\tfrac{1}{2}\int \mathrm{d}\mathbf{y}\,\frac{\delta}{\delta\,\phi(t,\mathbf{x})}\,[\,\phi(t,\mathbf{y})\boldsymbol{\nabla}^2\phi(t,\mathbf{y})\,] \\
=\;& \tfrac{1}{2}\int \mathrm{d}\mathbf{y}\,[\,\delta(\mathbf{x}-\mathbf{y})\boldsymbol{\nabla}^2\phi(t,\mathbf{y}) + \phi(t,\mathbf{y})\boldsymbol{\nabla}^2_{\mathbf{y}}\,\delta(\mathbf{x}-\mathbf{y})\,] \\
=\;& \tfrac{1}{2}\,\boldsymbol{\nabla}^2\phi(t,\mathbf{x}) + \tfrac{1}{2}\int \mathrm{d}\mathbf{y}\,\phi(t,\mathbf{y})\boldsymbol{\nabla}^2_{\mathbf{x}}\,\delta(\mathbf{x}-\mathbf{y}) \\
=\;& \tfrac{1}{2}\,\boldsymbol{\nabla}^2\phi(t,\mathbf{x}) + \tfrac{1}{2}\boldsymbol{\nabla}^2_{\mathbf{x}}\int \mathrm{d}\mathbf{y}\,\phi(t,\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y}) = \boldsymbol{\nabla}^2\phi(t,\mathbf{x})
\end{aligned}$$

Putting altogether we can eventually write

$$\dot{\Pi}(t,\mathbf{x}) = -\,\frac{\delta H[\phi,\Pi]}{\delta\,\phi(t,\mathbf{x})} = \boldsymbol{\nabla}^2\phi(t,\mathbf{x}) - m^2\,\phi(t,\mathbf{x}) - \frac{\lambda}{3!}\,\phi^3(t,\mathbf{x})$$

which endorses the Euler-Lagrange field equations. $\Box$

`Action and Lagrangian in Physical Units.` According to the classical mechanics of continuous systems we have

$$S[\phi] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\,L[\,\phi,\dot{\phi}\,] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \int \mathrm{d}\mathbf{x}\,\mathcal{L}[\,\phi(t,\mathbf{x}),\dot{\phi}(t,\mathbf{x})\,]$$

so that in the C. G. S. system of units $[\,S\,] = \mathrm{eV\ s}$. As a consequence the Lagrangian functional for a real self-interacting classical scalar field in physical units takes the form

$$\mathcal{L}[\,\phi(t,\mathbf{x}),\dot{\phi}(t,\mathbf{x})\,] = \frac{1}{2c^2}\,\dot{\phi}^2(t,\mathbf{x}) - \frac{1}{2}\,\mid\nabla\phi(t,\mathbf{x})\mid^2 - \frac{m^2c^2}{2\hbar^2}\,\phi^2(t,\mathbf{x}) - \frac{\lambda}{4!\hbar c}\,\phi^4(t,\mathbf{x})$$

whence we can readily get

$$[\mathcal{L}] = \text{eV cm}^{-3} \qquad [\phi] = \sqrt{\text{eV/cm}} \qquad \lambda \geq 0$$

and from its very definition

$$\Pi(t, \mathbf{x}) = \delta\mathcal{L}[\phi(t, \mathbf{x}), \dot{\phi}(t, \mathbf{x})]/\delta\dot{\phi}(t, \mathbf{x}) = \dot{\phi}(t, \mathbf{x})/c^2 \qquad [\Pi] = \text{eV}^{\frac{1}{2}} \text{ s cm}^{-\frac{5}{2}}$$

so that $[\Pi\dot{\phi}] = \text{eV/cm}^3$. Thus we obtain the classical non-vanishing Poisson's brackets

$$\begin{aligned}\{\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})\} &= \int \mathrm{d}\mathbf{z}\, \frac{\delta\phi(t, \mathbf{x})}{\delta\phi(t, \mathbf{z})} \cdot \frac{\delta\Pi(t, \mathbf{y})}{\delta\Pi(t, \mathbf{z})} \\ &= \int \mathrm{d}\mathbf{z}\, \delta(\mathbf{x} - \mathbf{z})\, \delta(\mathbf{y} - \mathbf{z}) = \delta(\mathbf{x} - \mathbf{y})\end{aligned}$$

while the Hermitean quantum field operator valued distributions $\widehat{\phi}(t, \mathbf{x})$ and $\widehat{\Pi}(t, \mathbf{y})$ keep the very same physical dimensions so that the canonical commutation relations yields

$$[\,\widehat{\phi}(t, \mathbf{x})\,,\, \widehat{\Pi}(t, \mathbf{y})\,] = i\hbar\, \delta(\mathbf{x} - \mathbf{y}) = i\hbar\, \{\phi(t, \mathbf{x})\,,\, \Pi(t, \mathbf{y})\}$$

all the other commutators being null, according to the Dirac-Heisenberg correspondence principle of quantum theory.

It turns out that NOBODY STILL KNOWS ANY EXACT SOLUTION of the simplest classical field theoretic model: namely, a mass-less scalar field with ϕ_4^4 self-interaction in the *1+3* dimensional Minkowski space $\mathcal{M}$

$$\Box\,\phi(x) + \tfrac{1}{6}\,\lambda\,\phi^3(x) = 0 \qquad (\,x \in \mathcal{M}\,,\ \lambda > 0\,)$$

`Spontaneous Symmetry Breaking`. Consider the classical Lagrangian for a real scalar field

$$\mathcal{L} = \tfrac{1}{2}\,\partial_\mu\,\phi\,(x)\,\partial^\mu\,\phi\,(x) + \tfrac{1}{2}\,\mu^2\,\phi^2\,(x) - \tfrac{1}{4}\,\lambda\,\phi^4\,(x)$$

with $\mu \in \mathbb{R}$ and $\lambda > 0$. The invariance symmetry groups of the model are the non-homogeneous full Lorentz group IO(1,3) and the internal discrete symmetry group $\mathbb{Z}_2$. The canonical energy-momentum tensor of the model, which is symmetric, is provided by

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,\phi(x)}\;\partial^\nu\,\phi(x)\; - \;\mathcal{L}(x)\,g^{\mu\nu} \; = \; T^{\nu\mu}$$

in such a manner that we obtain the classical Hamiltonian functional

$$H = \frac{1}{2}\int \mathrm{d}\mathbf{x}\;\left[\,\Pi^2(t, \mathbf{x}) + |\nabla\phi(t, \mathbf{x})|^2 - \mu^2\,\phi^2(t, \mathbf{x}) + \tfrac{1}{2}\,\lambda\,\phi^4(t, \mathbf{x})\,\right] = cP_0$$

which represents the total energy of the system, as well as the total momentum

$$\mathbf{P} = -\int \mathrm{d}\mathbf{x}\; \Pi(t, \mathbf{x})\, \nabla\phi(t, \mathbf{x})$$

Notice that the mass term in the Hamiltonian just corresponds to an imaginary tachyon-like mass $i\mu$. The classical field configurations which minimize the total energy are required to satisfy

$$\Pi(t,\mathbf{x}) = \dot{\phi}(t,\mathbf{x}) = 0 \qquad\qquad \nabla\phi(t,\mathbf{x}) = 0$$

which yields $\phi =$ constant and thereby

$$\frac{\mathrm{d}H}{\mathrm{d}\phi} = -\mu^2\,\phi + \lambda\,\phi^3 = 0 \Rightarrow \phi = 0 \vee \phi^2 = \frac{\mu^2}{\lambda} \equiv v^2 > 0$$

in such a manner that

$$H(\phi = 0) = 0 \qquad\qquad H(v) = -\tfrac{1}{4}\,\mu^2\,v^2 \int \mathrm{d}\mathbf{x} < 0$$

that means that the two constant field configurations that minimize the energy density are $\phi_\pm = \pm\, v$ which are evidently $\mathbb{Z}_2$ symmetric. After setting

$$\phi(x) = v + \varphi(x) \qquad\qquad \dot{\varphi}(x) = \Pi(x)$$

we readily obtain

$$\begin{aligned}
\mathcal{L}_\varphi &= \tfrac{1}{2}\,\partial_\nu\,\varphi\,(x)\,\partial^\nu\,\varphi\,(x) + \tfrac{1}{2}\,\mu^2\,[\,\pm\,v + \varphi\,(x)\,]^2 - \tfrac{1}{4}\,\lambda\,[\,\pm\,v + \varphi\,(x)\,]^4 \\
&= \tfrac{1}{2}\,\partial_\nu\,\varphi\,(x)\,\partial^\nu\,\varphi\,(x) + \tfrac{1}{2}\,\mu^2\,v^2 + \tfrac{1}{2}\,\mu^2\,\varphi^2\,(x) \pm \mu^2\,v\,\varphi\,(x) \\
&- \tfrac{1}{4}\,\lambda\Big(v^4 \pm 4v^3\varphi\,(x) + 6v^2\varphi^2\,(x) \pm 4v\,\varphi^3\,(x) + \varphi^4\,(x)\Big) \\
&= \tfrac{1}{2}\,\partial_\nu\,\varphi\,(x)\,\partial^\nu\,\varphi\,(x) + \tfrac{1}{4}\,\lambda\,v^4 - \mu^2\,\varphi^2\,(x) \\
&\mp \lambda\,v\,\varphi^3\,(x) - \tfrac{1}{4}\,\lambda\,\varphi^4\,(x)
\end{aligned}$$

and thereby

$$\begin{aligned}
H &= H_0 + H_{\mathrm{int}} \\
H_0 &= \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\ \big[\,\Pi^2(t,\mathbf{x}) + |\boldsymbol{\nabla}\varphi(t,\mathbf{x})|^2 + 2\mu^2\,\varphi^2(t,\mathbf{x}) - \mu^4/2\lambda\,\big] \\
H_{\mathrm{int}} &= \int \mathrm{d}\mathbf{x}\ \Big[\,\tfrac{1}{4}\,\lambda\,\varphi^4(t,\mathbf{x}) \pm \mu\,\sqrt{\lambda}\,\varphi^3\,(t,\mathbf{x})\,\Big]
\end{aligned}$$

It follows that the shifted field $\varphi\,(t,\mathbf{x})$ has a mass $\mu\sqrt{2}$, while a negative zero-point energy density does appear at the classical level, *viz.*

$$\rho_0 = -\,\frac{\mu^4 c^5}{4\lambda\hbar^3} = -\,\frac{\lambda v^4}{4(\hbar c)^3}$$

where v is expressed in GeV, whereas the $\mathbb{Z}_2$ symmetry is definitely broken.

3.2 Normal Modes Expansion

Consider the non-interacting or free scalar field theory, in which the Action, the Lagrangian and the Hamiltonian are quadratic functional of the scalar field function, so that the Euler-Lagrange equations of motion becomes linear and exactly solvable

$$\begin{aligned} \mathcal{L}_0 &= \tfrac{1}{2}\, g^{\mu\nu}\,\partial_\mu \phi(x)\,\partial_\nu \phi(x) - \tfrac{1}{2}\, m^2\, \phi^2(x) & (3.18) \\ H_0 &= \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\left[\Pi^2(x) - \phi(x)\boldsymbol{\nabla}^2\phi(x) + m^2\,\phi^2(x)\right] & (3.19) \end{aligned}$$

$$\left(\Box + m^2\right)\,\phi(x) = 0 \qquad (3.20)$$

which is the well known Klein-Gordon relativistic wave equation. To solve it, let us introduce the Fourier decomposition

$$\phi(x) \equiv (2\pi)^{-3/2}\int \mathrm{d}k\;\tilde{\phi}(k)\,\exp\{-\,ik\cdot x\} \qquad \tilde{\phi}^*(k) = \tilde{\phi}(-k) \qquad (3.21)$$

so that

$$(k^2 - m^2)\,\tilde{\phi}(k) = 0$$

the most general solution of which reads

$$\tilde{\phi}(k) = f(k)\,\delta(k^2 - m^2)$$

$f(k)$ being an arbitrary complex function which is regular on the momentum space hyperbolic manifold $k^2 = m^2$ ($k \in \mathcal{M}$) and which fulfill the reality condition $f^*(k) = f(-k)$ to endorse the real nature of a neutral scalar field. Then we have

$$\phi(x) \equiv (2\pi)^{-3/2}\int \mathrm{d}k\; f(k)\,\delta(k^2 - m^2)\,\exp\{-\,ik\cdot x\} \qquad (3.22)$$

and from the well known tempered distribution identities

$$\delta(ax) = \frac{1}{|a|}\,\delta(x) \qquad \theta(x) + \theta(-x) = 1 \qquad (3.23)$$

one can readily obtain the decomposition

$$\delta(k^2 - m^2) = \frac{1}{2\omega_{\mathbf{k}}}\left[\,\theta(k_0)\,\delta(k_0 - \omega_{\mathbf{k}}) + \theta(-\,k_0)\,\delta(k_0 + \omega_{\mathbf{k}})\,\right]$$

in which the Klein-Gordon dispersion relation holds true

$$\omega_{\mathbf{k}} = \omega(\mathbf{k}) \equiv c\,\sqrt{\mathbf{k}^2 + m^2c^2/\hbar^2} = \omega_{-\mathbf{k}} \qquad (3.24)$$

Hence

$$\begin{aligned}
\phi(x) &= (2\pi)^{-3/2} \int \mathrm{d}k \ f(k) \exp\{-\,ik \cdot x\} \\
&\times \ \frac{1}{2\omega_{\mathbf{k}}} \left[\, \theta(k_0)\, \delta(k_0 - \omega_{\mathbf{k}}) + \theta(-\,k_0)\, \delta(k_0 + \omega_{\mathbf{k}}) \,\right] \\
&= (2\pi)^{-3/2} \int \frac{\mathrm{d}\mathbf{k}}{2\omega_{\mathbf{k}}} \ f(\omega_{\mathbf{k}}, \mathbf{k}) \exp\{-\,it\omega_{\mathbf{k}} + i\,\mathbf{k} \cdot \mathbf{x}\} \\
&+ (2\pi)^{-3/2} \int \frac{\mathrm{d}\mathbf{k}}{2\omega_{\mathbf{k}}} \ f(-\omega_{\mathbf{k}}, -\mathbf{k}) \exp\{it\omega_{\mathbf{k}} - i\,\mathbf{k} \cdot \mathbf{x}\} \\
&= \int \frac{\mathrm{d}\mathbf{k}}{[\, 2\omega_{\mathbf{k}}(2\pi)^3\,]^{1/2}} \cdot \frac{f(\omega_{\mathbf{k}}, \mathbf{k})}{\sqrt{(2\omega_{\mathbf{k}})}} \exp\{-\,it\omega_{\mathbf{k}} + i\,\mathbf{k} \cdot \mathbf{x}\} + \text{c.c.}
\end{aligned}$$

where we used the reality condition. It is very convenient to set

$$u_{\mathbf{k}}(x) \equiv \sqrt{c}\,[\, 2\omega_{\mathbf{k}}(2\pi)^3\,]^{-\frac{1}{2}} \exp\{i\,\mathbf{k} \cdot \mathbf{x} - it\omega_{\mathbf{k}}\} \tag{3.25}$$

so that $[\, u_{\mathbf{k}}\,] = \sqrt{\mathrm{cm}}$ while

$$\begin{aligned}
\phi(x) &= \sum_{\mathbf{k}} \Big[\, f_{\mathbf{k}}\, u_{\mathbf{k}}(x) + f^*_{\mathbf{k}}\, u^*_{\mathbf{k}}(x) \Big] \\
\Pi(x) &= \sum_{\mathbf{k}} i\omega_{\mathbf{k}} \Big[\, f^*_{\mathbf{k}}\, u^*_{\mathbf{k}}(x) - f_{\mathbf{k}}\, u_{\mathbf{k}}(x) \Big]
\end{aligned} \tag{3.26}$$

with the suitable notations

$$f_{\mathbf{k}} \equiv \frac{f(\omega_{\mathbf{k}}, \mathbf{k})}{\sqrt{(2\omega_{\mathbf{k}})}} \qquad\qquad \int \mathrm{d}\mathbf{k} \equiv \sum_{\mathbf{k}}$$

which endorse the fact that (3.26) is nothing but the normal modes expansion of the real scalar field. The wave functions $u_{\mathbf{k}}(x)$ ($\mathbf{k} \in \mathbb{R}^3$) do constitute a complete and orthogonal set, *i.e.* the normal modes of the real scalar free field: namely, they fulfill the orthogonality relations

$$(\, u_{\mathbf{h}}\,,\, u_{\mathbf{k}}\,) \equiv \int \mathrm{d}\mathbf{x}\ u^*_{\mathbf{h}}(x)\, i\, \overleftrightarrow{\partial}_0\, u_{\mathbf{k}}(x) = \delta(\mathbf{h} - \mathbf{k}) \tag{3.27}$$

$$(\, u^*_{\mathbf{h}}\,,\, u^*_{\mathbf{k}}\,) \equiv \int \mathrm{d}\mathbf{x}\ u_{\mathbf{h}}(x)\, i\, \overleftrightarrow{\partial}_0\, u^*_{\mathbf{k}}(x) = -\,\delta(\mathbf{h} - \mathbf{k}) \tag{3.28}$$

$$(\, u^*_{\mathbf{h}}\,,\, u_{\mathbf{k}}\,) \equiv \int \mathrm{d}\mathbf{x}\ u_{\mathbf{h}}(x)\, i\, \overleftrightarrow{\partial}_0\, u_{\mathbf{k}}(x) = 0 \tag{3.29}$$

$$(\, u_{\mathbf{h}}\,,\, u^*_{\mathbf{k}}\,) \equiv \int \mathrm{d}\mathbf{x}\ u^*_{\mathbf{h}}(x)\, i\, \overleftrightarrow{\partial}_0\, u^*_{\mathbf{k}}(x) = 0 \tag{3.30}$$

as well as the closure relations

$$\sum_{\mathbf{k}} u_{\mathbf{k}}(x)\, u^*_{\mathbf{k}}(y) = -\, iD^{\,(-)}(x-y) \tag{3.31}$$

$$\sum_{\mathbf{k}} u^*_{\mathbf{k}}(x)\, u_{\mathbf{k}}(y) = iD^{\,(+)}(x-y) \tag{3.32}$$

where I have set

$$D^{\,(\pm)}(x-y) \stackrel{\mathrm{def}}{=} \mp\, i \int \frac{\mathrm{d}^4 k}{(2\pi)^3}\; \theta\,(k_0)\; \delta\left(k^2 - m^2c^2/\hbar^2\right)\, \mathrm{e}^{\pm\, ik\cdot(x-y)}$$

From these orthogonality relations it is easy to invert the normal modes expansions that yields

$$\begin{aligned} (\, u_{\mathbf{h}}\,,\, \phi\,) &= (\, u_{\mathbf{h}}\,,\, \textstyle\sum_{\mathbf{k}} [\, f_{\mathbf{k}}\, u_{\mathbf{k}} + f^*_{\mathbf{k}}\, u^*_{\mathbf{k}}\,]\,) \\ &= \sum_{\mathbf{k}} \Big[\, f_{\mathbf{k}}\,(\, u_{\mathbf{h}}\,,\, u_{\mathbf{k}}\,) + f^*_{\mathbf{k}}\,(\, u_{\mathbf{h}}\,,\, u^*_{\mathbf{k}}\,)\Big] \\ &= \sum_{\mathbf{k}} f_{\mathbf{k}}\, \delta(\mathbf{h}-\mathbf{k}) = f_{\mathbf{h}} \qquad (3.33) \\ (\, u^*_{\mathbf{h}}\,,\, \phi\,) &= -\, f^*_{\mathbf{h}} \qquad (3.34) \end{aligned}$$

Notice that the amplitudes $f_{\mathbf{k}}$ of the normal modes have canonical dimensions $[\, f_{\mathbf{k}}\,] = \sqrt{\mathrm{eV}}\ \mathrm{cm}^2$ in physical units and $[\, f_{\mathbf{k}}\,] = \mathrm{cm}^{\frac{3}{2}}$ in natural units.

As it is well known the normal modes decomposition is what we need to set into diagonal form the energy-momentum vector of the mechanical system. As a matter of fact, from the equality in natural units

$$\begin{aligned} P_0 &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} \Big[\, \Pi^2\,(x) + \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) + m^2\phi^2(x)\Big] \\ &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} \Big\{\dot\phi(x)\,\Pi(x) - \phi(x)\Big[\,\boldsymbol{\nabla}^{\,2}\phi(x) - m^2\,\phi(x)\Big]\Big\} \\ &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} \Big\{\dot\phi(x)\,\Pi\,(x) - \phi(x)\,\dot\Pi(x)\Big\} \\ &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2}\,\Pi(x)\,\overleftrightarrow{\partial_0}\,\phi(x) \qquad (3.35) \\ \mathbf{P} &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} \Big\{\phi(x)\boldsymbol{\nabla}\dot\phi(x) - \dot\phi(x)\boldsymbol{\nabla}\phi(x)\Big\} \\ &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2}\,\phi(x)\,\overleftrightarrow{\partial_0}\boldsymbol{\nabla}\phi(x) \qquad (3.36) \end{aligned}$$

by substituting the normal modes expansions

$$\phi(x) = \sum_{\mathbf{k}} \Big[\, f_{\mathbf{k}}\, u_{\mathbf{k}}(x) + f_{\mathbf{k}}^{*}\, u_{\mathbf{k}}^{*}(x) \Big]$$
$$\Pi(x) = \sum_{\mathbf{p}} i\, \omega_{\mathbf{p}} \Big[\, f_{\mathbf{p}}^{*}\, u_{\mathbf{p}}^{*}(x) - f_{\mathbf{p}}\, u_{\mathbf{p}}(x) \Big]$$

from the orthogonality relations we obtain

$$\begin{aligned} P_0 &= \sum_{\mathbf{k}} \sum_{\mathbf{p}} \tfrac{1}{2}\, \omega_{\mathbf{p}} \int \mathrm{d}\mathbf{x} \Big[\, f_{\mathbf{p}}^{*}\, u_{\mathbf{p}}^{*}(x) - f_{\mathbf{p}}\, u_{\mathbf{p}}(x) \Big] i \overset{\leftrightarrow}{\partial}_0 \Big[\, f_{\mathbf{k}}\, u_{\mathbf{k}}(x) + f_{\mathbf{k}}^{*}\, u_{\mathbf{k}}^{*}(x) \Big] \\ &= \sum_{\mathbf{k}} \sum_{\mathbf{p}} \tfrac{1}{2}\, \omega_{\mathbf{p}} \Big[\, f_{\mathbf{p}}^{*}\, f_{\mathbf{k}}\, (\, u_{\mathbf{p}} \,, u_{\mathbf{k}} \,) - f_{\mathbf{p}}\, f_{\mathbf{k}}^{*}\, \big(\, u_{\mathbf{p}}^{*} \,, u_{\mathbf{k}}^{*} \,\big) \;\Big] \\ &= \sum_{\mathbf{k}} \tfrac{1}{2}\, \omega_{\mathbf{k}} \Big(f_{\mathbf{k}} f_{\mathbf{k}}^{*} + f_{\mathbf{k}}^{*} f_{\mathbf{k}} \Big) = \int \mathrm{d}\mathbf{k}\, \omega(\mathbf{k}) |\, f(\mathbf{k})\,|^{2} \end{aligned} \tag{3.37}$$

Moreover we find

$$\begin{aligned} \mathbf{P} &= \sum_{\mathbf{k}} \sum_{\mathbf{p}} \tfrac{1}{2}\, \mathbf{p} \int \mathrm{d}\mathbf{x} \Big[\, f_{\mathbf{k}}\, u_{\mathbf{k}}(x) + f_{\mathbf{k}}^{*}\, u_{\mathbf{k}}^{*}(x) \Big] i \overset{\leftrightarrow}{\partial}_0 \Big[\, f_{\mathbf{p}}\, u_{\mathbf{p}}(x) - f_{\mathbf{p}}^{*}\, u_{\mathbf{p}}^{*}(x) \Big] \\ &= \sum_{\mathbf{k}} \sum_{\mathbf{p}} \tfrac{1}{2}\, \mathbf{p} \Big[\, f_{\mathbf{p}}^{*}\, f_{\mathbf{k}}\, (\, u_{\mathbf{p}} \,, u_{\mathbf{k}} \,) - f_{\mathbf{p}}\, f_{\mathbf{k}}^{*}\, \big(\, u_{\mathbf{p}}^{*} \,, u_{\mathbf{k}}^{*} \,\big) \;\Big] \\ &= \sum_{\mathbf{k}} \tfrac{1}{2}\, \mathbf{k} \Big(f_{\mathbf{k}} f_{\mathbf{k}}^{*} + f_{\mathbf{k}}^{*} f_{\mathbf{k}} \Big) = \int \mathrm{d}\mathbf{k}\, \mathbf{k}\, |\, f(\mathbf{k})\,|^{2} \end{aligned} \tag{3.38}$$

in such a manner that we can definitely write

$$P_{\mu} = \sum_{\mathbf{k}} \tfrac{1}{2}\, k_{\mu} \Big(f_{\mathbf{k}} f_{\mathbf{k}}^{*} + f_{\mathbf{k}}^{*} f_{\mathbf{k}} \Big) = \int \mathrm{d}\mathbf{k}\, k_{\mu}\, |\, f(\mathbf{k})\,|^{2} \qquad (\, k_0 = \omega_{\mathbf{k}}\,) \tag{3.39}$$

The complex amplitudes of the normal modes are called sometimes the Holomorphic Coordinates of the real scalar Klein-Gordon field. It is also quite evident that by introducing the related real canonical coordinates

$$\begin{cases} Q_{\mathbf{k}} \equiv & (2m\omega_{\mathbf{k}})^{-1/2}\, (f_{\mathbf{k}} + f_{\mathbf{k}}^{*}) \\ P_{\mathbf{k}} \equiv & -\, i\, \big(\tfrac{1}{2}\, m\omega_{\mathbf{k}}\big)^{1/2}\, (f_{\mathbf{k}} - f_{\mathbf{k}}^{*}) \end{cases}$$

$$2 f_{\mathbf{k}} = Q_{\mathbf{k}}\, \sqrt{2m\omega_{\mathbf{k}}} + i P_{\mathbf{k}}\, \sqrt{\frac{2}{m\omega_{\mathbf{k}}}} \tag{3.40}$$

we can write eventually

$$\begin{aligned} P_0 = \sum_{\mathbf{k}} H_{\mathbf{k}} &= \frac{1}{2m}\int \mathrm{d}\mathbf{k}\,\left(P_{\mathbf{k}}^2 + m^2\omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^2\right) \\ &= \sum_{\mathbf{k}} \tfrac{1}{2}\,\omega_{\mathbf{k}}\Big(f_{\mathbf{k}} f_{\mathbf{k}}^* + f_{\mathbf{k}}^* f_{\mathbf{k}}\Big) \end{aligned} \tag{3.41}$$

with

$$[\,P_{\mathbf{k}}\,] = \mathrm{g\ s^{-1}\ cm^{\frac{5}{2}}} \qquad\qquad [\,Q_{\mathbf{k}}\,] = \mathrm{cm^{\frac{5}{2}}}$$

which explicitly shows that a real scalar field is dynamically fully equivalent to an assembly of an infinite number of decoupled harmonic oscillators of unit mass and principal angular frequencies $\omega_{\mathbf{k}} = c\,(\mathbf{k}^2 + m^2c^2/\hbar^2)^{1/2}$.

As a matter of fact, we can rewrite the field and conjugated momentum expansions (3.26) in the suggestive form

$$\begin{aligned} \phi(x) &= \sum_{\mathbf{k}} \Big[\, f_{\mathbf{k}}(t)\, u_{\mathbf{k}}(\mathbf{x}) + f_{\mathbf{k}}^*(t)\, u_{\mathbf{k}}^*(\mathbf{x}) \,\Big] \\ \Pi(x) &= \sum_{\mathbf{k}} (i\omega_{\mathbf{k}}/c^2)\Big[\, f_{\mathbf{k}}^*(t)\, u_{\mathbf{k}}^*(\mathbf{x}) - f_{\mathbf{k}}(t)\, u_{\mathbf{k}}(\mathbf{x}) \,\Big] \end{aligned} \tag{3.42}$$

where of course

$$\begin{aligned} & f_{\mathbf{k}}(t) = f_{\mathbf{k}}\ \exp\{-\,it\omega_{\mathbf{k}}\} \qquad f_{\mathbf{k}}^*(t) = f_{\mathbf{k}}^*\ \exp\{it\omega_{\mathbf{k}}\} \\ & u_{\mathbf{k}}(\mathbf{x}) = \sqrt{c}\,[\,2\omega_{\mathbf{k}}(2\pi)^3\,]^{-1/2}\ \exp\{i\,\mathbf{k}\cdot\mathbf{x}\} \end{aligned} \tag{3.43}$$

and from the standard canonical Poisson's brackets [1] for the linear oscillator

$$\{Q_{\mathbf{h}}\,,\,Q_{\mathbf{k}}\} = 0 \qquad \{P_{\mathbf{h}}\,,\,P_{\mathbf{k}}\} = 0$$

$$\{Q_{\mathbf{h}}\,,\,P_{\mathbf{k}}\} = \delta\,(\,\mathbf{h} - \mathbf{k})$$

from (3.40) it is very simple to derive the canonical Hamiltonian equations for the time dependent holomorphic coordinates $f_{\mathbf{k}}(t)$: namely,

$$\{f_{\mathbf{h}}\,,\,f_{\mathbf{k}}\} = 0 = \{f_{\mathbf{h}}^*\,,\,f_{\mathbf{k}}^*\} \qquad \{f_{\mathbf{h}}\,,\,f_{\mathbf{k}}^*\} = -\,i\,\delta(\mathbf{h} - \mathbf{k}) \tag{3.44}$$

$$\dot{f}_{\mathbf{k}}(t) = \{f_{\mathbf{k}}(t)\,,\,H\} = -\,i\,\omega_{\mathbf{k}}\,f_{\mathbf{k}}(t) \tag{3.45}$$

the solution of which is just provided by (3.43).

[1] We recall the definition of the Poisson's brackets for two analytic functions F and G on the phase space of a mechanical system

$$\{F\,,\,G\} \equiv \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right)$$

3.3 Klein-Gordon Quantum Field

Once that the dynamical treatment of the free real scalar field has been developed within the canonical Hamiltonian formulation, the transition to the quantum theory of the system will directly follow in accordance with the Bohr-Dirac-Heisenberg correspondence principle – see any book of quantum mechanics *e.g.* [18]. According to the rules of correspondence which lead to the quantum theory of a linear harmonic oscillator, we shall introduce for each normal mode of the real scalar field the corresponding linear operators acting on the related Hilbert space and the associated algebra, *i.e.*

$$Q_{\mathbf{k}} \longmapsto \widehat{Q}_{\mathbf{k}} = \widehat{Q}_{\mathbf{k}}^{\dagger} \qquad P_{\mathbf{k}} \longmapsto \widehat{P}_{\mathbf{k}} = \widehat{P}_{\mathbf{k}}^{\dagger}$$

$$\widehat{P}_{\mathbf{k}} = -\,i\hbar\,\frac{\mathrm{d}}{\mathrm{d}Q_{\mathbf{k}}}$$

$$\{Q_{\mathbf{k}}\,,\,P_{\mathbf{k}'}\} \longmapsto \frac{1}{i\hbar}\,[\,\widehat{Q}_{\mathbf{k}}\,,\,\widehat{P}_{\mathbf{k}'}\,]$$

$$H_{\mathbf{k}} \longmapsto \widehat{H}_{\mathbf{k}} = \frac{1}{2m}\left(\widehat{P}_{\mathbf{k}}^{\,2} + m^2\omega_{\mathbf{k}}^2\widehat{Q}_{\mathbf{k}}^{\,2}\right) \tag{3.46}$$

and the Poisson's brackets among the holomorphic coordinates turn into the commutators among the creation destruction operators

$$f_{\mathbf{k}} \longmapsto \sqrt{\hbar c}\,a_{\mathbf{k}} \qquad f_{\mathbf{k}}^{*} \longmapsto a_{\mathbf{k}}^{\dagger}\,\sqrt{\hbar c} \tag{3.47}$$

$$[\,a_{\mathbf{k}}\,,\,a_{\mathbf{k}'}\,] = 0 \qquad [\,a_{\mathbf{k}}^{\dagger}\,,\,a_{\mathbf{k}'}^{\dagger}\,] = 0 \tag{3.48}$$

$$[\,a_{\mathbf{k}}\,,\,a_{\mathbf{k}'}^{\dagger}\,] = \delta(\mathbf{k}-\mathbf{k}') \tag{3.49}$$

As a consequence the scalar field function $\phi(x)$ together with its conjugated momentum $\Pi(x)$ will turn, after the transition to the quantum theory, into operator valued tempered distributions, the normal modes expansions of which can be obtained in a straightforward way from (3.26) and (3.47), that is

$$\phi(x) = \sum_{\mathbf{k}}\left[\,a_{\mathbf{k}}\,u_{\mathbf{k}}(x) + a_{\mathbf{k}}^{\dagger}\,u_{\mathbf{k}}^{*}(x)\,\right]\sqrt{\hbar c}$$

$$c\,\Pi(x) = i\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}}{c}\left[\,a_{\mathbf{k}}^{\dagger}\,u_{\mathbf{k}}^{*}(x) - a_{\mathbf{k}}\,u_{\mathbf{k}}(x)\,\right]\sqrt{\hbar c} \tag{3.50}$$

$$u_{\mathbf{k}}(x) \equiv \sqrt{c}\,[\,2\omega_{\mathbf{k}}(2\pi)^3\,]^{-\frac{1}{2}}\ \exp\{\,i\,\mathbf{k}\cdot\mathbf{x} - i\omega_{\mathbf{k}}t\} \tag{3.51}$$

$$[\,\phi(t,\mathbf{x})\,,\,\phi(t,\mathbf{y})\,] = 0 = [\,\Pi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\,]$$

$$[\,\phi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\,] \;=\; i\hbar\,\delta\,(\mathbf{x}-\mathbf{y}) \tag{3.52}$$

together with the inversion formulæ – see eq.s (3.33) and (3.34)

$$(\, u_{\mathbf{k}}\,,\, \phi\,) = a_{\mathbf{k}}\,\sqrt{\hbar c} \qquad\qquad (\, u^{*}_{\mathbf{k}}\,,\, \phi\,) \;\;=\;\; -\, a^{\dagger}_{\mathbf{k}}\,\sqrt{\hbar c} \tag{3.53}$$

This means that the classical expressions (3.39) and (3.41) of the energy and momentum of the free real scalar field will turn into the quantum operator expressions

$$\begin{aligned} cP_0 &= \sum_{\mathbf{k}} \tfrac{1}{2}\left(\widehat{P}^{\,2}_{\mathbf{k}} + \omega^2_{\mathbf{k}}\widehat{Q}^{\,2}_{\mathbf{k}}\right) \\ &= \sum_{\mathbf{k}} \tfrac{1}{2}\,\hbar\omega_{\mathbf{k}}\Big(a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}} + a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}\Big) \\ &= \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\,a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}} + \delta(\mathbf{0})\sum_{\mathbf{k}} \tfrac{1}{2}\,\hbar\omega_{\mathbf{k}} \end{aligned} \tag{3.54}$$

$$\begin{aligned} \mathbf{P} &= \sum_{\mathbf{k}} \tfrac{1}{2}\,\hbar\mathbf{k}\,\Big(a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}} + a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}\Big) \\ &= \sum_{\mathbf{k}} \hbar\mathbf{k}\,a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}} \end{aligned} \tag{3.55}$$

It is worthwhile to spend some few words concerning the divergent quantity

$$\begin{aligned} cU_0 &= \delta(\mathbf{0})\sum_{\mathbf{k}} \tfrac{1}{2}\,\hbar\omega_{\mathbf{k}} \equiv V\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\;\tfrac{1}{2}\,\hbar\omega_{\mathbf{k}} \\ &= V\hbar c\int_0^K \mathrm{d}k\;\frac{k^2}{4\pi^2}\,\sqrt{k^2 + m^2c^2/\hbar^2} \end{aligned} \tag{3.56}$$

where V is the volume of a very large box and $\hbar K \gg mc$ is a very large wave-number. The latter is called the vacuum energy or zero-point energy of the real scalar field. Since we know that a free real scalar field is dynamically equivalent to an infinite (continuous) set of linear oscillators, we can roughly understand the divergent quantity U_0 to be generated by summing up the quantum fluctuations of the canonical pair of operators $\phi(t,\mathbf{x})$ and $\Pi(t,\mathbf{x})$, *alias* $\widehat{Q}_{\mathbf{k}}$ and $\widehat{P}_{\mathbf{k}}$ ($\mathbf{k} \in \mathbb{R}^3$) at each point $\mathbf{x} \in V$ of a very large box in the three dimensional space.

It turns out to be more appropriate to discuss the vacuum energy density, *i.e.* the regularized vacuum energy *per* unit volume of the quantum real scalar field. The vacuum state vector or vacuum state amplitude of the actual quantum mechanical system under investigation is nothing but the ground state of the system and physically corresponds to the absence of field quanta, *i.e.* spin-less massive particles of rest mass m. It is defined by

$$a_{\mathbf{k}}\,|\,0\,\rangle = 0 = \langle\,0\,|\,a^{\dagger}_{\mathbf{k}} \qquad \forall\,\mathbf{k} \in \mathbb{R}^3 \tag{3.57}$$

so that, after setting

$$\frac{c}{V}\,\langle 0\,|\,P_0\,|\,0\,\rangle \equiv \langle\,\rho\,\rangle\, c^2 = \frac{\hbar c}{4\pi^2}\int_0^K k^2\,\mathrm{d}\,k\ \sqrt{k^2+(mc/\hbar)^2}$$

together with $\xi \equiv \hbar K/mc$, then we have[2]

$$\begin{aligned}
\langle\,\rho\,\rangle &= \frac{m^4c^3}{4\pi^2\hbar^3}\int_0^{\xi} x^2\sqrt{1+x^2}\ \mathrm{d}\,x \\
&= \frac{m^4c^3}{16\pi^2\hbar^3}\left[\,\xi(1+\xi^2)^{3/2}-\frac{1}{2}\,\xi(1+\xi^2)^{1/2}-\frac{1}{2}\,\ln\left(\xi+\sqrt{1+\xi^2}\right)\right] \\
&= \frac{\hbar K^4}{16\pi^2 c}+\frac{m^2cK^2}{16\pi^2\hbar}-\frac{m^4c^3}{32\pi^2\hbar^3}\left[\ln\left(\frac{\hbar K}{mc}\right)-\frac{1}{4}+\ln 2+O\left(\frac{mc}{\hbar K}\right)^2\right] \\
&\approx \frac{K^3}{16\pi^2}\left(\frac{\hbar K}{c}\right) \qquad (\hbar K \gg mc) \qquad\qquad (3.58)
\end{aligned}$$

The cosmological constant puzzle. If we trust in General Relativity and in Quantum Field Theory up to the Planck scale

$$\begin{aligned}
M_P &= \sqrt{\hbar c/G_N} = 1.220\ 93(7)\times 10^{\,19}\ \mathrm{GeV/c^2} \\
&= 2.176\ 51(13)\times 10^{\,-8}\ \mathrm{kg} \\
1\ \mathrm{GeV/c^2} &= 1.782\ 661\ 845(39)\times 10^{\,-\,27}\ \mathrm{kg}
\end{aligned}$$

where G_N is the Newtonian gravitational constant

$$G_N = 6.673\ 84(80)\times 10^{\,-\,11}\ \mathrm{m^3\ kg^{\,-1}\ s^{\,-2}}$$

then we might take $K \ \simeq\ \ell_P^{-1}$ which eventually gives

$$\langle\,\rho_P\,\rangle \approx \frac{M_P}{16\pi^2\,\ell_P^3} \approx 2\times 10^{\,121}\ (\mathrm{GeV}/c^2\,)\,\mathrm{m}^{\,-\,3} \approx 3\times 10^{\,94}\ \mathrm{kg\ m}^{\,-\,3} \qquad (3.59)$$

in which $\ell_P = \sqrt{\hbar G_N/c^3} = 1.616\ 20(10)\times 10^{-35}$ m denotes the Planck length. This mass density value is a truly enormously large one, if we remember that the mass density of the ordinary nuclear matter is $1.8\times 10^{\,18}$ kg m$^{\,-\,3}$. So it is no surprise that Paul Adrian Maurice Dirac soon suggested that this zero-point energy must be simply discarded, as it turns out to be irrelevant for any laboratory experiment in which solely energy differences are indeed observable. However, soon afterwards Wolfgang Pauli recognized that this vacuum energy surely couples to Einstein's gravity and it would then give rise to a large cosmological constant, so large that the size of the Universe could not even reach the earth-moon distance. On the contrary, the present day observed value of the so called DARK ENERGY DENSITY of the Universe is

$$\rho_\Lambda = \Omega_\Lambda\,\rho_c = \frac{3H_0^2}{8\pi G_N}\,\Omega_\Lambda = 1.053\ 75(13)\times 10^{-5}\,\Omega_\Lambda^3\ (\,\mathrm{GeV}/c^2\,)\,\mathrm{cm}^{\,-\,3}$$

[2]See for example [12] eq. (**2.272** 2.) p.105

where $\Omega_\Lambda = 0.685^{+0.017}_{-0.016}$ is the dark energy density fraction, $H_0 = 100\,\Omega_\Lambda$ Km s^{-1} Mpc^{-1} is the present day Hubble expansion rate, while

$$\rho_c = 3H_0^2/8\pi G_N = 1.878\ 47(23) \times 10^{-26}\ h^2 \quad \mathrm{kg\ m}^{-3}$$

is the critical mass density of the Universe, that is the observed total mass density of the Universe which in turn entails a flat Euclidean Universe, $h = 0.673(12)$ being the scale factor for Hubble expansion rate. This leads to the cosmological constant order of magnitude, *viz.*

$$\Lambda = 3\Omega_\Lambda H_0^2/c^2 = 8\pi\Omega_\Lambda G_N\,\rho_c/c^2 \simeq 10^{-52}\ \mathrm{m}^{-2} \tag{3.60}$$

which is extremely small but non-vanishing [3]. This eventually means that the ratio

$$\frac{\langle\rho_P\rangle}{\rho_\Lambda} = 1.14 \times 10^{121}$$

between the dark energy density and the vacuum energy density of any quantum field in the Universe is of 121 orders of magnitude! This is the COSMOLOGICAL CONSTANT PUZZLE, a somewhat paradoxical feature that still deserves some discussion. The solution of the above apparent paradox lies in the inclusion of the Einstein theory of the gravitational field, *i.e.* the General Relativity, where the quantum mechanical matter fields act as the source of the classical gravitational field, which gives rise to a curved space-time other than Minkowski space. In other words, it is mandatory to consider the quantum field theory in a curved space and the related renormalization issue - to this concern see N.D. Birrell & P.C.W. Davies (1982) *Quantum fields in curved space*, Cambridge University Press, Cambridge (UK) chapter 6, § **6.1-2**, pp. 150-173.

Leaving aside this intriguing topic, we now turn back to the realm of Galileian laboratory experiments and endorse the Dirac's point of view. To this concern, we shall introduce the useful concept of an operator written in normal form as well as the concept of the normal product of operators [5]. The normal form of an operator involving products of creation and annihilation operators is said to be the form in which in each term all the creation operators are written to the left of all the annihilation operators. Consider the simplest example: we write down in normal form the product of the two Hermitean operators

$$\begin{aligned} F(x)\,G(y) \;&\equiv\; \sum_{\mathbf{k}} \left\{ F^*_{\mathbf{k}}(x)\,a^\dagger_{\mathbf{k}} + F_{\mathbf{k}}(x)\,a_{\mathbf{k}} \right\} \\ &\times\; \sum_{\mathbf{h}} \left\{ G^*_{\mathbf{h}}(y)\,a^\dagger_{\mathbf{h}} + G_{\mathbf{h}}(y)\,a_{\mathbf{h}} \right\} \end{aligned}$$

[3]This tiny but non-vanishing measured value of the cosmological constant does actually provide a striking phenomenological evidence against super-symmetry (SUSY), the existence of which has been nowadays definitely ruled out by experimental data of high energy physics, astro-particle physics and cosmology.

$$
\begin{aligned}
= & \sum_{\mathbf{k}}\sum_{\mathbf{h}} F^*_{\mathbf{k}}(x)\, G^*_{\mathbf{h}}(y)\, a^\dagger_{\mathbf{k}}\, a^\dagger_{\mathbf{h}} \;+\; \text{h.c.} \\
+ & \sum_{\mathbf{k}}\sum_{\mathbf{h}} \Big(F^*_{\mathbf{k}}(x)\, G_{\mathbf{h}}(y) + F_{\mathbf{h}}(x)\, G^*_{\mathbf{k}}(y) \Big)\, a^\dagger_{\mathbf{k}}\, a_{\mathbf{h}} \\
+ & \sum_{\mathbf{k}} F_{\mathbf{k}}(x)\, G^*_{\mathbf{k}}(y) \qquad (3.61)
\end{aligned}
$$

The sum of terms **not involving** any ordinary c −number functions is called the normal product of the original operators. The normal product may also be defined as the original product reduced to its normal form with all the commutator functions being taken equal to zero in the process of reduction. The normal product of the operators $F(x)$ and $G(y)$ is denoted by the symbol

$$
: F(x)\, G(y) : \stackrel{\text{def}}{=} F(x)\, G(y) - \sum_{\mathbf{k}} F_{\mathbf{k}}(x)\, G^*_{\mathbf{k}}(y)
$$

We now agree by definition to express all dynamical variables which are bi-linear in the quantum operators with the same arguments, such as the energy, momentum and angular momentum of the radiation fields, in the form of normal products. For example, we shall write the energy-momentum tetra-vector quantum operator

$$
P_\nu = \int \mathrm{d}\mathbf{x} : T_{0\nu}(t, \mathbf{x}) :
$$

$$
cP_0 = \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} : c^2\, \Pi^2(x) - \phi(x)\, \dot{\Pi}(x)\, c^{-2} : \qquad (3.62)
$$

$$
= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} : c^2\, \Pi^2 - \phi \bigtriangleup \phi + (mc/\hbar)^2 \phi^2 : = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\, a^\dagger_{\mathbf{k}}\, a_{\mathbf{k}} \qquad (3.63)
$$

$$
\mathbf{P} = -\int \mathrm{d}\mathbf{x}\ : \Pi(x) \boldsymbol{\nabla}\phi(x) : = \sum_{\mathbf{k}} \hbar\mathbf{k}\, a^\dagger_{\mathbf{k}}\, a_{\mathbf{k}} \qquad (3.64)
$$

Now, if we keep the definition of the vacuum state to be given by (3.57) it follows that the expectation values of all the dynamical variables vanish for the vacuum state, *e.g.* $\langle\, P_\mu \,\rangle \equiv 0$. By this method we exclude from the theory at the outset the so called zero-point quantities of the type of the zero-point energy, which usually arise in the process of setting up of the quantum field theories and turn out to be, strictly speaking, mathematically ill-defined divergent quantities.

As a final remark, from the canonical equal time commutation relations

$$
[\, \phi(x), \phi(y)\,]_{x_0=y_0} = 0 = [\, \Pi(x), \Pi(y)\,]_{x_0=y_0} \qquad [\, \phi(x), \Pi(y)\,]_{x_0=y_0} = i\hbar\delta(\mathbf{x}-\mathbf{y})
$$

it immediately follows that

$$[\,\phi(x), P_\mu\,] = i\hbar\partial_\mu\phi(x) \qquad [\,\phi(x), L_{\mu\nu}\,] = i\hbar\{x_\mu\partial_\nu\phi(x) - x_\nu\partial_\mu\phi(x)\} \quad (3.65)$$

as it can be readily checked. It is then also easy to check that the definition of the conjugate momentum field $\Pi(x)$ and the Klein-Gordon field equation can be recast into the canonical Heisenberg form, *viz.*,

$$\dot{\phi}(x) = \frac{c}{i\hbar}\,[\,\phi(x)\,,\,P_0\,] = \Pi(x)\,c^2$$

$$\dot{\Pi}(x)/c^2 = \frac{1}{i\hbar c}\,[\,\Pi(x)\,,\,P_0\,] = \left(\triangle - m^2c^2/\hbar^2\right)\,\phi(x)$$

3.4 The Fock Space

The quantum theory of the non-interacting Klein-Gordon scalar field leads to the concept of spin-less neutral massive particle. As a matter of fact, it appears to be clear that the non-negative number operators $a^\dagger_{\mathbf{k}}\, a_{\mathbf{k}}$ do possess integer eigenvalues $N_{\mathbf{k}} = 0, 1, 2, \ldots$ which are interpreted as the numbers of particles of a given wave number $\mathbf{k}$ and a given energy $\omega_{\mathbf{k}}$. The energy or Hamiltonian operator turns out to be positive semi-definite and we can readily derive, from the canonical commutators (3.52), the continuous and unbounded spectra and the common eigenstates of the energy-momentum commuting operators P_μ : namely,

$$E\left(\left\{N_{\mathbf{k}} \mid \mathbf{k} \in \mathbb{R}^3\right\}\right) = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\, N_{\mathbf{k}} \qquad N_{\mathbf{k}} = 0, 1, 2, \ldots \tag{3.66}$$

$$\mathbf{P}\left(\left\{N_{\mathbf{k}} \mid \mathbf{k} \in \mathbb{R}^3\right\}\right) = \sum_{\mathbf{k}} \hbar\mathbf{k}\, N_{\mathbf{k}} \qquad N_{\mathbf{k}} = 0, 1, 2, \ldots \tag{3.67}$$

3.4.1 The Many-Particle States

The setting up of the eigenstates leads to the well known construction of the so called Fock space :

Vladimir Alexandrovich Fock
Sankt Petersburg 22.12.1898 – 27.12.1974
Konfigurationsraum und Zweite Quantelung
Zeitschrift der Physik A **75**, 622–647 (1932).

Actually, in order to describe $N-$particle states, consider the state

$$|\,\mathbf{k}_1\,\mathbf{k}_2\,\ldots\,\mathbf{k}_N\rangle = (N!)^{-\frac{1}{2}} a^\dagger_{\mathbf{k}_1}\, a^\dagger_{\mathbf{k}_2}\, \cdots\, a^\dagger_{\mathbf{k}_N}\,|\,0\,\rangle \tag{3.68}$$

with $\mathbf{k}_1 \neq \mathbf{k}_2 \neq \cdots \neq \mathbf{k}_N$ which satisfies

$$P^\mu\,|\,\mathbf{k}_1\,\mathbf{k}_2\,\ldots\,\mathbf{k}_N\rangle = \left(\textstyle\sum_{\jmath=1}^N k^\mu_\jmath\right)|\,\mathbf{k}_1\,\mathbf{k}_2\,\ldots\,\mathbf{k}_N\rangle \tag{3.69}$$

$$k^\mu_\jmath = \left(\omega_{\mathbf{k}_\jmath}/c, \mathbf{k}_\jmath\right) \qquad (\,\jmath = 1, 2, \ldots, N\,) \tag{3.70}$$

It is very important to gather that, owing to the commutation relation $[\,a^\dagger_{\mathbf{h}}\,,\,a^\dagger_{\mathbf{k}}\,] = 0$, by its very construction any $N-$particle state is completely symmetric under the exchange of any wave numbers, *i.e.* the neutral spin-less particles obey the Bose-Einstein statistics.

In particular, the 1-particle energy-momentum eigenstates are given by $|\,\mathbf{k}\,\rangle = a^\dagger_{\mathbf{k}}|\,0\,\rangle$ and satisfy

$$P^\mu|\,\mathbf{k}\,\rangle \;=\; \hbar k^\mu|\,\mathbf{k}\,\rangle \qquad k^\mu = (\omega_{\mathbf{k}}/c, \mathbf{k})$$

The 1-particle wave functions in the coordinate representation, for a given wave number, are defined in terms of the matrix elements of the field operator (3.50) and read

$$u_{\mathbf{k}}(x) \equiv \langle\, 0 \,|\, \phi(x) \,|\, \mathbf{k} \,\rangle = \sqrt{\frac{c}{2\omega_{\mathbf{k}}(2\pi)^3}}\ \exp\{-\, i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}\} \tag{3.71}$$

Notice that they turn out to be normalized in such a way to satisfy the orthogonality and closure relations

$$(\, u_{\mathbf{k}} \,,\, u_{\mathbf{h}} \,) \equiv \int \mathrm{d}\mathbf{x}\ u^*_{\mathbf{k}}(x)\, i\, \overleftrightarrow{\partial}_0\, u_{\mathbf{h}}(x) \ = \ \delta\,(\mathbf{k} - \mathbf{h}) \tag{3.72}$$

$$\sum_{\mathbf{k}}\ u^*_{\mathbf{k}}(y)\, u_{\mathbf{k}}(x) \ \equiv\ (-\, i\,)\, D^{\,(-)}(x-y) \tag{3.73}$$

$$i\, D^{(\pm)}(x) = \frac{\pm 1}{(2\pi)^3} \int \mathrm{d}k\ \delta\left(k^2 - m^2\right)\ \theta\,(k_0)\ \exp\{\pm\, i\, k\cdot x\} \tag{3.74}$$

The 1-particle wave functions (3.71) satisfy by construction the Klein-Gordon wave equation

$$\left(\Box + m^2\right) u_{\mathbf{k}}(x) = 0 \qquad \forall\, \mathbf{k} \in \mathbb{R}^3$$

and do thereby represent a complete orthogonal basis, with respect to the inner product, in the 1-particle Hilbert space

$$\mathcal{H}_1 = \overline{\mathrm{V}}_1 \qquad \mathrm{V}_1 = \{\, |\, \mathbf{k} \,\rangle = a^\dagger_{\mathbf{k}} \,|\, 0 \,\rangle \quad \mathbf{k} \in \mathbb{R}^{\,3} \}$$

More precisely, eq. (3.71) does explicitly realize the isomorphism between the Fock space representation $\mathcal{H}_1$ of the space of 1-particle states and the space-time coordinate representation $L^2(\mathbb{R}^3)$ of the 1-particle wave functions with respect to (3.72).

It turns out that the previously introduced 1-particle energy-momentum eigenstates $|\, \mathbf{k} \,\rangle = a^\dagger_{\mathbf{k}} \,|\, 0 \,\rangle$ are improper states and have to be normalized, in the generalized sense of the tempered distributions, according to

$$\langle\, \mathbf{h} \,|\, \mathbf{k} \,\rangle = \delta\,(\mathbf{h} - \mathbf{k}\,) \tag{3.75}$$

Moreover they satisfy the closure or completeness relation in the 1-particle Hilbert space $\mathcal{H}_1$ that reads

$$\sum_{\mathbf{k}}\ |\, \mathbf{k} \,\rangle\langle\, \mathbf{k} \,| = \mathbb{I}_1 \tag{3.76}$$

It is important to remark that these orthogonality and closure relations, as well as the normal modes decomposition (3.50), are not manifestly Lorentz covariant. As a consequence, the insofar developed quantum procedure for a real scalar field is set up in a particular class of inertial reference frames connected by spatial rotations belonging to the group $SO(3)$.

The $N-$particle wave functions in the coordinates representation can be readily obtained in the completely symmetric form

$$\begin{aligned} u_N(x_1, x_2, \ldots, x_N) &= (N!)^{-1/2} \prod_{j=1}^{N} u_{\mathbf{k}_j}(x_j) \\ &\equiv \langle x_1 x_2 \ldots x_N \,|\, \mathbf{k}_1 \mathbf{k}_2 \ldots \mathbf{k}_N \rangle \end{aligned} \tag{3.77}$$

Furthermore, we can write the generic normalized element of the $N-$particle completely symmetric Hilbert space – the closure of the symmetric product of 1–particle Hilbert spaces

$$\mathcal{H}_N \equiv \overline{\mathrm{V}}_N \qquad \mathrm{V}_N = \{\underbrace{\mathcal{H}_1 \overset{\mathrm{s}}{\otimes} \mathcal{H}_1 \overset{\mathrm{s}}{\otimes} \ldots \overset{\mathrm{s}}{\otimes} \mathcal{H}_1}_{\text{N times}}\} \equiv \mathcal{H}_1^{\overset{\mathrm{s}}{\otimes} n} \tag{3.78}$$

in the form

$$|\varphi_N\rangle \equiv \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \cdots \sum_{\mathbf{k}_N} \varphi_N(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N)\ |\mathbf{k}_1 \mathbf{k}_2 \ldots \mathbf{k}_N\rangle$$

with

$$\sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \cdots \sum_{\mathbf{k}_N} |\,\varphi_N(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N)\,|^2 = 1$$

To end up, we are now able to write the generic normalized element of the Fock space of the spin-less neutral scalar particle states

$$\mathcal{F} \;\equiv\; \mathbf{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_N \oplus \ldots \;=\; \bigoplus_{n=0}^{\infty} \mathcal{H}_1^{\overset{\mathrm{s}}{\otimes} n} \tag{3.79}$$

in the form

$$|\,\Phi\,\rangle \;=\; \sum_{N=0}^{\infty} C_N \,|\,\varphi_N\,\rangle \qquad\qquad \sum_{N=0}^{\infty} |\,C_N\,|^2 = 1$$

which summarizes the setting up of the Fock space of the particle states, the structure of which is characterized by the canonical quantum algebra (3.52) and the energy momentum operators (3.64). From the above construction of the Fock space of the states of a free real scalar field, it appears quite evident that all the quantum states can be generated by linear combinations of repeated applications of the creation operators on the vacuum state. This property is known as the cyclic property of the vacuum state.

`Many-Particle States.` Sometimes, in order to avoid too many dummy $\delta-$factors in the calculations, it is more convenient to restrict, without any loss of generality, the normal modes $u_{\mathbf{k}}(x)$ to the interior of a space-like symmetric 3-torus of very large side L, i.e. to choose periodic boundary conditions in the three dimensional ordinary space. Then we get

$$u_{\mathbf{k}}(x) = [\,(2\pi)^3\omega_{\mathbf{k}}\,]^{-\frac{1}{2}}\ \exp\{\,i\mathbf{k}\cdot\mathbf{x} - it\omega_{\mathbf{k}}\} \tag{3.80}$$

where $\mathbf{k} = 2\pi\mathbf{n}/L\,,\ \mathbf{n} \in \mathbb{Z}^3\,,\ \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$, in such a manner that states an wave functions are properly normalized in terms of the Kronecker symbols

$$(u_{\mathbf{k}'}, u_{\mathbf{k}}) = \langle\mathbf{k}'|\mathbf{k}\rangle = \delta_{\mathbf{nn}'}\left(\frac{L}{2\pi}\right)^3 \equiv \delta_{\mathbf{kk}'} \tag{3.81}$$

$$\langle\mathbf{k}\,|\,\mathbf{k}\rangle = \delta^{(3)}(0) \quad\longrightarrow\quad \left(\frac{L}{2\pi}\right)^3 = \frac{\mathrm{Vol}}{(2\pi)^3}$$

Hence, a generic $(N, r)-$particle improper state can be written in the admittedly rather cumbersome notation

$$|\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\rangle_N \;=\; \prod_{\jmath=1}^{r}\left(\nu_{\mathbf{k}_\jmath}!\right)^{-\frac{1}{2}}\left(a^\dagger_{\mathbf{k}_\jmath}\right)^{\nu_{\mathbf{k}_\jmath}}|0\rangle \qquad \sum_{\jmath=1}^{r}\nu_{\mathbf{k}_\jmath} = N \tag{3.82}$$

where $\{\mathbf{k}_\jmath \in \mathbb{R}^3\,,\ \nu_{\mathbf{k}_\jmath} \in \mathbb{N}\,|\,\jmath = 1, 2, \ldots, r\}$ so that we can suitably rewrite the normalization of the many-particle states in the full Kronecker form, albeit suitably keeping the same canonical engineering dimensions of the improper continuum case,

$$\begin{aligned}&{}_{N'}\langle\nu'_{\mathbf{k}_1}\,\ldots\,,\nu'_{\mathbf{k}_s}\,|\,\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\rangle_N \;=\\ &\delta_{NN'}\,\delta_{rs}\sum_{[\,\mathrm{perm}\,]}\delta_{\mathbf{k}_1\mathbf{k}'_{\mathrm{p}(1)}}\cdots\delta_{\mathbf{k}_r\mathbf{k}'_{\mathrm{p}(s)}}\,\delta_{\nu_1\nu'_{\mathrm{p}(1)}}\cdots\delta_{\nu_r\nu'_{\mathrm{p}(s)}}\end{aligned}$$

in which the sum is over the $s!$ permutations of the integers

$$\{1, 2, \ldots, s\} \quad\stackrel{\mathrm{p}}{\longrightarrow}\quad \{\,\mathrm{p}(1), \mathrm{p}(2), \ldots, \mathrm{p}(s)\}$$

while the wave vector is omitted in the Kronecker symbol for the particle multiplicity $\nu_{\mathbf{k}_\jmath}$ for the sake of brevity and in order to lighten the expressions. From the creation and destruction parts of the Klein-Gordon quantum field operator

$$\phi^{(+)}(x) = \sqrt{\hbar c}\sum_{\mathbf{k}} a^\dagger_{\mathbf{k}}\,u^*_{\mathbf{k}}(x) = [\,\phi^{(-)}(x)\,]^\dagger \qquad\qquad \phi^{(-)}(x) = \sqrt{\hbar c}\sum_{\mathbf{k}} a_{\mathbf{k}}\,u_{\mathbf{k}}(x) \tag{3.83}$$

$$[\,\phi^{(+)}(x)\,]^* = \sqrt{\hbar c}\sum_{\mathbf{k}} a^\dagger_{\mathbf{k}}\,u_{\mathbf{k}}(x)$$

which separately satisfy the Klein-Gordon wave equation, we can suitably build up the self-adjoint number operator

$$\begin{aligned}\frac{1}{\hbar c}\,\left(\phi^{(+)\,*}\,,\,\phi^{(-)}\right) \;&\equiv\; \sum_{\mathbf{k}}\sum_{\mathbf{k}'} a^\dagger_{\mathbf{k}}\,a_{\mathbf{k}'}\int \mathrm{d}\mathbf{x}\,u^*_{\mathbf{k}}(t,\mathbf{x})\,\overset{\leftrightarrow}{\partial}_0\,u_{\mathbf{k}'}(t,\mathbf{x})\\ &=\; \sum_{\mathbf{k}}\sum_{\mathbf{k}'} a^\dagger_{\mathbf{k}}\,a_{\mathbf{k}'}\,(u_{\mathbf{k}}\,,\,u_{\mathbf{k}'}) = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}}\,a_{\mathbf{k}} \equiv \mathrm{N}\end{aligned} \tag{3.84}$$

which admits the set of the many-particle states as a complete and orthogonal basis

$$\mathrm{N}\,|\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\rangle_N = N\,|\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\rangle_N \tag{3.85}$$

as one can readily check by direct inspection by making a repeated use of the canonical commutation relations among particle creation and destruction operators. Moreover, since any pair of many-particle states with a different number of quanta are always mutually orthogonal, it is a rather simple exercise to show that the matrix elements of the number operator are actually non-vanishing in each subspace $\mathcal{H}_N$ of the Fock space

$$\begin{aligned}&{}_{N'}\langle\nu'_{\mathbf{k}_1}\,\ldots\,,\nu'_{\mathbf{k}_s}\,|\,\mathrm{N}\,|\,\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\rangle_N \;=\\ &N\,\delta_{\,NN'}\,\delta_{\,rs}\sum_{[\,\mathrm{perm}\,]}\delta_{\,\mathbf{k}_1\mathbf{k}'_{\mathrm{p}(1)}}\cdots\delta_{\,\mathbf{k}_r\mathbf{k}'_{\mathrm{p}(s)}}\,\delta_{\,\nu_1\nu'_{\mathrm{p}(1)}}\cdots\delta_{\,\nu_r\nu'_{\mathrm{p}(s)}}\end{aligned} \tag{3.86}$$

In such a circumstance, one can eventually show that *e.g.* the positive frequency Wightman distribution is replaced by

$$\begin{aligned}&\frac{1}{\hbar c}\left(\frac{2\pi}{L}\right)^{3r}\;{}_N\langle\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\,|\,\phi(x')\phi(x)\,|\,\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\rangle_N\\ =\;\; &-iD^{(-)}(x'-x)+\sum_{\imath=1}^{r}\nu_{\mathbf{k}_\imath}\left[\,u_{\,\mathbf{k}_\imath}(x')u^*_{\mathbf{k}_\imath}(x)+u^*_{\mathbf{k}_\imath}(x')u_{\,\mathbf{k}_\imath}(x)\,\right]\end{aligned} \tag{3.87}$$

If the number r of the involved wave numbers and the side L of the 3-torus are very large, then we can pass to the continuum limit and replace the right-hand side of the above expression with

$$\frac{1}{\hbar c}\left(\frac{2\pi}{L}\right)^{3r}\;{}_N\langle\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\,|\,\phi(x')\phi(x)\,|\,\nu_{\mathbf{k}_1}\,\ldots\,,\nu_{\mathbf{k}_r}\rangle_N \tag{3.88}$$

$$=\;\; -iD^{(-)}(x'-x)+\int\mathrm{d}\mathbf{k}\;\mathrm{n}_{\mathbf{k}}\,[\,u_{\,\mathbf{k}}(x')u^*_{\mathbf{k}}(x)+u^*_{\mathbf{k}}(x')u_{\,\mathbf{k}}(x)\,] \tag{3.89}$$

where $\mathrm{n}_{\mathbf{k}}$ does now represent the density of the states, i.e., the real number of the 1-particle states with the wave number between $\mathbf{k}$ and $\mathbf{k}+\mathrm{d}\mathbf{k}$ in such a manner that

$$\int\mathrm{d}\mathbf{k}\;\mathrm{n}_{\mathbf{k}} = N$$

3.4.2 The Lorentz Covariance Properties

We end up this section by discussing the behavior of the $N-$particle states and of the field operators under Lorentz transformations. Firstly we recall that the above introduced 1-particle states of the basis

$$\left\{\,|\,\mathbf{k}\,\rangle = a^\dagger_{\mathbf{k}}\,|\,0\,\rangle\,|\,\mathbf{k}\in\mathbb{R}^{\,3}\,\right\}$$

in the Hilbert space $\mathcal{H}_1$ do not satisfy Lorentz covariant orthogonality and completeness relations. To remedy this, let us first consider the completeness relation and the trivial identity

$$
\begin{aligned}
\mathbb{I}_{\mathcal{H}_1} &= \sum_{\mathbf{k}} |\,\mathbf{k}\,\rangle\langle\,\mathbf{k}\,| = \int \mathrm{d}\mathbf{k}\; a^{\dagger}_{\mathbf{k}}\,|\,0\,\rangle\langle\,0\,|\,a_{\mathbf{k}} \\
&= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\,2\,\omega_{\mathbf{k}}}\;[\,(2\pi)^3\,2\,\omega_{\mathbf{k}}\,]^{\frac12}\,a^{\dagger}_{\mathbf{k}}\,|\,0\,\rangle\langle\,0\,|\,a_{\mathbf{k}}\,[\,(2\pi)^3\,2\,\omega_{\mathbf{k}}\,]^{\frac12} \\
&\stackrel{\mathrm{def}}{=} \int \mathrm{D}k\; a^{\dagger}(k)\,|\,0\,\rangle\langle\,0\,|\,a(k) \equiv \int \mathrm{D}k\;|\,k\,\rangle\langle\,k\,|
\end{aligned}
$$

whence we get

$$
\begin{aligned}
\int \mathrm{D}k &\equiv \int \frac{c\,\mathrm{d}\mathbf{k}}{(2\pi)^3\,2\,\omega_{\mathbf{k}}} = \frac{1}{(2\pi)^3}\int \mathrm{d}^4k\;\theta(k_0)\,\delta\left(k^2 - m^2c^2/\hbar^2\right) \\
|\,k\,\rangle &\equiv [\,(2\pi)^3\,2\,\omega_{\mathbf{k}}/c\,]^{\frac12}\,a^{\dagger}_{\mathbf{k}}\,|\,0\,\rangle = a^{\dagger}(k)\,|\,0\,\rangle
\end{aligned}
$$

The Lorentz invariant completeness relation for the 1-particle Hilbert space $\mathcal{H}_1$ can also be written in the two equivalent forms

$$
\sum_{\mathbf{k}} |\,\mathbf{k}\,\rangle\,\langle\,\mathbf{k}\,| \;=\; \mathbb{I}_{\mathcal{H}_1} \;=\; \int \mathrm{D}k\;|\,k\,\rangle\langle\,k\,|
$$

Now it is clear that the 1-particle states of the new basis, which will be named covariant 1-particle states,

$$
\mathcal{H}_1 = \left\{\,|\,k\,\rangle = 2(2\pi)^3\,\frac{\omega_{\mathbf{k}}}{c}\,a^{\dagger}_{\mathbf{k}}\,|\,0\,\rangle\;|\;\mathbf{k}\in\mathbb{R}^3\,\right\} \tag{3.90}
$$

fulfill manifestly Lorentz covariant orthogonality relations

$$
\langle\,k'\,|\,k\,\rangle \;=\; 2(2\pi)^3\,\frac{\omega_{\mathbf{k}}}{c}\,\delta\,(\mathbf{k}-\mathbf{k}') \tag{3.91}
$$

Consider in fact any complex scalar field in momentum space $\varphi'(k') = \varphi(k)$, where $k'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\,k^{\,\nu}$, which is supposed to be regular on the upper mass shell hyperboloid $\{k^{\,2} = m^{\,2}\,|\,k_0 > 0\}$. Then we can write

$$
\begin{aligned}
&\int \mathrm{D}\,k\,\langle\,p\,|\,k\,\rangle\,\varphi(k) = \frac{1}{(2\pi)^3}\int \mathrm{d}^4k\;\theta(k_0)\,\delta\left(k^2 - m^2\right)\,\langle\,p\,|\,k\,\rangle\,\varphi(k) \\
= &\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\;2(2\pi)^3\;\omega_{\mathbf{k}}\;\delta(\mathbf{k}-\mathbf{p})\,\varphi(\mathbf{k}, k_0 = \omega_{\mathbf{k}}) = \varphi(p) \qquad (\,p_0 = \omega_{\mathbf{p}}\,)
\end{aligned}
$$

Hence, from the manifest Lorenz invariance of the measure $\int \mathrm{D}k$ and of the on-mass-shell couple of complex scalar fields $\varphi(k), \varphi(p)$, the manifest Lorentz invariance of the quantity $\langle\,p\,|\,k\,\rangle$ immediately follows. $\square$

Furthermore we can write

$$\phi(x) \equiv \int \mathrm{D}k \left[a(k)\, e^{-ik\cdot x} + a^{\dagger}(k)\, e^{ik\cdot x} \right]_{k_0 = \omega_{\mathbf{k}}} \tag{3.92}$$

$$\Pi(x) \equiv i \int \mathrm{D}k\ k_0 \Big[a^{\dagger}(k)\, e^{ik\cdot x} - a(k)\, e^{-ik\cdot x} \Big]_{k_0 = \omega_{\mathbf{k}}} \tag{3.93}$$

so that the 1-particle wave functions in the coordinate representation, which correspond to the 1-particle state $|\,k\,\rangle$ and are still defined in terms of the matrix elements of the field operator (3.50), will coincide in natural units with the plane waves

$$u_k\,(x) \equiv \langle\, 0\,|\,\phi(x)\,|\,k\,\rangle = \exp\{-\,i\,k\cdot x\} \qquad (k_0 = \omega_{\mathbf{k}}) \tag{3.94}$$

which are normalized in such a way to satisfy

$$(\,u_k\,,\,u_h\,) \equiv \int \mathrm{d}\mathbf{x}\ u_k^*\,(x)\, i\,\overleftrightarrow{\partial}_0\, u_h\,(x) = (2\pi)^3\, 2\omega_{\mathbf{k}}\,\delta(\mathbf{k}-\mathbf{h}) \tag{3.95}$$

$$(\,u_k^*\,,\,u_h^*\,) = -\,(\,u_k\,,\,u_h\,) \qquad (\,u_k\,,\,u_h^*\,) = (\,u_k^*\,,\,u_h\,) = 0 \tag{3.96}$$

In turn we have the manifestly covariant inversion formulæ

$$(\,u_k\,,\,\phi\,) = a(k) \qquad a^{\dagger}(k) = (\phi\,,\,u_k^*) = -(\,u_k^*\,,\,\phi\,) \tag{3.97}$$

It is worthwhile to gather that we have the following canonical engineering dimensions, both in physical and natural units: namely,

$$[\,|\,k\rangle\,] = [\,a(k)\,] = \mathrm{cm} \qquad [\,\mathrm{D}k\,] = \mathrm{cm}^{-2} \tag{3.98}$$

Now it becomes clear that to each element of the restricted Poincaré group, which is uniquely specified by the ten canonical coordinates

$$\big(\omega^{\,\mu\nu}, \mathrm{a}^{\lambda}\big) = \big(\boldsymbol{\alpha}, \boldsymbol{\eta}, \mathrm{a}^{\lambda}\big) = (\omega, \mathrm{a})$$

there will correspond a UNITARY OPERATOR such that

$$U(\omega,\mathrm{a})\,|\,0\,\rangle = \exp\left\{ \frac{i}{\hbar}\,\mathrm{a}^{\mu}\,P_{\mu} - \frac{i}{2\hbar}\,\omega^{\,\rho\sigma} L_{\rho\sigma} \right\} |\,0\,\rangle = |\,0\,\rangle \tag{3.99}$$

$$\langle\,0\,|\,U^{\dagger}(\omega,\mathrm{a}) = \langle\,0\,|\,\exp\left\{ -\,\frac{i}{\hbar}\mathrm{a}^{\mu}\,P_{\mu} + \frac{i}{2\hbar}\,\omega^{\,\rho\sigma} L_{\rho\sigma} \right\} = \langle\,0\,| \tag{3.100}$$

which means that the vacuum state is Poincaré invariant, as expected, or, in other words, that $IO(1,3)$ acts trivially on the one dimensional ray of the Fock space generated by the vacuum state. In the case of the Klein-Gordon neutral field, the explicit expressions for the Hermitean generators are provided by – see the proof and details at the end of this paragraph

$$
\begin{aligned}
P_0 &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} : \Pi^2(x) + \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) + m^2\phi^2(x) : = \sum_{\mathbf{k}}\ \omega_{\mathbf{k}}\, a_{\mathbf{k}}^{\dagger}\, a_{\mathbf{k}} \\
P_k &= \int \mathrm{d}\mathbf{x}\ : \Pi(x)\,\partial_k\phi(x) : = \sum_{\mathbf{k}}\ \mathbf{k}\, a_{\mathbf{k}}^{\dagger}\, a_{\mathbf{k}} \\
L_{\imath\jmath} &= \int \mathrm{d}\mathbf{x}\ : x_\imath\,\Pi(x)\,\partial_\jmath\phi(x) - x_\jmath\,\Pi(x)\,\partial_\imath\phi(x) : \\
&= \sum_{\mathbf{k}}\ \frac{i}{2}\left(k_\imath\, a_{\mathbf{k}}^{\dagger}\,\frac{\overleftrightarrow{\partial}}{\partial k^\jmath}\, a_{\mathbf{k}} \ -\ k_\jmath\, a_{\mathbf{k}}^{\dagger}\,\frac{\overleftrightarrow{\partial}}{\partial k^\imath}\, a_{\mathbf{k}} \right) \\
L_{0k} &= x_0\,P_k - m\,X_k(t) = \frac{i}{2}\sum_{\mathbf{p}}\omega_{\mathbf{p}}\ a_{\mathbf{p}}^{\dagger}\,\frac{\overleftrightarrow{\partial}}{\partial\,p^k}\, a_{\mathbf{p}} \\
X^k(t) &= \frac{1}{2m}\int \mathrm{d}\mathbf{x}\; x^k : \Pi^2(x) + \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) + m^2\phi^2(x) :
\end{aligned}
$$

Moreover, under a passive Poincaré transformation the creation-annihilation operators undergo the changes

$$a'(k) \equiv U(\omega, \mathrm{a})\, a(k)\, U^{-1}(\omega, \mathrm{a}) \ =\ a(k')\ \exp\{-\,i\,k\cdot \mathrm{a}\} \tag{3.101}$$

$$k'_\mu = \Lambda_\mu^{\ \nu}\, k_\nu \qquad k_0 = \omega_{\mathbf{k}} \qquad g^{\mu\nu}\, k'_\mu\, k'_\nu = k'^2 = k^2 = m^2 \tag{3.102}$$

$$a'^{\dagger}(k) \ \equiv\ U(\omega, \mathrm{a})\, a^{\dagger}(k)\, U^{-1}(\omega, \mathrm{a}) \ =\ \exp\{\,i\,k\cdot \mathrm{a}\}\, a^{\dagger}(k') \tag{3.103}$$

which endorses the Lorentz invariance, up to a phase factor, of the creation and annihilation operators of the Klein-Gordon quantum scalar field.

`Proof.` To start up, consider an infinitesimal Poincaré transformation

$$U(\delta\omega, \delta\mathrm{a})\, a\,(k)\, U^{\dagger}(\delta\omega, \delta\mathrm{a}) \simeq a\,(k) + i\left[\,\delta\mathrm{a}^{\mu}\, P_\mu - \tfrac{1}{2}\,\delta\,\omega^{\rho\sigma} L_{\rho\sigma}\,,\ a\,(k)\,\right]$$

From the canonical commutation relations (3.65) it is immediate to show that

$$
\begin{aligned}
[\,a(k)\,,\ P_\mu\,] &= [\,(\,u_k\,,\ \phi\,)\,,\ P_\mu\,] = i\,(\,u_k\,,\ \partial_\mu\,\phi\,) \\
&= \int \mathrm{D}p\ p_\mu \Big[\,a\,(p)\ (\,u_k\,,\ u_p) - a^{\dagger}(p)\ \left(\,u_k\,,\ u_p^{*}\right)\,\Big] = k_\mu\, a\,(k)
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
[\,a(k)\,,\ L_{\mu\nu}\,] &= [\,(\,u_k\,,\ \phi\,)\,,\ L_{\mu\nu}\,] = \int \mathrm{d}\mathbf{x}\ u_k^{*}(t,\mathbf{x})\, i\overleftrightarrow{\partial}_0[\,\phi(t,\mathbf{x})\,,\ L_{\mu\nu}\,] \\
&= -\int \mathrm{d}\mathbf{x}\ u_k^{*}(t,\mathbf{x})\,\overleftrightarrow{\partial}_0\Big(x_\mu\,\partial_\nu\,\phi(t,\mathbf{x}) - x_\nu\,\partial_\mu\,\phi(t,\mathbf{x})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \int \mathrm{D}p\; p_\nu \left[a^\dagger(p)\, \frac{i\partial}{\partial p^\mu}\, (\, u_k\,,\, u_p^*) + a\,(p)\, \frac{i\partial}{\partial p^\mu}\, (\, u_k\,,\, u_p) \,\right] - \mu \leftrightarrow \nu \\
&= -\int \mathrm{D}p\; p_\nu \left[a^\dagger(p)\, \frac{i\partial}{\partial k^\mu}\, (\, u_k\,,\, u_p^*) + a\,(p)\, \frac{i\partial}{\partial k^\mu}\, (\, u_k\,,\, u_p) \,\right] - \mu \leftrightarrow \nu \\
&= -\frac{i\partial}{\partial k^\mu} \int \mathrm{D}p\; p_\nu\, a\,(p)\, (\, u_k\,,\, u_p) - \mu \leftrightarrow \nu \\
&= -\frac{i\partial}{\partial k^\mu}\, (k_\nu\, a(k)) - \mu \leftrightarrow \nu = i\, k_\mu\, \frac{\partial}{\partial k^\nu}\, a(k) - i\, k_\nu\, \frac{\partial}{\partial k^\mu}\, a(k)
\end{aligned}
$$

where use has been made of the inversion formulæ. Hence, under an infinitesimal Poincaré transformation we get

$$
\begin{aligned}
& U(\delta\omega, \delta\mathrm{a})\, a\,(k)\, U^{-1}(\delta\omega, \delta\mathrm{a}) \simeq a\,(k) - i\left[\, a\,(k)\,,\; \delta\mathrm{a}^\mu\, P_\mu - \tfrac{1}{2}\,\delta\,\omega^{\rho\sigma}\, L_{\rho\sigma}\,\right] \\
&= \left\{ 1 - i\,\delta\mathrm{a}^\mu\, k_\mu - \tfrac{1}{2}\,\delta\,\omega^{\mu\nu} \left(k_\mu\, \frac{\partial}{\partial k^\nu} - k_\nu\, \frac{\partial}{\partial k^\mu} \right) \right\} a\,(k)
\end{aligned}
$$

so that we eventually find

$$
a'(k) - a(k) \,\simeq\, \delta a(k) \,\simeq\, \left\{ \tfrac{1}{2}\,\epsilon^{\mu\nu} \left(k_\nu\, \frac{\partial}{\partial k^\mu} - k_\mu\, \frac{\partial}{\partial k^\nu} \right) - i\, k_\mu\, \epsilon^\mu \right\} a\,(k) \tag{3.104}
$$

where I have identified as usual $\delta\mathrm{a}^\mu \equiv \epsilon^\mu\,,\ \delta\,\omega^{\mu\nu} \equiv \epsilon^{\mu\nu}$. The action of an infinitesimal Lorentz transformation on the wave tetra-vector yields

$$
a(\Lambda\, k) - a(k) \,\simeq\, a(k + \delta k) - a(k) = \delta k^\mu\, \frac{\partial}{\partial k^\mu}\, a(k) = \epsilon^{\mu\nu}\, k_\nu\, \frac{\partial}{\partial k^\mu}\, a(k)
$$

in such a manner that we can finally get the finite transformation rule

$$
\begin{aligned}
U(\omega, \mathrm{a})\, a\,(k)\, U^{-1}(\omega, \mathrm{a}) \;&=\; \exp\{-\, i\, k\cdot \mathrm{a}\}\, a\,(\Lambda\, k) \\
&\simeq\; (1 - i\, k_\mu\, \mathrm{a}^\mu + \cdots) \left(1 - \omega^{\mu\nu}\, k_\nu\, \frac{\partial}{\partial k^\mu} + \cdots \right) a(k) \\
&=\; \left\{ 1 - i\, k_\mu\, \mathrm{a}^\mu + \tfrac{1}{2}\,\omega^{\mu\nu} \left(k_\nu\, \frac{\partial}{\partial k^\mu} - k_\mu\, \frac{\partial}{\partial k^\nu} \right) \right\} a(k) + \cdots
\end{aligned}
$$

and consequently

$$
a'(k) \,\equiv\, U(\omega, \mathrm{a})\, a\,(k)\, U^{-1}(\omega, \mathrm{a}) \,=\, a(\Lambda\, k) \exp\{-\, i\, k\cdot \mathrm{a}\} \qquad k_0 = \omega_{\mathbf{k}}
$$

that completes the proof. □

It is worthwhile to notice that by repeating the very same steps for the inverse Poincaré unitary similarity transformation we obtain

$$
a'(k) \,\equiv\, U^{-1}(\omega, \mathrm{a})\, a\,(k)\, U(\omega, \mathrm{a}) \,=\, a(\Lambda^{-1} k) \exp\{\, i\, k\cdot \mathrm{a}\} \qquad (\, k_0 = \omega_{\mathbf{k}}\,)
$$

with

$$
U^{-1}(\omega, \mathrm{a}) = U(-\,\omega, -\,\mathrm{a}) = U^\dagger(\omega, \mathrm{a})
$$

which implies in turn

$$\begin{aligned} a'(k') &\equiv U^{-1}(\omega,\mathrm{a})\, a(\Lambda\, k)\, U(\omega,\mathrm{a}) \\ &= a(k)\exp\{\, i\, k\cdot \mathrm{a}\} \qquad\qquad (\, k_0=\omega_{\mathbf{k}}\,) \end{aligned} \tag{3.105}$$

showing that for any homogeneous Lorentz active transformation the creation and annihilation operators are invariant.

Turning now to the 1-particle states, if we define

$$|\, k'\,\rangle \equiv a'^{\dagger}(k')|\, 0\,\rangle$$

then we can write the equality

$$\begin{aligned} |\, k'\,\rangle &\equiv U^{-1}(\omega,\mathrm{a})\, a^{\dagger}(\Lambda\cdot k)\, U(\omega,\mathrm{a})|\, 0\,\rangle \\ &= U^{-1}(\omega,\mathrm{a})\, a^{\dagger}(\Lambda\cdot k)\,|\, 0\,\rangle \\ &= \exp\{-\, i\, k\cdot \mathrm{a}\}\; a^{\dagger}(k)\,|\, 0\,\rangle \;=\; \exp\{-\, i\, k\cdot \mathrm{a}\}\;|\, k\,\rangle \end{aligned}$$

so that by virtue of the manifestly covariant orthogonality relation (3.91) we can write

$$\langle\, p'\,|\, k'\,\rangle = \langle\, p\,|\, k\,\rangle = \delta(\mathbf{p}-\mathbf{k}\,)\,(2\pi)^3\, 2\omega_{\mathbf{k}}$$

Hence, from the basic formulæ

$$U(\omega,\mathrm{a})\,|\, 0\,\rangle = \exp\,\{\, (i/\hbar)\,\mathrm{a}^{\mu}\, P_{\mu} - (i/2\hbar)\,\omega^{\,\rho\sigma} L_{\rho\sigma}\}\,|\, 0\,\rangle = |\, 0\,\rangle$$

$$|\, k'\,\rangle = U^{\dagger}(\omega,\mathrm{a})\,|\,\Lambda\cdot k\,\rangle = \exp\{-\, i\, k\cdot \mathrm{a}\}\;|\, k\,\rangle \qquad \forall\,|\, k\,\rangle \in \mathcal{H}_1$$

it follows that the vacuum state transforms according to a one dimensional trivial representation of the Poincaré group, while the 1-particle Hilbert space $\mathcal{H}_1$ actually carries an irreducible, unitary, infinite dimensional representation of $IO(1,3)$. In turn, owing to the completely symmetric property of the many-particle states, also the Hilbert spaces (3.78) carry irreducible, unitary, infinite dimensional representations of the Poincaré group, so that the Fock space definitely decomposes according to (3.79).

From the transformation law (3.105) of the creation-destruction operators we readily obtain the following identity between operator valued tempered distributions

$$\phi'(x'\,) \equiv U^{\dagger}(\omega,\mathrm{a})\,\phi(x'\,)\, U(\omega,\mathrm{a}) = \phi(x) \tag{3.106}$$

which looks the very same as that one for the classical scalar field. Thus we eventually obtain the Poincaré transformation rule for the Klein-Gordon operator valued tempered distribution: namely,

$$\begin{aligned} \phi'(x'\,) &\equiv U^{-1}(\omega,\mathrm{a})\,\phi\,(x'\,)\, U(\omega,\mathrm{a}) \\ &= \int \mathrm{D}\, k'\,\Big[\, a'(k'\,)\,\exp\{-i\, k'\cdot x'\} + \mathrm{h.\,c.}\Big]_{k_0'=\omega_{\mathbf{k}'}} \\ &= \int \mathrm{D}\,(\Lambda k)\,\Big[\,\mathrm{e}^{ik\cdot \mathrm{a}}\, a(k)\,\mathrm{e}^{-\,ik\cdot(x+\mathrm{a})} + \mathrm{h.\,c.}\Big]_{k_0=\omega_{\mathbf{k}}} \;=\; \phi(x) \end{aligned}$$

in accordance with (3.106). Notice that the above relation, as expected from the general principles of Quantum Mechanics and Quantum Field Theory, is equivalent to set

$$\phi(x')\,|\,\Upsilon'\,\rangle = \phi(x')\,U(\omega,\mathrm{a})\,|\,\Upsilon\,\rangle = U(\omega,\mathrm{a})\,\phi(x)\,|\,\Upsilon\,\rangle \tag{3.107}$$

for any state vector $|\,\Upsilon\,\rangle$ belonging to the Fock space $\mathcal{F}$.

Let us now collect the ten conserved dynamical quantities related to the spin-less charge-less quantum field

$$\begin{aligned}
P_0 &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} : \Pi^2(x) + \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) + m^2\phi^2(x) : \\
P_k &= \int \mathrm{d}\mathbf{x}\ :\Pi(x)\,\partial_k\,\phi(x) : \\
L_{jk} &= \int \mathrm{d}\mathbf{x}\ : x_j\,\Pi(x)\,\partial_k\,\phi(x) - x_k\,\Pi(x)\,\partial_j\,\phi(x) : \\
L_{0k} &= x_0\,P_k - m\,X_k(t) \qquad \dot{L}_{0k} = 0 \quad \Longleftrightarrow \quad \mathbf{X}(t) = \mathbf{X}(0) + \frac{\mathbf{P}}{m}\,t \\
X_k(t) &= \frac{1}{2m}\int \mathrm{d}\mathbf{x}\ x_k : \Pi^2(x) + \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) + m^2\phi^2(x) :
\end{aligned}$$

Owing to the self-adjointness of the field operators

$$\phi(x) = \phi^\dagger(x) \qquad\qquad \Pi(x) = \Pi^\dagger(x)$$

all the above ten conserved dynamical quantities turn out to be self-adjoint operators corresponding to physical observable quantities. For example

$$\begin{aligned}
\mathbf{P}^\dagger &= -\int \mathrm{d}\mathbf{x}\ :[\,\boldsymbol{\nabla}\,\phi^\dagger(x)\,]\,\Pi^\dagger(x) : \\
&= -\int \mathrm{d}\mathbf{x}\ :[\,\boldsymbol{\nabla}\,\phi(x)\,]\,\Pi(x) : \\
&= -\int \mathrm{d}\mathbf{x}\ :\Pi(x)\boldsymbol{\nabla}\,\phi(x) :
\end{aligned} \tag{3.108}$$

thanks to normal ordering. It appears thereby evident that the self-adjoint operators $(P_\mu\,,\,L_{\rho\sigma})$ acting on the Fock space are the generators of a unitary infinite dimensional representation of the Poincaré group on $\mathcal{F}$. From the canonical commutation relations (3.52) it is immediate to show that

$$[\,\phi(x)\,,\,P_\mu\,] = i\hbar\,\partial_\mu\,\phi(x) \tag{3.109}$$

$$[\,\phi(x)\,,\,L_{\mu\nu}\,] = i\hbar\,x_\mu\,\partial_\nu\,\phi(x) - \ i\hbar\,x_\nu\,\partial_\mu\,\phi(x) \tag{3.110}$$

By direct inspection, it is straightforward to verify that, using the canonical commutation relations (3.52) and the normal ordering prescription, the self-adjoint operators $(P_\mu\,,\,L_{\rho\sigma})$ do actually fulfill the Poincaré Lie algebra (1.47) so that, in any neighborhood of the unit element $\omega^{\,\rho\sigma} = 0 = a^{\mu}$, we can safely write

$$U\ :\ \mathcal{F}\ \to\ \mathcal{F} \qquad U(\omega, \mathrm{a}) = \exp\left\{\, i\,\mathrm{a}^{\mu}\,P_\mu - \tfrac{1}{2}\,i\,\omega^{\,\rho\sigma}L_{\rho\sigma}\right\}$$

This is the way in which the infinite dimensional unitary representation of the Poincaré group on the Fock space does explicitly emerge from the quantum theory of the real Klein-Gordon field with mass m and spin zero. In fact, for *e.g.* an infinitesimal passive Poincaré transformation we have

$$\begin{aligned}
\phi^{\prime}\,(x) &\equiv U(\delta\,\omega, \delta\,a)\,\phi\,(x)\,U^{\,\dagger}(\delta\,\omega, \delta\,a) \\
&= \phi\,(x) + i\,\epsilon^{\,\mu}\,[\,P_\mu\,,\,\phi\,(x)\,] - \tfrac{1}{2}\,i\,\epsilon^{\,\rho\sigma}\,[\,L_{\rho\sigma}\,,\,\phi\,(x)\,] \\
&= \phi\,(x) + (\,\epsilon^{\,\mu} + \epsilon^{\,\mu\nu}\,x_\nu\,)\,\partial_\mu\,\phi\,(x) \\
&= \phi\,(x + \delta\,x)
\end{aligned} \tag{3.111}$$

where we have denoted the infinitesimal parameters by $\delta a^{\mu} = \epsilon^{\mu}$ and $\delta\omega^{\,\rho\sigma} = \epsilon^{\,\rho\sigma}$ as usual.

2. The Angular Momentum Operator of a Scalar Quantum Field. For a spin-less Klein-Gordon field the angular momentum operator is provided by the Nöther theorem, up to the normal ordering quantum prescription. It's of a purely orbital nature and reads

$$L_{\,\mu\nu} = \int \mathrm{d}\mathbf{x}\ \ :x_{\,\mu}\,T_{0\,\nu}\,(t, \mathbf{x}) - x_{\,\nu}\,T_{0\,\mu}\,(t, \mathbf{x}):$$

where

$$T_{0\,\mu} = \Pi\,\partial_\mu\phi + \tfrac{1}{2}\,g_{0\mu}\Big(\,|\,\nabla\phi\,|^2 + m^2\phi^2 - \Pi^2\Big) \qquad \Pi = \dot{\phi}$$

Consider first the constant spatial components of the orbital angular momentum operator

$$\mathbf{L} = (L_{23}, L_{31}, L_{12}) \qquad L_{\,\imath\jmath} = \int \mathrm{d}^3\mathrm{x}\ \ :\mathrm{x}^{\,\imath}\,T^{\,0\jmath}(0, \mathbf{x}) - \mathrm{x}^{\,\jmath}\,T^{\,0\imath}(0, \mathbf{x}):$$

By substituting the normal mode expansions

$$\phi(x) = \sum_{\mathbf{p}}\Big[\,a_{\mathbf{p}}\,u_{\mathbf{p}}(x) + a_{\mathbf{p}}^{\dagger}\,u_{\mathbf{p}}^{*}(x)\,\Big]$$

$$\Pi(x) = \sum_{\mathbf{p}} i\omega_{\,\mathbf{p}}\Big[\,a_{\mathbf{p}}^{\dagger}\,u_{\mathbf{p}}^{*}(x) - a_{\mathbf{p}}\,u_{\mathbf{p}}(x)\,\Big]$$

we obtain

$$\begin{aligned}
L_{\,\imath\jmath} &= \int \mathrm{d}^3\mathrm{x}\ \ :\mathrm{x}^{\,\jmath}\,\Pi(0, \mathbf{x})\nabla_{\imath}\,\phi(0, \mathbf{x}) - \mathrm{x}^{\,\imath}\,\Pi(0, \mathbf{x})\nabla_{\jmath}\,\phi(0, \mathbf{x}): \\
&= \int \mathrm{d}^3\mathrm{x}\,\sum_{\mathbf{p}}\sum_{\mathbf{k}}\ :\Big[\,a_{\mathbf{p}}\,u_{\mathbf{p}}(0, \mathbf{x}) - a_{\mathbf{p}}^{\dagger}\,u_{\mathbf{p}}^{*}(0, \mathbf{x})\,\Big]
\end{aligned}$$

$$
\begin{aligned}
&\times \quad \Big[a_{\mathbf{k}}\, u_{\mathbf{k}}(0,\mathbf{x}) - a^{\dagger}_{\mathbf{k}}\, u^{*}_{\mathbf{k}}(0,\mathbf{x}) \Big] : \mathrm{x}^{\jmath}\,\mathrm{k}_{\imath}\,\omega_{\mathbf{p}} \;-\; \{\imath \longleftrightarrow \jmath\} \\
&= \quad \sum_{\mathbf{p}} \sum_{\mathbf{k}} \; \mathrm{k}_{\imath}\,\omega_{\mathbf{p}}\, a^{\dagger}_{\mathbf{k}}\, a_{\mathbf{p}}\, (\,i\,\partial/\partial \mathrm{k}^{\jmath})\int \mathrm{d}\mathbf{x}\, u_{\mathbf{p}}(0,\mathbf{x})\, u^{*}_{\mathbf{k}}(0,\mathbf{x}) \\
&- \quad \sum_{\mathbf{p}} \sum_{\mathbf{k}} \; \mathrm{k}_{\imath}\,\omega_{\mathbf{p}}\, a^{\dagger}_{\mathbf{p}}\, a_{\mathbf{k}}\, (\,i\,\partial/\partial \mathrm{k}^{\jmath})\int \mathrm{d}\mathbf{x}\, u^{*}_{\mathbf{p}}(0,\mathbf{x})\, u_{\mathbf{k}}(0,\mathbf{x}) \\
&- \quad \sum_{\mathbf{p}} \sum_{\mathbf{k}} \; \mathrm{k}_{\imath}\,\omega_{\mathbf{p}}\, a^{\dagger}_{\mathbf{p}}\, a^{\dagger}_{\mathbf{k}}\, (\,i\,\partial/\partial \mathrm{k}^{\jmath})\int \mathrm{d}\mathbf{x}\, u^{*}_{\mathbf{p}}(0,\mathbf{x})\, u^{*}_{\mathbf{k}}(0,\mathbf{x}) \\
&+ \quad \sum_{\mathbf{p}} \sum_{\mathbf{k}} \; \mathrm{k}_{\imath}\,\omega_{\mathbf{p}}\, a_{\mathbf{k}}\, a_{\mathbf{p}}\, (\,i\,\partial/\partial \mathrm{k}^{\jmath})\int \mathrm{d}\mathbf{x}\, u_{\mathbf{p}}(0,\mathbf{x})\, u_{\mathbf{k}}(0,\mathbf{x}) \\
&- \quad \{\, \imath \longleftrightarrow \jmath \,\}
\end{aligned}
$$

Now, from the basic integrals

$$
\begin{aligned}
\int \mathrm{d}\mathbf{x}\, u_{\mathbf{p}}(0,\mathbf{x})\, u_{\mathbf{k}}(0,\mathbf{x}) &= (\,2\,\omega_{\mathbf{p}}\,)^{-1}\,\delta(\mathbf{k}+\mathbf{p}) \\
\int \mathrm{d}\mathbf{x}\, u^{*}_{\mathbf{p}}(0,\mathbf{x})\, u_{\mathbf{k}}(0,\mathbf{x}) &= (\,2\,\omega_{\mathbf{p}}\,)^{-1}\,\delta(\mathbf{k}-\mathbf{p})
\end{aligned}
$$

we obtain

$$
\begin{aligned}
L_{\imath\jmath} \quad &= \quad -\sum_{\mathbf{p}} \sum_{\mathbf{k}} \; \tfrac{1}{2}\,\mathrm{k}_{\imath}\Big(a^{\dagger}_{\mathbf{p}}\, a_{\mathbf{k}} - a^{\dagger}_{\mathbf{k}}\, a_{\mathbf{p}} \Big)\, (\,i\,\partial/\partial \mathrm{k}^{\jmath})\, \delta(\mathbf{k}-\mathbf{p}) \\
&- \quad \sum_{\mathbf{p}} \sum_{\mathbf{k}} \; \tfrac{1}{2}\,\mathrm{k}_{\imath}\Big(a^{\dagger}_{\mathbf{p}}\, a^{\dagger}_{\mathbf{k}} - a_{\mathbf{p}}\, a_{\mathbf{k}} \Big)\, (\,i\,\partial/\partial \mathrm{k}^{\jmath})\, \delta(\mathbf{k}+\mathbf{p}) \\
&- \quad \{\, \imath \longleftrightarrow \jmath \,\} \qquad\qquad (\,\imath \neq \jmath\,)
\end{aligned}
$$

where derivatives are understood in the sense of the distributions, in such a manner that we eventually get

$$
L_{\imath\jmath} \quad = \quad \sum_{\mathbf{k}} \; \left(\tfrac{1}{2}\,\mathrm{k}_{\imath}\, a^{\dagger}_{\mathbf{k}}\, \frac{i\,\overleftrightarrow{\partial}}{\partial \mathrm{k}^{\jmath}}\, a_{\mathbf{k}} \;-\; \tfrac{1}{2}\,\mathrm{k}_{\jmath}\, a^{\dagger}_{\mathbf{k}}\, \frac{i\,\overleftrightarrow{\partial}}{\partial \mathrm{k}^{\imath}}\, a_{\mathbf{k}} \right)
$$

because the remaining integrals do vanish. As a matter of fact, consider for example the case $\imath = 1$, $\jmath = 2$, $\mathbf{k} = (k_x, k_y, k_z)$ so that

$$
\begin{aligned}
& \iint \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{k}\;\, k_x\, a_{\mathbf{p}}\, a_{\mathbf{k}}\, \frac{i\partial}{\partial k_y}\; \delta(\mathbf{k}+\mathbf{p}) \\
= \quad & -\iint \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{k}\;\, \delta(\mathbf{k}+\mathbf{p})\; k_x\, a_{\mathbf{p}}\, \frac{i\partial}{\partial k_y}\; a_{\mathbf{k}} \\
= \quad & -\int \mathrm{d}\mathbf{k}\, k_x\, a_{-\mathbf{k}}\, \frac{i\partial}{\partial k_y}\; a_{\mathbf{k}}
\end{aligned}
$$

Now, if we send $\mathbf{k}$ into $-\mathbf{k}$ in the very last integral we obtain

$$
\begin{aligned}
& -\int \mathrm{d}\mathbf{k}\, k_x\, a_{-\mathbf{k}}\, \frac{i\partial}{\partial k_y}\; a_{\mathbf{k}} = -\int \mathrm{d}\mathbf{k}\, k_x\, a_{\mathbf{k}}\, \frac{i\partial}{\partial k_y}\; a_{-\mathbf{k}} \\
& = \int \mathrm{d}\mathbf{k}\, k_x \left(\frac{i\partial}{\partial k_y}\; a_{\mathbf{k}} \right) a_{-\mathbf{k}} = \int \mathrm{d}\mathbf{k}\, k_x\, a_{-\mathbf{k}}\, \frac{i\partial}{\partial k_y}\; a_{\mathbf{k}}
\end{aligned}
$$

because $[\, a_{\mathbf{k}}\,,\, a_{\mathbf{p}}\,] = 0\,,\ \forall\,\mathbf{k}\,,\mathbf{p} \in \mathbb{R}^3$, so that consequently

$$[\,\partial a_{\mathbf{k}}/\partial k^{\jmath}\,,\, a_{\mathbf{p}}\,] = 0 \qquad\qquad (\,\jmath = 1,2,3\,)$$

and thereby

$$\sum_{\mathbf{p}}\sum_{\mathbf{k}}\ \tfrac{1}{2}\,\mathrm{k}_{\imath}\Big(a_{\mathbf{p}}^{\dagger}\,a_{\mathbf{k}}^{\dagger} - a_{\mathbf{p}}\,a_{\mathbf{k}}\Big)\,(\,i\,\partial/\partial \mathrm{k}^{\jmath})\,\delta(\mathbf{k}+\mathbf{p}) \equiv 0$$

as previously claimed.

The spatial-temporal components of the relativistic angular momentum operator of a Klein-Gordon scalar quantum field read

$$L^{0k} = \textstyle\int \mathrm{d}\mathbf{x}\ :x_0\,T^{\,0k}\,(t,\mathbf{x}) - x^{\,k}\,T^{\,0\,0}\,(t,\mathbf{x}):$$

or equivalently

$$\hbar\boldsymbol{\Upsilon}_t = ct\,\mathbf{P} - mc\,\mathbf{X}_{\,t}$$

where

$$\hbar\boldsymbol{\Upsilon} = (L^{\,01}, L^{\,02}, L^{\,03})$$

$$P^{\,k} = \textstyle\int \mathrm{d}\mathbf{x}\ :T^{\,0k}\,(t,\mathbf{x}): =\ -\int \mathrm{d}\mathbf{x}\ :\Pi(t,\mathbf{x})\,\nabla_k\phi(t,\mathbf{x}):$$

Notice that the operator

$$\mathbf{X}_{\,t} = \frac{1}{2mc^2}\int \mathrm{d}^3\mathrm{x}\,\mathbf{x} : c^{\,2}\,\Pi^{\,2}(t,\mathbf{x}) + \nabla\phi(t,\mathbf{x})\cdot\nabla\phi(t,\mathbf{x}) + \phi^2(t,\mathbf{x})(mc/\hbar)^2 :$$

corresponds to the instantaneous **position operator of the center of the energy**, the quantum and relativistic generalization of the center of gravity of the Newtonian Mechanics. It is worthwhile to gather that the operator $\hbar\boldsymbol{\Upsilon}_t = ct\,\mathbf{P} - mc\,\mathbf{X}_{\,t}$ is explicitly time dependent, so that its evolution equation in the Heisenberg picture turns out to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Upsilon}_t \equiv \dot{\boldsymbol{\Upsilon}}_t = \frac{\partial}{\partial t}\,\boldsymbol{\Upsilon}_t + \frac{1}{i\hbar}[\,\boldsymbol{\Upsilon}_t\,,\,H\,] = \frac{\partial}{\partial t}\,\boldsymbol{\Upsilon}_t + \frac{c}{i\hbar}[\,\boldsymbol{\Upsilon}_t\,,\,P_0\,]$$

according to the general principles of Quantum Mechanics. Since $[\,P_0, \mathbf{P}\,] = 0$ we can definitely write

$$\hbar\dot{\boldsymbol{\Upsilon}}_t = c\,\mathbf{P} - \frac{i}{\hbar}[\,H, \mathbf{X}_{\,t}\,]\,mc = c\,\mathbf{P} - mc\,\dot{\mathbf{X}}_{\,t}$$

After turning to natural units we find

$$\begin{aligned}
[\,H,\,mX_{\,t}^{\,k}\,] &= [\,P_0, L_{\,0k}\,] = \tfrac{1}{2}\textstyle\int \mathrm{d}^3\mathrm{x}\,x^{\,k}\,[\,H, :\Pi^{\,2}(x) + \nabla\phi(x)\cdot\nabla\phi(x) + m^2\phi^2(x):\,] \\
&= i\int \mathrm{d}^3\mathrm{x}\ x_k :\Pi(x)\dot{\Pi}(x) + \nabla\phi(x)\cdot\nabla\,\Pi(x) + m^2\phi(x)\Pi(x): \\
&= i\int \mathrm{d}^3\mathrm{x}\ x_k :\Pi(x)(\triangle - m^2)\phi(x) + \nabla\phi(x)\cdot\nabla\,\Pi(x) + m^2\phi(x)\Pi(x): \\
&= i\int \mathrm{d}^3\mathrm{x}\ x_k :\Pi(x)\triangle\phi(x) + \nabla\phi(x)\cdot\nabla\,\Pi(x): \\
&= i\int \mathrm{d}^3\mathrm{x}\ \nabla_{\jmath}\,(\,x_k\,\{:\Pi(x)\nabla_{\jmath}\,\phi(x):\,) + :\Pi(x)\,\nabla_k\,\phi(x):\} = iP_k
\end{aligned}$$

Notice that this result is perfectly consistent with the commutator

$$[\, P_\mu , L_{\,\rho\sigma}\,] = i\hbar\, g_{\mu\rho}\, P_\sigma - i\hbar\, g_{\mu\sigma}\, P_\rho$$

for $\mu = \rho = 0$ and $\sigma = k$, as required by the Poincaré Lie algebra, as well as the quantum counterpart of the Nöther theorem, *viz.,*

$$\dot{\mathbf{P}} = \dot{\boldsymbol{\Upsilon}} = 0$$

or even

$$\dot{\boldsymbol{\Upsilon}} = 0 \qquad \Longleftrightarrow \qquad \dot{\mathbf{X}} = \frac{\mathbf{P}}{m} \;\Longleftrightarrow\; \ddot{\mathbf{X}} = 0$$

that proves the inertial motion of the center of energy operator and just provides in turn an explicit expression in terms of the field operators: namely,

$$\begin{aligned}
\mathbf{X}_{\,t} \;&=\; \frac{\mathbf{P}}{m}\, t + \mathbf{X}_{\,0} = \,-\,\frac{t}{m}\int \mathrm{d}^3\mathrm{x}\; \mathbf{x} : \Pi(0,\mathbf{x})\nabla\phi(0,\mathbf{x}) : \\
&+\; \frac{1}{2mc^{\,2}}\int \mathrm{d}^3\mathrm{x}\; \mathbf{x} : c^{\,2}\,\Pi^{\,2}(0,\mathbf{x}) + \nabla\phi(0,\mathbf{x})\cdot\nabla\phi(0,\mathbf{x}) + \phi^2(0,\mathbf{x})(mc/\hbar)^2 :
\end{aligned}$$

Let us now substitute the normal modes expansions

$$\begin{aligned}
\Pi(0,\mathbf{x}) &= \sum_{\mathbf{k}}\; i\omega_{\mathbf{k}}\left[\, a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x}) - a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x})\,\right] \\
\nabla\phi(0,\mathbf{x}) &= \sum_{\mathbf{k}}\; i\mathbf{k}\left[\, a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x}) - a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x})\,\right] \\
\phi(0,\mathbf{x}) &= \sum_{\mathbf{k}}\; \left[\, a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x}) + a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x})\,\right]
\end{aligned}$$

with $u_{\mathbf{k}}(0,\mathbf{x}) = [\,(2\pi)^3\, 2\omega_{\mathbf{k}}\,]^{-1/2} \exp\{\, i\mathbf{k}\cdot\mathbf{x}\}$, whence we obtain

$$\begin{aligned}
&\int \mathrm{d}^3\mathrm{x}\; \mathbf{x} : \Pi^{\,2}(0,\mathbf{x}) : \\
=\;& \int \mathrm{d}^3\mathrm{x} \sum_{\mathbf{k}}\; \omega_{\mathbf{k}} : \left[\, a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x}) - a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x})\,\right] \mathbf{x} \sum_{\mathbf{p}}\; \omega_{\mathbf{p}}\left[\, a_{\mathbf{p}} u_{\mathbf{p}}(0,\mathbf{x}) - a_{\mathbf{p}}^{\dagger} u_{\mathbf{p}}^{*}(0,\mathbf{x})\,\right] : \\
=\;& -\int \mathrm{d}^3\mathrm{x} \sum_{\mathbf{k},\mathbf{p}}\; i\omega_{\mathbf{k}}\omega_{\mathbf{p}} : \left[\, a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x}) - a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x})\,\right]\left[\, a_{\mathbf{p}}^{\dagger}\nabla_{\mathbf{p}} u_{\mathbf{p}}^{*}(0,\mathbf{x}) + a_{\mathbf{p}}\nabla_{\mathbf{p}} u_{\mathbf{p}}(0,\mathbf{x})\,\right] :
\end{aligned}$$

Now we find

$$\begin{aligned}
&\tfrac{1}{2}\, i\sum_{\mathbf{k},\mathbf{p}}\; \sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}\; a_{\mathbf{k}}^{\dagger} a_{\mathbf{p}}^{\dagger}\nabla_{\mathbf{p}}\int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\; \exp\{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{k})\} \\
=\;& \tfrac{1}{2}\, i\int \mathrm{d}\mathbf{p}\; \sqrt{\omega_{\mathbf{p}}}\; a_{\mathbf{p}}^{\dagger}\,\nabla_{\mathbf{p}}\int \mathrm{d}\mathbf{k}\; a_{\mathbf{k}}^{\dagger}\,\sqrt{\omega_{\mathbf{k}}}\;\delta(\mathbf{p}+\mathbf{k}) \\
=\;& \tfrac{1}{2}\, i\int \mathrm{d}\mathbf{p}\; \sqrt{\omega_{\mathbf{p}}}\; a_{\mathbf{p}}^{\dagger}\,\nabla_{\mathbf{p}}\left(a_{-\mathbf{p}}^{\dagger}\,\sqrt{\omega_{\mathbf{p}}}\right) \\
=\;& \tfrac{1}{2}\, i\int \mathrm{d}\mathbf{p}\;\left(\omega_{\mathbf{p}}\; a_{\mathbf{p}}^{\dagger}\,\nabla_{\mathbf{p}}\, a_{-\mathbf{p}}^{\dagger} + \tfrac{1}{2}\, a_{\mathbf{p}}^{\dagger}\, a_{-\mathbf{p}}^{\dagger}\;\mathbf{p}/\omega_{\mathbf{p}}\right) = \tfrac{1}{2}\, i\int \mathrm{d}\mathbf{p}\;\omega_{\mathbf{p}}\; a_{\mathbf{p}}^{\dagger}\,\nabla_{\mathbf{p}}\, a_{-\mathbf{p}}^{\dagger}
\end{aligned}$$

the other term being evidently null owing to symmetric integration. In a quite analogous way we immediately get

$$\begin{aligned}
& - \quad \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} \sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}\; a_{\mathbf{k}} a_{\mathbf{p}} \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\; \exp\{-\, i\mathbf{x}\cdot(\mathbf{p}+\mathbf{k})\} \\
& = \quad -\tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p}\; \omega_{\mathbf{p}}\; a_{\mathbf{p}}\; \nabla_{\mathbf{p}}\, a_{-\mathbf{p}}
\end{aligned}$$

The remaining two terms yield

$$\begin{aligned}
& \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} \sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}\; a_{\mathbf{k}}^{\dagger} a_{\mathbf{p}} \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\; \exp\{-\, i\mathbf{x}\cdot(\mathbf{p}-\mathbf{k})\} \\
= \quad & \tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p} \sqrt{\omega_{\mathbf{p}}}\; : a_{\mathbf{p}} \nabla_{\mathbf{p}} \int \mathrm{d}\mathbf{p}\; \sqrt{\omega_{\mathbf{k}}}\; a_{\mathbf{k}}^{\dagger} : \delta(\mathbf{p}-\mathbf{k}) \\
= \quad & \tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p} \sqrt{\omega_{\mathbf{p}}}\; : a_{\mathbf{p}} \nabla_{\mathbf{p}} \left(\sqrt{\omega_{\mathbf{p}}}\; a_{\mathbf{p}}^{\dagger}\right) : \\
= \quad & \tfrac{1}{2}\, i \int \mathrm{d}\mathbf{k} \left\{ \omega_{\mathbf{k}} \left(\nabla_{\mathbf{k}}\, a_{\mathbf{k}}^{\dagger}\right) a_{\mathbf{k}} + \tfrac{1}{2}\, a_{\mathbf{k}}^{\dagger}\, a_{\mathbf{k}}\, \mathbf{k}/\omega_{\mathbf{k}} \right\} \\
- \quad & \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} \sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}\; a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\; \exp\{\, i\mathbf{x}\cdot(\mathbf{p}-\mathbf{k})\} \\
= \quad & -\tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p} \sqrt{\omega_{\mathbf{p}}}\, a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}} \int \mathrm{d}\mathbf{k}\; \sqrt{\omega_{\mathbf{k}}}\; a_{\mathbf{k}}\, \delta(\mathbf{p}-\mathbf{k}) \\
= \quad & -\tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p} \left(\omega_{\mathbf{p}}\, a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{\mathbf{p}} + \tfrac{1}{2}\, a_{\mathbf{p}}^{\dagger}\, a_{\mathbf{p}}\, \mathbf{p}/\omega_{\mathbf{p}}\right)
\end{aligned}$$

so that we definitely obtain in natural units the following result for the first addendum of the above expression (3.112), *viz.,*

$$\begin{aligned}
& \int \mathrm{d}^3\mathrm{x}\; \mathbf{x} : \Pi^2(0,\mathbf{x}) : \\
= \quad & \tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p}\, \omega_{\mathbf{p}} \left[a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{\mathbf{p}} - \left(\nabla_{\mathbf{p}}\, a_{\mathbf{p}}^{\dagger}\right) a_{\mathbf{p}} - a_{\mathbf{p}}^{\dagger}\, \nabla_{\mathbf{p}}\, a_{-\mathbf{p}}^{\dagger} + a_{\mathbf{p}}\, \nabla_{\mathbf{p}}\, a_{-\mathbf{p}} \right]
\end{aligned}$$

Next we find

$$\begin{aligned}
& \int \mathrm{d}^3\mathrm{x}\; \mathbf{x} : \phi^2(0,\mathbf{x}) : \\
= \quad & \int \mathrm{d}^3\mathrm{x} \sum_{\mathbf{k},\mathbf{p}}\; : \left[a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x}) + a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x}) \right] \mathbf{x} \left[a_{\mathbf{p}} u_{\mathbf{p}}(0,\mathbf{x}) + a_{\mathbf{p}}^{\dagger} u_{\mathbf{p}}^{*}(0,\mathbf{x}) \right] : \\
= \quad & i \int \mathrm{d}^3\mathrm{x} \sum_{\mathbf{k},\mathbf{p}}\; : \left[a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x}) + a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x}) \right] \left[a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, u_{\mathbf{p}}^{*}(0,\mathbf{x}) - a_{\mathbf{p}} \nabla_{\mathbf{p}}\, u_{\mathbf{p}}(0,\mathbf{x}) \right] : \\
= \quad & \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} \left(\omega_{\mathbf{k}}\omega_{\mathbf{p}}\right)^{-1/2} a_{\mathbf{k}}^{\dagger} a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\; \exp\{-i\mathbf{x}\cdot(\mathbf{k}+\mathbf{p})\} \\
- \quad & \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} \left(\omega_{\mathbf{k}}\omega_{\mathbf{p}}\right)^{-1/2} a_{\mathbf{k}} a_{\mathbf{p}} \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\; \exp\{\, i\mathbf{x}\cdot(\mathbf{k}+\mathbf{p})\}
\end{aligned}$$

$$
\begin{aligned}
+ &\quad \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} \ (\omega_{\mathbf{k}}\omega_{\mathbf{p}})^{-1/2}\, a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\ \exp\{\, i\mathbf{x}\cdot(\mathbf{k}-\mathbf{p})\} \\
- &\quad \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} \ (\omega_{\mathbf{k}}\omega_{\mathbf{p}})^{-1/2}\, a_{\mathbf{k}}^{\dagger} a_{\mathbf{p}} \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\ \exp\{-i\mathbf{x}\cdot(\mathbf{k}-\mathbf{p})\} \\
= &\quad \tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p}\ \omega_{\mathbf{p}}^{-1/2} a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}} \left(\omega_{\mathbf{p}}^{-1/2} a_{-\mathbf{p}}^{\dagger}\right)\ +\ \mathrm{H.c.} \\
+ &\quad \tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p}\ \omega_{\mathbf{p}}^{-1/2} a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}} \left(\omega_{\mathbf{p}}^{-1/2} a_{\mathbf{p}}\right) \\
- &\quad \tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p}\ \omega_{\mathbf{p}}^{-1/2} : a_{\mathbf{p}} \nabla_{\mathbf{p}} \left(\omega_{\mathbf{p}}^{-1/2} a_{\mathbf{p}}^{\dagger}\right) : \\
= &\quad \tfrac{1}{2}\, i \int \mathrm{d}\mathbf{p}\ \omega_{\mathbf{p}}^{-1} a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{-\mathbf{p}}^{\dagger}\ +\ \mathrm{H.c.} \\
+ &\quad \frac{i}{2} \int \mathrm{d}\mathbf{p}\, \omega_{\mathbf{p}}^{-1} \left[\, a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{\mathbf{p}} - \left(\nabla_{\mathbf{p}}\, a_{\mathbf{p}}^{\dagger}\right) a_{\mathbf{p}} \,\right]
\end{aligned}
$$

so that

$$
\begin{aligned}
&m^2 \int \mathrm{d}^3\mathrm{x}\ \mathbf{x} : \phi^2(0,\mathbf{x}) : \\
= &\int \mathrm{d}\mathbf{p}\ \frac{im^2}{2\omega_{\mathbf{p}}}\ \left[\, a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{-\mathbf{p}}^{\dagger} - a_{\mathbf{p}} \nabla_{\mathbf{p}}\, a_{-\mathbf{p}} + a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{\mathbf{p}} - \left(\nabla_{\mathbf{p}}\, a_{\mathbf{p}}^{\dagger}\right) a_{\mathbf{p}} \right]
\end{aligned}
$$

Finally we have to calculate

$$
\begin{aligned}
&\int \mathrm{d}^3\mathrm{x}\ \mathbf{x} : \nabla\phi(0,\mathbf{x}) \cdot \nabla\phi(0,\mathbf{x}) : \\
= &\int \mathrm{d}^3\mathrm{x}\ \mathbf{x} : \sum_{\mathbf{k},\mathbf{p}} \ \left[a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x}) - a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x}) \right] (\mathbf{k}\cdot\mathbf{p}) \left[\, a_{\mathbf{p}}^{\dagger} u_{\mathbf{p}}^{*}(0,\mathbf{x}) - a_{\mathbf{p}} u_{\mathbf{p}}(0,\mathbf{x}) \,\right] : \\
= &\ i \int \mathrm{d}^3\mathrm{x} : \sum_{\mathbf{k},\mathbf{p}} \ \left[a_{\mathbf{k}} u_{\mathbf{k}}(0,\mathbf{x}) - a_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(0,\mathbf{x}) \right] (\mathbf{k}\cdot\mathbf{p}) \left[\, a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, u_{\mathbf{p}}^{*}(0,\mathbf{x}) + a_{\mathbf{p}} \nabla_{\mathbf{p}}\, u_{\mathbf{p}}(0,\mathbf{x}) \,\right] : \\
= &\ \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} (\mathbf{k}\cdot\mathbf{p})(\omega_{\mathbf{k}}\omega_{\mathbf{p}})^{-1/2} : a_{\mathbf{k}} a_{\mathbf{p}}^{\dagger} : \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\ \exp\{\, i\mathbf{x}\cdot(\mathbf{k}-\mathbf{p})\}\ +\ \mathrm{H.c.} \\
+ &\ \tfrac{1}{2}\, i \sum_{\mathbf{k},\mathbf{p}} (\mathbf{k}\cdot\mathbf{p})(\omega_{\mathbf{k}}\omega_{\mathbf{p}})^{-1/2} a_{\mathbf{k}} a_{\mathbf{p}} \nabla_{\mathbf{p}} \int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^3}\ \exp\{\, i\mathbf{x}\cdot(\mathbf{k}+\mathbf{p})\}\ +\ \mathrm{H.c.} \\
= &\int \mathrm{d}\mathbf{p}\ \frac{i\mathbf{p}^2}{2\omega_{\mathbf{p}}} \left[a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{-\mathbf{p}}^{\dagger} - a_{\mathbf{p}} \nabla_{\mathbf{p}}\, a_{-\mathbf{p}} + a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{\mathbf{p}} - \left(\nabla_{\mathbf{p}}\, a_{\mathbf{p}}^{\dagger}\right) a_{\mathbf{p}} \right]
\end{aligned}
$$

Summing up altogether we eventually obtain

$$
\boldsymbol{\Upsilon} = \frac{i}{2c} \int \mathrm{d}\mathbf{p}\ \omega_{\mathbf{p}} \left[\, a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{\mathbf{p}} - \left(\nabla_{\mathbf{p}}\, a_{\mathbf{p}}^{\dagger}\right) a_{\mathbf{p}} \,\right]
$$

$$
\mathbf{X}_t = \frac{\mathbf{P}}{m}\, t - \frac{\hbar\boldsymbol{\Upsilon}}{mc} = \sum_{\mathbf{p}} \left\{ \hbar t\, \frac{\mathbf{p}}{m}\, a_{\mathbf{p}}^{\dagger}\, a_{\mathbf{p}} - \frac{i\hbar\omega_{\mathbf{p}}}{2mc^2} \left[\, a_{\mathbf{p}}^{\dagger} \nabla_{\mathbf{p}}\, a_{\mathbf{p}} - \left(\nabla_{\mathbf{p}}\, a_{\mathbf{p}}^{\dagger}\right) a_{\mathbf{p}} \,\right] \right\}
$$

which actually endorses the inertial motion of the center of the energy.

3.5 Special Distributions

We have already met the positive and negative frequency scalar distributions

$$D^{(\pm)}(x) = \pm\,\frac{1}{i}\int\frac{\mathrm{d}^4k}{(2\pi)^3}\;\exp\{\pm\,i\,k\cdot x\}\;\delta\left(k^2 - m^2c^2/\hbar^2\right)\,\theta\,(k_0)$$

The latter ones are characterized by

$$\frac{i}{\hbar c}\,\langle\,0\,|\,\phi\,(x)\,\phi\,(y)\,|\,0\,\rangle = D^{\,(-)}\,(x-y) = \frac{i}{\hbar c}\sum_{\mathbf{k}} u_{\mathbf{k}}(x)\,u^{*}_{\mathbf{k}}(y)$$

$$D^{\,(+)}\,(x-y) = -\,D^{\,(-)}\,(y-x) = -\,\frac{i}{\hbar c}\sum_{\mathbf{k}} u^{*}_{\mathbf{k}}(x)\,u_{\mathbf{k}}(y)$$

$$[\,D^{\,(\pm)}\,(x)\,]^{*} \;=\; D^{\,(\mp)}\,(x)$$

From the normal modes expansion (3.50) and the canonical commutation relations (3.52) we obtain the commutator between two real scalar free field operator at arbitrary points, which is known as the Pauli-Jordan distribution

$$\frac{i}{\hbar c}\,[\,\phi(x)\,,\,\phi(y)\,] \equiv D\,(x-y) = \frac{i}{\hbar c}\sum_{\mathbf{k}}\Big(u_{\mathbf{k}}(x)\,u^{*}_{\mathbf{k}}(y) - u^{*}_{\mathbf{k}}(x)\,u_{\mathbf{k}}(y)\Big)$$

where

$$\begin{aligned} D\,(x) &\overset{\text{def}}{=} i\int\frac{\mathrm{d}\,k}{(2\pi)^3}\;\exp\{-\,i\,k\cdot x\}\,\delta\left(k^2 - m^2\right)\,\mathrm{sgn}\,(k_0)\\ &\equiv D^{(-)}\,(x) \;+\; D^{(+)}\,(x) \end{aligned} \tag{3.112}$$

where the sign distribution is defined to be

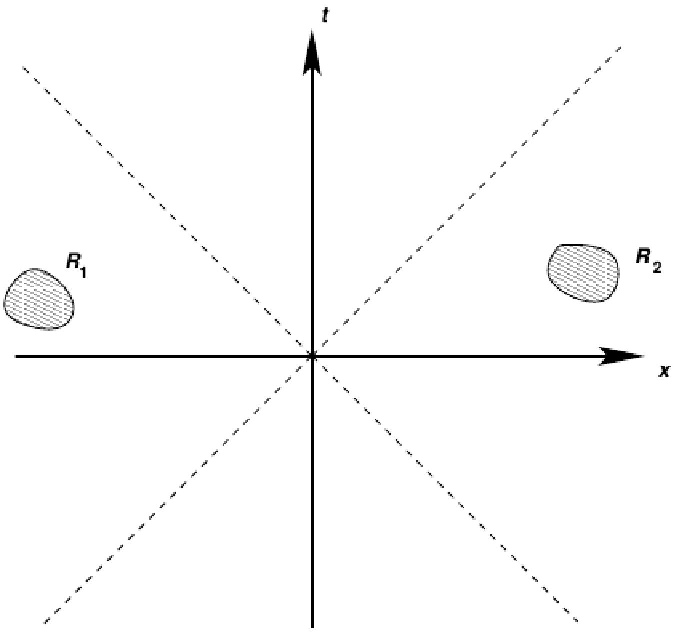
$$\mathrm{sgn}\,(k_0) = \theta(k_0) - \theta(-k_0) = \begin{cases} +1 & \text{for } k_0 > 0\\ -1 & \text{for } k_0 < 0\end{cases}$$

The Pauli-Jordan distribution is a Poincaré invariant solution of the Klein-Gordon wave equation

$$\left(\Box_x + m^2c^2/\hbar^2\right)\,D\,(x-y) = 0$$

with the initial conditions

$$\lim_{x_0\to y_0} D\,(x-y) = 0 \qquad\qquad \lim_{x_0\to y_0}\frac{\partial}{\partial x_0}\,D\,(x-y) \;=\; \delta\,(\mathbf{x}-\mathbf{y})$$



Two regions R_1 and R_2 that are causally disconnected

in such a manner that

$$\lim_{x_0 \to y_0} \left[\Pi(x) \, , \, \phi(y) \right] = \lim_{x_0 \to y_0} \frac{\hbar \partial}{i \partial x_0} D(x-y) = -i\hbar \, \delta(\mathbf{x}-\mathbf{y})$$

in accordance with the canonical equal-time commutation relations.

The Pauli-Jordan distribution is REAL AND ODD

$$D^*(x) = D(x) \qquad\qquad D(-x) = -D(x)$$

and enjoys as well the very important property of vanishing for space-like separations, that is

$$D(x-y) = 0 \qquad \text{for} \quad (x_0 - y_0)^2 < (\mathbf{x}-\mathbf{y})^2 \tag{3.113}$$

The above feature is known as the MICRO-CAUSALITY PROPERTY.

A further very important distribution related to causality is the causal Green function or Feynman propagator. It is defined as follows:

$$D_F(x-y) \;=\; \begin{cases} \langle 0 | \phi(x) \, \phi(y) | 0 \rangle & \text{for } x_0 > y_0 \\ \langle 0 | \phi(y) \, \phi(x) | 0 \rangle & \text{for } x_0 < y_0 \end{cases}$$

$$
\begin{aligned}
&= \theta(x_0 - y_0)\,\langle 0\,|\,\phi(x)\,\phi(y)\,|\,0\,\rangle \\
&+ \theta(y_0 - x_0)\,\langle 0\,|\,\phi(y)\,\phi(x)\,|\,0\,\rangle \\
&\equiv \langle 0\,|\,T\,\phi(x)\,\phi(y)\,|\,0\,\rangle
\end{aligned}
\tag{3.114}
$$

the last line just defining the chronological product of operators in terms of the time ordering symbol T that prescribes the place for the operators that follow in the order with the latest to the left. It is easy to check, by applying the Klein-Gordon differential operator $\square_x + m^2$ to the Feynman propagator and taking (3.52) into account, that the causal Green function is a solution of the non-homogeneous equation

$$
(\,\square_x + m^2c^2/\hbar^2\,)\,D_F(x-y) = -\,i\hbar c\,\delta^{(4)}(x-y)
$$

so that its Fourier representation reads

$$
D_F(x-y) = \frac{i\hbar c}{(2\pi)^4}\int \frac{\exp\{-\,i\,k\cdot(x-y)\}}{k^2-(mc/\hbar)^2+i\varepsilon}\ \mathrm{d}^4k \tag{3.115}
$$

`Proof.` Consider first the time derivative

$$
\begin{aligned}
\frac{\partial}{\partial x_0}\,D_F(x-y) &= \delta(x_0-y_0)\,\langle 0\,|\,\phi(x)\,\phi(y)\,|\,0\,\rangle + \theta(x_0-y_0)\,\langle 0\,|\,\Pi(x)\,\phi(y)\,|\,0\,\rangle \\
&- \delta(y_0-x_0)\,\langle 0\,|\,\phi(y)\,\phi(x)\,|\,0\,\rangle + \theta(y_0-x_0)\,\langle 0\,|\,\phi(y)\,\Pi(x)\,|\,0\,\rangle \\
&= \theta(x_0-y_0)\,\langle 0\,|\,\Pi(x)\,\phi(y)\,|\,0\,\rangle + \theta(y_0-x_0)\,\langle 0\,|\,\phi(y)\,\Pi(x)\,|\,0\,\rangle \\
&+ \delta(x_0-y_0)\,\langle 0\,|\,[\,\phi(x_0,\mathbf{x})\,,\,\phi(x_0,\mathbf{y})\,]\,|\,0\,\rangle \\
&= \theta(x_0-y_0)\,\langle 0\,|\,\Pi(x)\,\phi(y)\,|\,0\,\rangle + \theta(y_0-x_0)\,\langle 0\,|\,\phi(y)\,\Pi(x)\,|\,0\,\rangle
\end{aligned}
$$

owing to the equal time canonical commutation relations. Next we find

$$
\begin{aligned}
\frac{\partial^2}{\partial x_0^2}\,D_F(x-y) &= \delta(x_0-y_0)\,\langle 0\,|\,\Pi(x)\,\phi(y)\,|\,0\,\rangle - \delta(y_0-x_0)\,\langle 0\,|\,\phi(y)\,\Pi(x)\,|\,0\,\rangle \\
&+ \theta(x_0-y_0)\,\langle 0\,|\,\dot\Pi(x)\,\phi(y)\,|\,0\,\rangle + \theta(y_0-x_0)\,\langle 0\,|\,\phi(y)\,\dot\Pi(x)\,|\,0\,\rangle \\
&= \delta(x_0-y_0)\,\langle 0\,|\,[\,\Pi(x_0,\mathbf{x})\,,\,\phi(x_0,\mathbf{y})\,]\,|\,0\,\rangle \\
&+ \theta(x_0-y_0)\,\langle 0\,|\,\dot\Pi(x)\,\phi(y)\,|\,0\,\rangle + \theta(y_0-x_0)\,\langle 0\,|\,\phi(y)\,\dot\Pi(x)\,|\,0\,\rangle \\
&= \theta(x_0-y_0)\,\langle 0\,|\,\dot\Pi(x)\,\phi(y)\,|\,0\,\rangle + \theta(y_0-x_0)\,\langle 0\,|\,\phi(y)\,\dot\Pi(x)\,|\,0\,\rangle \\
&- i\delta(x_0-y_0)\,\delta(\mathbf{x}-\mathbf{y})
\end{aligned}
$$

thanks again to the equal time canonical commutation relations. Thus we eventually obtain

$$
\begin{aligned}
(\square_x + m^2)\,D_F(x-y) &= \theta(x_0-y_0)\,\langle 0\,|\,\{\ddot\phi(x) - \boldsymbol{\nabla}^2\phi(x) + m^2\phi(x)\}\,\phi(y)\,|\,0\,\rangle \\
&+ \theta(y_0-x_0)\,\langle 0\,|\,\phi(y)\{\ddot\phi(x) - \boldsymbol{\nabla}^2\phi(x) + m^2\phi(x)\}\,|\,0\,\rangle \\
&- i\delta(x_0-y_0)\,\delta(\mathbf{x}-\mathbf{y})\ =\ -\,i\delta(x-y)
\end{aligned}
$$

for the operator valued Hermitean scalar field distribution $\phi(x) = \phi^\dagger(x)$ does satisfy the Klein-Gordon wave equation. Moreover, from the integral representation of the Heaviside step distribution

$$\theta\,(z_0) \;=\; \lim_{\varepsilon\,\to\,0^+}\;\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{p_0 - i\,\varepsilon}\;\exp\{\,i\,p_0\,z_0\} \tag{3.116}$$

and from the normal modes expansion of the Hermitean Klein-Gordon field

$$\phi(x) = \sum_{\mathbf{k}}\left[\,a_{\mathbf{k}}\,u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}}\,u^*_{\mathbf{k}}(x)\,\right]\sqrt{\hbar c}$$
$$u_{\mathbf{k}}(x) \equiv [\,(2\pi)^3\,2\omega_{\mathbf{k}}/c\,]^{-1/2}\;\exp\{-\,i\,\omega_{\mathbf{k}}t + i\,\mathbf{k}\cdot\mathbf{x}\}$$

by making use of the canonical commutation relations (3.52) and turning to natural units we can write

$$\begin{aligned}
\langle\,0\,|\,T\,\phi(x)\,\phi(y)\,|\,0\,\rangle \;&=\; \frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}k_0\;\exp\{i(k_0-\omega_{\mathbf{k}})(x_0-y_0)\}\\
&\times\;\sum_{\mathbf{k}}\,(2\pi)^{-3}\,[\,2\omega_{\mathbf{k}}(k_0-i\varepsilon)\,]^{-1}\\
&\times\;\exp\{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\}\\
&+\;\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}k_0\;\exp\{-\,i(k_0-\omega_{\mathbf{k}})(x_0-y_0)\}\\
&\times\;\sum_{\mathbf{k}}\,(2\pi)^{-3}\,[\,2\omega_{\mathbf{k}}(k_0-i\varepsilon)\,]^{-1}\\
&\times\;\exp\{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\}
\end{aligned}$$

Changing the integration variable from k_0 to $k_0' = -\,k_0$ in the first integral of the right hand side of the previous equality we obtain

$$\begin{aligned}
D_F(x-y) \;&=\; -\,\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\,k_0\;\exp\{-\,i\,(k_0+\omega_{\mathbf{k}})\,(x_0-y_0)\}\\
&\times\;\sum_{\mathbf{k}}\,(2\pi)^{-3}\,[\,2\,\omega_{\mathbf{k}}\,(k_0+i\,\varepsilon)\,]^{-1}\\
&\times\;\exp\{\,i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\}\\
&+\;\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\,k_0\;\exp\{-\,i\,(k_0-\omega_{\mathbf{k}})\,(x_0-y_0)\}\\
&\times\;\sum_{\mathbf{k}}\,(2\pi)^{-3}\,[\,2\,\omega_{\mathbf{k}}\,(k_0-i\,\varepsilon)\,]^{-1}\\
&\times\;\exp\{\,i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\}
\end{aligned}$$

and a translation with respect to the k_0 integration variable yields

$$\begin{aligned}
\langle T\,\phi(x)\,\phi(y)\rangle \;&=\; \frac{i}{(2\pi)^4}\int\mathrm{d}\,k\;\exp\{-\,i\,k\cdot(x-y)\}\,[\,2\,\omega_{\mathbf{k}}\,]^{-1}\\
&\times\;\left(\frac{1}{k_0-\omega_{\mathbf{k}}+i\,\varepsilon}\;-\;\frac{1}{k_0+\omega_{\mathbf{k}}-i\,\varepsilon}\right)\\
&=\;\int\frac{\mathrm{d}\,k}{(2\pi)^4}\;\mathrm{e}^{-\,i\,k\cdot(x-y)}\;\frac{i}{k^2-m^2+i\,\varepsilon}
\end{aligned}$$

which proves the Fourier representation (3.115). □

The latter one just involves the $(+i\varepsilon)$ prescription in momentum space that corresponds to causality in coordinate space. As a matter of fact, if we define the creation $\phi^{(+)}(x)$ and destruction $\phi^{(-)}(x)$ parts of the Hermitean Klein-Gordon field operator according to

$$\phi^{(-)}(x) = \sum_{\mathbf{k}} a_{\mathbf{k}}\, u_{\mathbf{k}}(x) \qquad (3.117)$$

$$\phi^{(+)}(x) = \sum_{\mathbf{k}} a^{\dagger}_{\mathbf{k}}\, u^{*}_{\mathbf{k}}(x) \qquad (3.118)$$

it turns out that we have

$$\begin{aligned} \langle\, T\,\phi(x)\,\phi(y)\,\rangle_0 \;&=\; \theta(x_0 - y_0)\,\langle\,\phi^{(-)}(x)\,\phi^{(+)}(y)\,\rangle_0 \\ &+\; \theta(y_0 - x_0)\,\langle\,\phi^{(-)}(y)\,\phi^{(+)}(x)\,\rangle_0 \\ &=\; i\,\theta(y_0 - x_0)\,D^{\,(+)}\,(x-y) - i\,\theta(x_0 - y_0)\,D^{\,(-)}\,(x-y) \end{aligned} \qquad (3.119)$$

which shows that, first, a particle is created out of the vacuum by the creation part of the free real scalar field operator and then, later, it is annihilated by the destruction part of the free real scalar field operator: the opposite never occurs, what precisely endorses the causality requirement in coordinate space.

3.5.1 Euclidean Formulation

The causal Green's function $D_F(x)$ enjoys a nice and quite useful feature: the opportunity to turn to the Euclidean formulation. In the non-relativistic quantum mechanics the substitution

$$t \quad \longleftrightarrow \quad -\,i\hbar\beta = \frac{-i\hbar}{k_B T}$$

where $k_B = 8.617\,343(15) \times 10^{-5}$ eV K^{-1} denotes the Boltzmann constant, allows to make the transition from the evolution operator to the canonical ensemble operator

$$\mathrm{e}^{-\,iHt/\hbar} \quad \longleftrightarrow \quad \mathrm{e}^{-\,\beta H}$$

Hence the Feynman propagating kernel

$$G(t; \mathbf{r}, \mathbf{r}\,') = \langle\, \mathbf{r} | \mathrm{e}^{-\,iHt/\hbar} | \mathbf{r}\,' \,\rangle$$

is turned into the Boltzmann heat kernel

$$G(\beta; \mathbf{r}, \mathbf{r}\,') = \langle\, \mathbf{r} | \mathrm{e}^{-\,\beta H} | \mathbf{r}\,' \,\rangle$$

the latter being positive definite. Notice that the Fourier representation of the Boltzmann heat kernel for a free particle is given by

$$G(\beta;\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi\hbar)^3}\int \mathrm{d}\mathbf{p}\ \exp\left\{\frac{i}{\hbar}\,\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}') - \frac{\beta\,\mathbf{p}^2}{2m}\right\}$$

so that its value at coincident points $\mathbf{r}=\mathbf{r}'$ is nothing but

$$G(\beta;0) = \lambda_T^{-3} \qquad\qquad \lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$$

the latter being the thermal wave-length of the particle. This can be suitably generalized to the quantum field theory. As a matter of fact, consider once again the Fourier representation (3.115) and change the energy integration variable as $k_0 = -\,i\,k_4$ so that

$$D_F\,(x) \;=\; \frac{i\hbar c}{(2\pi)^4}\int_{-\infty}^{\infty}\mathrm{d}\,(-\,i\,k_4)\sum_{\mathbf{k}}\frac{\exp\{-\,k_4x_0 + i\,\mathbf{k}\cdot\mathbf{x}\}}{-k_4^2-\mathbf{k}^2-m^2c^2/\hbar^2} \tag{3.120}$$

where the $+i\varepsilon$ prescription has been dropped since the denominator is now positive definite. If we further set

$$ct \;=\; x_0 \;\equiv\; -\,i\,x_4 \;=\; -\,i\,\beta\hbar c$$

then we finally get

$$\begin{aligned} -\,D_F\,(-\,i\,x_4,\mathbf{x}) \;&=\; \frac{\hbar c}{(2\pi)^4}\int \mathrm{d}\,k_E\ \exp\{i\,k_{E\mu}\,x_{E\mu}\}\ \left(k_E^2 + m^2c^2/\hbar^2\right)^{-1} \\ &\equiv\; D_E\,(x_E\,) \end{aligned} \tag{3.121}$$

where we use the notation

$$\begin{aligned} &x_E = x_{E\mu} = (\mathbf{x},x_4) \qquad k_E = k_{E\mu} = (\mathbf{k},k_4) \qquad (\,j,k=1,2,3\,) \\ &k_E\cdot x_E = k_{E\mu}\,x_{E\mu} = k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 \\ &\int \mathrm{d}\,k_E = \int_{-\infty}^{\infty}\mathrm{d}\,k_4\sum_{\mathbf{k}} \end{aligned}$$

The location of the poles of the Feynman propagator in the complex energy plane, that corresponds to the causal $+i\varepsilon$ prescription, is such that the above substitutions lead to a positive definite denominator $(k_E^2 + m^2c^2/\hbar^2\,)^{-1}$ which does not require any prescription. It is precisely this crucial aspect that encodes the causality requirement in momentum space.

Notice that in the mass-less case at the spatial origin we get

$$\frac{1}{\hbar c}\lim_{m\to 0} D_E(0,\beta\hbar c) = \left(\frac{k_B T}{2\pi\hbar c}\right)^2 \tag{3.122}$$

so that we can define a further relativistic thermal wave-length

$$\ell_T \equiv \frac{hc}{k_B T} \simeq 32.5 \text{ nm} \qquad \text{at} \quad T = 300\ K \tag{3.123}$$

In configuration space we turn to the Euclidean formulation, according to which the Action and the Lagrangian in the Minkowski space are transformed into the purely imaginary Euclidean Action and Lagrangian. As a matter of fact, if we change the time integration variable according to $x_0 = -ix_4$, then we readily obtain turning to natural units

$$\begin{aligned} S[\,\phi\,] \ &\mapsto \ S_E[\,\phi_E\,] \\ &= \ \frac{i}{2}\int \mathrm{d}x_E \,\Big(\partial_\mu\phi_E(x_E)\,\partial_\mu\phi_E(x_E) + m^2\,\phi_E^2(x_E)\Big) \end{aligned} \tag{3.124}$$

with the Euclidean indexes always lower case so that

$$\partial_\mu\phi_E \equiv \frac{\partial\phi_E}{\partial x_{E\mu}} \qquad\qquad (\mu = 1,2,3,4)$$

If we assume the asymptotic behavior for the Euclidean scalar field

$$\lim_{x_E\to\infty} \phi_E(x_E)\sqrt{x_E^2} \ = \ 0 \tag{3.125}$$

then we can also write

$$\begin{aligned} S_E[\,\phi_E\,] \ &= \ \frac{i}{2}\int \mathrm{d}x_E\ \phi_E(x_E)\,(-\,\partial_E^2 + m^2)\,\phi_E(x_E) \\ &= \ \frac{i}{2}\int \mathrm{d}k_E\ \tilde{\phi}_E(k_E)(k_E^2 + m^2)\,\tilde{\phi}_E(-\,k_E) \end{aligned} \tag{3.126}$$

in which I have set by definition

$$\phi_E(x_E) \ \equiv \ \frac{1}{(2\pi)^4}\int \mathrm{d}k_E\ \tilde{\phi}_E(k_E)\ \exp\{i\,k_{E\mu}\,x_{E\mu}\}$$

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3.6 Problems

1. **The complex scalar field.** Consider the field theory of a complex valued scalar field with the classical Lagrange density

$$\mathcal{L}(x) = \partial_\mu \Phi^*(x)\, \partial^\mu \Phi(x) - m^2\, \Phi^*(x)\, \Phi(x)$$

It is easier to analyze the theory by considering $\Phi(x)$ and $\Phi^*(x)$ as the independent variables in configuration space rather than the real and imaginary parts of the complex scalar field function.

(**a**) Find the Hamiltonian and the canonical equations of motion

Solution. The Action is given by

$$S[\,\Phi\,] = \int \mathrm{d}x\ \mathcal{L}[\,\Phi(x)\,] \qquad \Phi(x) = u(x) + i\,v(x)$$

which leads to the conjugated canonical momenta

$$\frac{\delta S}{\delta\,\dot{\Phi}(x)} \equiv \Pi(x) = \dot{\Phi}^*(x) \qquad \frac{\delta S}{\delta\,\dot{\Phi}^*(x)} \equiv \Pi^*(x) = \dot{\Phi}(x)$$

and to the classical Hamiltonian functional

$$\begin{aligned} H[\,\Pi, \Phi\,] &= \int \mathrm{d}\mathbf{x}\ \Big(\Pi(x)\,\dot{\Phi}(x) + \Pi^*(x)\,\dot{\Phi}^*(x) - \mathcal{L}(x)\Big) \\ &= \int \mathrm{d}\mathbf{x}\ \Big(\,|\,\Pi(x)\,|^2 + |\,\boldsymbol{\nabla}\Phi(x)\,|^2 + m^2\,|\,\Phi(x)\,|^2\Big) \end{aligned}$$

The Poisson brackets are evidently given by

$$\{\Phi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\} = \delta\,(\mathbf{x}-\mathbf{y}) = \{\Phi^*(t,\mathbf{x})\,,\,\Pi^*(t,\mathbf{y})\}$$

all the others being equal to zero, so that the Hamilton equations read

$$\begin{cases} \dot{\Phi}(x) = \{\Phi(x)\,,\,H\} = \Pi^*(x) \\ \ddot{\Phi}(x) = \dot{\Pi}^*(x) = \{H\,,\,\Pi^*(x)\} \end{cases}$$

$$\begin{cases} \dot{\Phi}^*(x) = \{\Phi^*(x), H\} = \Pi(x) \\ \ddot{\Phi}^*(x) = \dot{\Pi}(x) = \{H, \Pi(x)\} \end{cases}$$

whence we immediately find the Klein-Gordon wave equations

$$\ddot{\Phi}(x) = \boldsymbol{\nabla}^2\Phi(x) - m^2\,\Phi(x) \qquad (\,\Box + m^2\,)\,\Phi(x) = 0$$

(**b**) Find the diagonal Hamiltonian operator introducing creation and annihilation operators. Show that the charged scalar quantum field contains two types of massive spinless particles of rest mass m.

Solution. The normal modes decomposition of the complex scalar free field can be easily obtained by a straightforward generalization of the treatment for the real scalar free field (3.50). The result is evidently

$$\begin{aligned}
\Phi(x) &= \sum_{\mathbf{k}} \Big[a_{\mathbf{k}}\, u_{\mathbf{k}}(x) + b_{\mathbf{k}}^{\dagger}\, u_{\mathbf{k}}^{*}(x) \Big] \\
\Pi(x) &= \dot{\Phi}^{\dagger}(x) = \sum_{\mathbf{p}} i\omega_{\mathbf{p}} \Big[-\, b_{\mathbf{p}}\, u_{\mathbf{p}}(x) + a_{\mathbf{p}}^{\dagger}\, u_{\mathbf{p}}^{*}(x) \Big] \\
u_{\mathbf{k}}(x) &\equiv [\, 2\omega_{\mathbf{k}}(2\pi)^3\,]^{-1/2}\ \exp\{-\,i\,x^0\omega_{\mathbf{k}} + i\,\mathbf{k}\cdot\mathbf{x}\}
\end{aligned}$$

where $\omega_{\mathbf{k}} = (\,\mathbf{k}^2 + m^2\,)^{1/2}$ together with the canonical commutation relations

$$\begin{aligned}
[\,\Phi(t,\mathbf{x})\,,\,\Phi(t,\mathbf{y})\,] = 0 &= [\,\Pi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\,] \\
[\,\Phi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\,]\ &=\ i\,\delta\,(\mathbf{x}-\mathbf{y})
\end{aligned}$$

It is clear that the main difference with respect to the real case is the appearance of two kinds of creation and destruction operators, as the reality conditions no longer hold true, which satisfy the algebra

$$\begin{aligned}
&[\,a_{\mathbf{k}}\,,a_{\mathbf{p}}\,] = [\,b_{\mathbf{k}}\,,b_{\mathbf{p}}\,] = 0 \\
&[\,a_{\mathbf{k}}\,,b_{\mathbf{p}}\,] = [\,a_{\mathbf{k}}^{\dagger}\,,b_{\mathbf{p}}\,] = 0 \\
&[\,a_{\mathbf{k}}\,,a_{\mathbf{p}}^{\dagger}\,] = [\,b_{\mathbf{k}}\,,b_{\mathbf{p}}^{\dagger}\,] = \delta\,(\mathbf{k}-\mathbf{p})
\end{aligned}$$

Then the normal ordered Hamiltonian and momentum operator takes the diagonal form

$$\begin{aligned}
H\,[\,\Pi,\Phi\,] &= \int \mathrm{d}\mathbf{x}\ :|\,\Pi(x)\,|^2 + |\,\boldsymbol{\nabla}\Phi(x)\,|^2 + m^2\,|\,\Phi(x)\,|^2: \\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \right) = P_0 \\
\mathbf{P} &= -\int \mathrm{d}\mathbf{x}\ \ :\Pi(x)\boldsymbol{\nabla}\Phi(x) + \Pi^{\dagger}(x)\boldsymbol{\nabla}\Phi^{\dagger}(x): \\
&= \sum_{\mathbf{k}} \mathbf{k} \left(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \right)
\end{aligned}$$

It follows therefrom that the complex scalar free field describe two kinds of particles with the very same value m of the rest mass.

(**c**) Rewrite the conserved Noether charge

$$Q = iq \int \mathrm{d}\mathbf{x} \left[\Phi^\dagger(x)\Pi^\dagger(x) - \Phi(x)\Pi(x) \right]$$

in terms of creation and annihilation operators and evaluate the charge of the particles of each type.

Solution. From the invariance of the classical Lagrangian under $U(1)$ phase transformations $\Phi(x) \mapsto \Phi'(x) = \mathrm{e}^{iq\theta}\Phi(x)$ we immediately get the Noether current

$$\begin{aligned} \Delta\Phi(x) = iq\,\Phi(x)\,\delta\theta \quad &\Longrightarrow \quad \mathrm{T} = -iq \\ \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi(x)}\,\mathrm{T}\,\Phi(x) &= -\,q\,\Phi(x)\,i\partial^\mu\Phi^*(x) \\ J^\mu(x) &= q\,\Phi^*(x)\,i\overleftrightarrow{\partial}^\mu\,\Phi(x) \end{aligned}$$

Moreover, from the normal modes expansion and the normal ordering prescription we readily obtain

$$\begin{aligned} Q &= iq \int \mathrm{d}\mathbf{x}\ :\Phi^\dagger(x)\,\Pi^\dagger(x) - \Phi(x)\,\Pi(x): \\ &= q \int \mathrm{d}\mathbf{x} \sum_{\mathbf{k}\,,\,\mathbf{p}} \left[b_{\mathbf{p}}\,u_{\mathbf{p}}(x) + a^\dagger_{\mathbf{p}}\,u^*_{\mathbf{p}}(x) \right] i\overleftrightarrow{\partial}_0 \left[a_{\mathbf{k}}\,u_{\mathbf{k}}(x) + b^\dagger_{\mathbf{k}}\,u^*_{\mathbf{k}}(x) \right] \\ &= q \sum_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} - b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \right) \end{aligned}$$

which is understood so that each particle normal mode carries one unit of positive charge, whereas each antiparticle normal mode carries one unit of negative charge, the sign of the charge being conventional.

2. **Poincaré covariance**

The four energy-momentum operators P_μ ($\mu = 0, 1, 2, 3$) together with the six angular momentum operators $L_{\mu\nu} = -\,L_{\nu\mu}$ are the generators of the Poincaré group, for the infinite dimensional unitary representation acting on the Fock space of an Hermitean scalar quantum field $\phi(x)$.

(**a**) Show that $[\,\phi(x)\,,\,L_{\mu\nu}\,] = i\,(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu)\,\phi(x)$

Solution. We have

$$T^{\,0}_{\ \mu}(x) = \Pi\,(x)\,\partial_\mu\,\phi\,(x) - \delta^0_\mu\,\mathcal{L}\,(x)$$

and the equal–time canonical commutation relations

$$[\,\phi\,(t,\mathbf{x})\,,\,\mathcal{L}\,(t,\mathbf{y})\,] = \Pi\,(t,\mathbf{y})\,[\,\phi\,(t,\mathbf{x})\,,\,\Pi\,(t,\mathbf{y})\,] = i\,\Pi\,(t,\mathbf{x})\,\delta\,(\mathbf{x}-\mathbf{y})$$

$$[\,\phi\,(t,\mathbf{x})\,,\Pi\,(t,\mathbf{y})\,\partial_\nu\,\phi\,(t,\mathbf{y})\,] = i\,\Big(\partial_\nu\,\phi\,(t,\mathbf{y}) + \delta_\nu^0\,\Pi\,(t,\mathbf{y})\Big)\,\delta\,(\mathbf{x}-\mathbf{y})$$

Then we obtain for $x_0 = y_0 = t$

$$\begin{aligned}
[\,\phi\,(x)\,,\,L_{\mu\nu}\,] &= \int \mathrm{d}\mathbf{y}\;y_\mu\,[\,\phi\,(t,\mathbf{x})\,,\,T^{\,0}_{\;\;\nu}\,(t,\mathbf{y})\,] - \mu \leftrightarrow \nu \\
&= i\,x_\mu\,\partial_\nu\,\phi\,(t,\mathbf{x}) - i\,x_\nu\,\partial_\mu\,\phi\,(t,\mathbf{x})
\end{aligned}$$

and thereby

$$\frac{i}{2}\,[\,\phi\,(x)\,,\,L_{\mu\nu}\,]\,\delta\,\omega^{\,\mu\nu} = x_\nu\,\partial_\mu\,\phi\,(x)\,\epsilon^{\,\mu\nu} = \delta\,x^{\,\mu}\,\partial_\mu\,\phi\,(x)$$

It follows that the finite passive Lorentz transformations for the spin-less and charge-less Klein-Gordon quantum field read

$$\phi'(x') = U^{\,\dagger}(\omega)\,\phi(\Lambda\,x)\,U(\omega) = \phi(x)$$

where

$$U(\omega) = \exp\left\{-\,\frac{i}{2}\,\omega^{\,\mu\nu}\,L_{\mu\nu}\right\}$$

(**b**) Show that $[\,L_{\mu\nu}\,,\,P_\rho\,] = -\,i\,g_{\mu\rho}\,P_\nu + i\,g_{\nu\rho}\,P_\mu\,.$

Solution.

Let us first calculate the commutator $[\,L_{\mu\nu}\,,\,P_0\,]$. To this aim, for any analytic functional of the scalar field $\phi\,(x)$ and conjugated momentum $\Pi\,(x)$ operators we have

$$[\,\mathfrak{F}\,(\phi\,(x),\Pi\,(x))\,\,,\,P_\mu\,] = i\,\partial_\mu\,\mathfrak{F}$$

In fact, for example,

$$[\,\phi^2\,(x)\,,\,P_\mu\,] = 2\phi\,(x)\,[\,\phi\,(x)\,,\,P_\mu\,] = 2\phi\,(x)\,i\,\partial_\mu\,\phi\,(x) = i\,\partial_\mu\,\phi^2\,(x)$$

and iterating we obviously get $\forall\,n\in\mathbb{N}$

$$[\,\phi^{\,n}\,(x)\,,\,P_\mu\,] = i\,\partial_\mu\,\phi^{\,n}\,(x)$$

$$[\,\Pi^{\,n}\,(x)\,,\,P_\mu\,] = i\,\partial_\mu\,\Pi^{\,n}\,(x)$$

so that the above statement holds true. Then we find

$$
\begin{aligned}
[\, L_{\mu\nu}\,,\, P_0\,] &= \int \mathrm{d}\mathbf{x}\,\Big(x_\mu\,[\, T^{\,0}_{\ \nu}(x)\,,\, P_0\,] - x_\nu\,[\, T^{\,0}_{\ \mu}(x)\,,\, P_0\,]\Big) \\
&= \int \mathrm{d}\mathbf{x}\,\Big(x_\mu\, i\partial_0\, T^{\,0}_{\ \nu}(x) - x_\nu\, i\partial_0\, T^{\,0}_{\ \mu}(x)\Big) \\
&= \int \mathrm{d}\mathbf{x}\,\Big(-\; x_\mu\, i\partial_\jmath\, T^{\,\jmath}_{\ \nu}(x) + x_\nu\, i\partial_\jmath\, T^{\,\jmath}_{\ \mu}(x)\Big) \\
&\doteq i\int \mathrm{d}\mathbf{x}\,\Big(g_{\mu\jmath}\, T^{\,\jmath}_{\ \nu}(x) - g_{\nu\jmath}\, T^{\,\jmath}_{\ \mu}(x)\Big) \\
&= i\int \mathrm{d}\mathbf{x}\,\Big(T_{\mu\nu}(x) - g_{\mu 0}\, T^{\,0}_{\ \nu}(x) - T_{\nu\mu}(x) - g_{\nu 0}\, T^{\,0}_{\ \mu}(x)\Big) \\
&= -\,i\, g_{\mu 0}\, P_\nu + i\, g_{\nu 0}\, P_\mu
\end{aligned}
$$

in which, as usual, a boundary term has been neglected. Furthermore we find

$$
\begin{aligned}
[\, L_{\mu\nu}\,,\, P_\jmath\,] &= \int \mathrm{d}\mathbf{x}\,\Big(x_\mu\,[\, T^{\,0}_{\ \nu}(x)\,,\, P_\jmath\,] - x_\nu\,[\, T^{\,0}_{\ \mu}(x)\,,\, P_\jmath\,]\Big) \\
&= \int \mathrm{d}\mathbf{x}\,\Big(x_\mu\, i\partial_\jmath\, T^{\,0}_{\ \nu}(x) - x_\nu\, i\partial_\jmath\, T^{\,0}_{\ \mu}(x)\Big) \\
&\doteq -\,i\int \mathrm{d}\mathbf{x}\,\Big(g_{\mu\jmath}\, T^{\,0}_{\ \nu}(x) - g_{\nu\jmath}\, T^{\,0}_{\ \mu}(x)\Big) \\
&= -\,i\, g_{\mu\jmath}\, P_\nu + i\, g_{\nu\jmath}\, P_\mu
\end{aligned}
$$

up to a boundary term, which completes the proof.

(**c**) A fully detailed check of the canonical commutation relations

$$
[\, L_{\mu\nu}\,,\, L_{\rho\sigma}\,] = -\,i\, g_{\mu\rho}\, L_{\nu\sigma} + \,i\, g_{\nu\rho}\, L_{\mu\sigma} + \,i\, g_{\mu\sigma}\, L_{\nu\rho} - \,i\, g_{\nu\sigma}\, L_{\mu\rho}
$$

is straightforward although somewhat tedious and can be found in [21], **7.8**, pp. 144 - 147.

3. **Special distributions in space-time coordinates.**

(**a**) Evaluate the scalar distribution

$$
\frac{i}{\hbar c}\,\langle\, 0\,|\,\phi(x)\,\phi(y)\,|\,0\,\rangle = \frac{i}{\hbar c}\,[\,\phi^{(-)}(x)\,,\phi^{(+)}(y)\,] \equiv D^{(-)}(x-y)
$$

explicitly in terms of Bessel functions.

Solution. Let us consider the positive and negative parts of the Pauli-Jordan distribution (3.112) in the four dimensional Minkowski space

$$
\begin{aligned}
D^{(\pm)}(x) &\equiv \frac{\pm 1}{(2\pi)^3 i}\int \exp\{\pm\, i k\cdot x)\}\,\delta\,(k^2-m^2)\,\theta(k_0)\,\mathrm{d}^4k \\
&= \frac{\pm 1}{(2\pi)^3\, 2i}\int \mathrm{d}\mathbf{k}\;(\mathbf{k}^2+m^2)^{-1/2} \\
&\times\; \exp\{\pm\, i\, x^0(\mathbf{k}^2+m^2)^{1/2}\mp i\,\mathbf{k}\cdot\mathbf{x}\} \\
&\equiv \frac{\pm 1}{4i\pi^2\, r}\int_0^\infty \mathrm{d}k\;\frac{k\,\sin(kr)}{(k^2+m^2)^{1/2}} \\
&\times\; \exp\left\{\pm\, ict\,\sqrt{k^2+m^2}\,\right\} \\
&= \frac{\pm\, i}{4\pi^2\, r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}\int_0^\infty \mathrm{d}k\;\frac{\cos(kr)}{(k^2+m^2)^{1/2}} \\
&\times\; \exp\left\{\pm\, i\, x_0\,(k^2+m^2)^{1/2}\,\right\}
\end{aligned}
$$

where $r\equiv\mid\mathbf{x}\mid,\;k=\mid\mathbf{k}\mid$, whence it is clear that the positive and negative parts of the Pauli-Jordan commutator are complex conjugate quantities

$$
[\,D^{(\pm)}(x)\,]^* = D^{(\mp)}(x)
$$

Then we can write

$$
\begin{aligned}
D^{(+)}(x) &= \frac{i}{8\pi^2\, r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}\int_{-\infty}^\infty \mathrm{d}k\;\frac{\cos(kr)}{(k^2+m^2)^{1/2}} \\
&\times\; \exp\left\{i\, x^0\,(k^2+m^2)^{1/2}\,\right\} \\
&= \frac{i}{8\pi^2\, r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}\int_{-\infty}^\infty \mathrm{d}k\;(k^2+m^2)^{-1/2} \\
&\times\; \exp\left\{ikr+it(k^2+m^2)^{1/2}\right\}
\end{aligned}
$$

Sometimes the distribution $D^{(-)}(x-y)$ is also called the Wightman distribution of positive frequency and denoted by $W_+(x-y)$. Consider now the integral

$$
\begin{aligned}
I(t,r) &= \int_{-\infty}^\infty \mathrm{d}k\;(k^2+m^2)^{-1/2} \\
&\times\; \exp\{ikr+it(k^2+m^2)^{1/2}\} \qquad (t>0)
\end{aligned}
$$

and perform the change of variable $k=m\sinh\eta$, so that $(k^2+m^2)^{1/2}=m\cosh\eta$. Then we obtain

$$
I(x^0,r) = \int_{-\infty}^\infty \mathrm{d}\eta\;\exp\{im\,(x^0\cosh\eta+r\sinh\eta)\}
$$

Here $x^0 > 0$ so that two cases should be distinguished, *i.e.* $0 < x^0 < r$ and $x^0 > r$. By setting $\lambda \equiv (x^0)^2 - \mathbf{x}^2$ it is convenient to carry out respectively the substitutions

$$\begin{cases} x^0 = \sqrt{-\lambda}\,\sinh\xi\,, & r = \sqrt{-\lambda}\,\cosh\xi\,, \quad 0 < x^0 < r \\ x^0 = \sqrt{\lambda}\,\cosh\xi\,, & r = \sqrt{\lambda}\,\sinh\xi\,, \qquad x^0 > r \end{cases}$$

in such a way that we can write

$$\begin{aligned} I(x^0, r) &= \theta(-\lambda)\int_{-\infty}^{\infty} \mathrm{d}\eta\ \exp\{im\sqrt{-\lambda}\,\sinh(\xi+\eta)\} \\ &+ \theta(\lambda)\int_{-\infty}^{\infty} \mathrm{d}\eta\ \exp\{im\sqrt{\lambda}\,\cosh(\xi+\eta)\} \\ &= \theta(-\lambda)\int_{-\infty}^{\infty} \mathrm{d}\eta\ \exp\{im\sqrt{-\lambda}\,\sinh\eta\} \\ &+ \theta(\lambda)\int_{-\infty}^{\infty} \mathrm{d}\eta\ \exp\{im\sqrt{\lambda}\,\cosh\eta\} \qquad (x^0 > 0) \end{aligned}$$

Now we can use the integral representations of the cylindrical Bessel functions of real and imaginary arguments [12] eq.s **8.421**1. p. 965 and **8.432**4. p. 969 that yield for $\lambda = (ct - r)(ct + r)$

$$\begin{aligned} I(t, r) &= \theta(t)\left[2\theta(-\lambda)\,K_0(m\sqrt{-\lambda}) + \theta(\lambda)\,\pi i\,H_0^{(1)}(m\sqrt{\lambda})\right] \\ &+ \theta(-t)\left[2\theta(-\lambda)\,K_0(m\sqrt{-\lambda}) - \theta(\lambda)\,\pi\, i\,H_0^{(2)}(m\sqrt{\lambda})\right] \\ &= 2\theta(-\lambda)\,K_0(m\sqrt{-\lambda}) \\ &+ \pi i\,\theta(\lambda)\left[\mathrm{sgn}(t)\,J_0(m\sqrt{\lambda}) + iN_0(m\sqrt{\lambda})\right] = I(t, \lambda) \end{aligned}$$

and finally

$$D^{(-)}(x) = \frac{i}{8\pi^2 r}\,\frac{\partial}{\partial r}\,I(t, r) = \frac{1}{4\,i\,\pi^2}\,\frac{\partial}{\partial\lambda}\,I(t, \lambda)$$

We note that in the neighborhood of the origin the cylindrical Bessel functions of real and imaginary arguments may be represented in the form

$$\begin{aligned} J_0(z) &= 1 - \left(\frac{z}{2}\right)^2 + O(z^4) \\ N_0(z) &= \frac{2}{\pi}\left[1 - \left(\frac{z}{2}\right)^2\right]\,\ln\frac{z}{2} + \frac{2}{\pi}\,\mathbf{C} + O(z^2) \\ K_0(z) &= -\left[1 + \left(\frac{z}{2}\right)^2\right]\,\ln\frac{z}{2} - \mathbf{C} + O(z^2) \end{aligned}$$

where **C** is the Mascheroni's constant. By replacing the differentiation with respect to x with differentiation with respect to λ and taking into account the discontinuity of the function $I(t,\lambda)$ on the light-cone manifold $\lambda = 0$ we obtain the following expression for the positive and negative frequency parts of the canonical commutator: namely,

$$\begin{aligned} D^{(-)}(x) &= \frac{1}{4\pi}\,\mathrm{sgn}(t)\,\delta(\lambda) + i\,\theta(-\lambda)\,\frac{m}{4\pi^2\sqrt{-\lambda}}\,K_1(m\sqrt{-\lambda}) \\ &+ i\,\theta(\lambda)\,\frac{m}{8\pi\sqrt{\lambda}}\,\Big[\,N_1(m\sqrt{\lambda}) + i\,\mathrm{sgn}(t)\,J_1(m\sqrt{\lambda})\,\Big] \\ D^{(+)}(x) &= \frac{1}{4\pi}\,\mathrm{sgn}(t)\,\delta(\lambda) - i\,\theta(-\lambda)\,\frac{m}{4\pi^2\sqrt{-\lambda}}\,K_1(m\sqrt{-\lambda}) \\ &- i\,\theta(\lambda)\,\frac{m}{8\pi\sqrt{\lambda}}\,\Big[\,N_1(m\sqrt{\lambda}) - i\,\mathrm{sgn}(t)\,J_1(m\sqrt{\lambda})\,\Big] \end{aligned}$$

so that the Pauli-Jordan commutator and the Feynman propagator are respectively expressed by

$$\begin{aligned} D(x) &= D^{(+)}(x) + D^{(-)}(x) \\ &= \frac{1}{2\pi}\,\mathrm{sgn}(t)\,\delta(\lambda) - \frac{m}{4\pi\sqrt{\lambda}}\,\theta(\lambda)\,\mathrm{sgn}(x^0)\,J_1(m\sqrt{\lambda}) \\ D_F(x) &= i\theta(-t)\,D^{(+)}(x) - i\theta(t)\,D^{(-)}(x) \\ &= \frac{1}{4\pi i}\,\delta(\lambda) + \frac{m\,\theta(-\lambda)}{4\pi^2\sqrt{-\lambda}}\,K_1(m\sqrt{-\lambda}) \\ &+ \theta(\lambda)\,\frac{m}{8\pi\sqrt{\lambda}}\,\Big[\,N_1(m\sqrt{\lambda}) + i\,J_1(m\sqrt{\lambda})\,\Big] \end{aligned}$$

in full agreement with the textbook by N.N. Bogoliubov and D.V. Shirkov (1959) *Introduction to the Theory of Quantized Fields*, John Wiley & Sons, New York (U.S.) §15.1, pp. 150-151, eq.s (15.10), (15.11), (15.13) and (15.14).

From the series expansions of the Basset and Bessel functions

$$\begin{aligned} K_1(z) &= \frac{1}{z} + \frac{z}{2}\left[\ln\frac{z}{2} + \frac{1}{2}\,\mathbf{C} - \frac{1}{2}\,\psi(2)\right] + O(z^3\ln z) \\ J_1(z) &= \frac{z}{2}\left[1 + \frac{z^2}{8} + O(z^4)\right] \\ N_1(z) &= -\,\frac{2}{\pi z} + \frac{z}{\pi}\left(\ln\frac{z}{2} + \mathbf{C}\right) + O(z) \end{aligned}$$

we can readily derive the simple and nice expression for the Wightman distributions of a mass-less scalar field *viz.*,

$$iD_0^{(-)}(x) = \frac{i}{4\pi}\,\mathrm{sgn}(t)\,\delta(\lambda) + \frac{1}{4\pi^2\lambda} = \frac{1}{4\pi^2[\,x^2 - i\epsilon\,\mathrm{sgn}(t)\,]}$$

(**b**) Evaluate the scalar causal 2-point Green function of order n in the $D-$dimensional Minkowskian space, which is defined to be

$$G_n^{(D)}(z) = \frac{i}{(2\pi)^D}\int \frac{\exp\{-\,i\,k\cdot z\}}{(k^2-m^2+i\varepsilon)^n}\,\mathrm{d}^D k$$

explicitly in terms of Bessel functions.

Solution. It is very instructive to first compute the integral

$$\begin{aligned} I_n^D(z) &\equiv \frac{i}{(2\pi)^D}\int \frac{\exp\{-\,i\,k\cdot z\}}{(k^2-m^2+i\varepsilon)^n}\,\mathrm{d}^D k \\ &= \frac{i\,(2m)^{1-n}}{(2\pi)^D\,(n-1)!}\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}m^{n-1}}\int \frac{\exp\{-\,i\,k\cdot z\}}{k^2-m^2+i\varepsilon}\,\mathrm{d}^D k\right) \end{aligned}$$

where $z = (z^0, z^1, \ldots, z^{D-1})$ and $k = (k^0, k^1, \ldots, k^{D-1})$ are coordinate and conjugate momentum in a D-dimensional Minkowskian space, so that $k\cdot z = k^0z^0 - k^1z^1 - \cdots - k^{D-1}z^{D-1} = k^0z^0 - \mathbf{k}\cdot\mathbf{z}$, while n is a sufficiently large natural number that will be better specified further on. Turning to a D-dimensional Euclidean space, after setting $z^0 = iz_D$, $k^0 = ik_D$ we immediately obtain

$$I_n^D(z) \equiv \frac{(-1)^n}{(2\pi)^D}\int \frac{\exp\{i\,k_E\cdot z_E\}}{(k_E^2+m^2)^n}\,\mathrm{d}^D k_E$$

with $k_E = (\mathbf{k}, k_D)$, $z_E = (\mathbf{z}, z_D)$. The spherical polar coordinates of k_E are $k, \phi, \theta_1, \theta_2, \ldots, \theta_{D-2}$ and we have

$$\begin{cases} k_1 = k\cos\theta_1 \\ k_2 = k\sin\theta_1\cos\theta_2 \\ k_3 = k\sin\theta_1\sin\theta_2\cos\theta_3 \\ \cdots\cdots\cdots \\ k_{D-1} = k\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-2}\cos\phi \\ k_D = k\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-2}\sin\phi \end{cases}$$

with $0 \le \theta_i \le \pi$ for $i = 1, 2, \ldots, D-2$ and $0 \le \phi \le 2\pi$ while $k = |k_E| = (\mathbf{k}^2 + k_D^2)^{1/2} \ge 0$. It turns out that

$$\frac{\partial\,(k_1, k_2, \cdots, k_D)}{\partial\,(k, \phi, \theta_1, \cdots, \theta_{D-2})} = k^{D-1}(\sin\theta_1)^{D-2}(\sin\theta_2)^{D-3}\cdots(\sin\theta_{D-2})$$

If we choose the Euclidean momentum Ok_1 axis along z_E we evidently obtain $k_E\cdot x_E = kz_E\cos\theta_1 \equiv kz_E\cos\theta$ and thereby we immediately obtain

$$
\begin{aligned}
I_n^D(z) &= (-1)^n\,(2\pi)^{-D}\int_0^\infty \mathrm{d}k\ k^{D-1}\,(k^2+m^2)^{-n} \\
&\times \int_0^\pi \mathrm{d}\theta\ (\sin\theta)^{D-2}\ \exp\{i\,kz_E\cos\theta\} \\
&\times (2\pi)\prod_{j=2}^{D-2}\int_0^\pi \mathrm{d}\theta_j\ (\sin\theta_j)^{D-j-1}
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\int_0^\pi \mathrm{d}\theta_j\ (\sin\theta_j)^{D-j-1} &= 2\int_0^1 \mathrm{d}t_j\ (1-t_j^2)^{(D-j-2)/2} \\
&= \int_0^1 \mathrm{d}y\ y^{-1/2}(1-y)^{(D-j)/2-1} \\
&= B(1/2, D/2-j/2) \\
&= \sqrt{\pi}\ \frac{\Gamma(D/2-j/2)}{\Gamma(D/2-j/2+1/2)}
\end{aligned}
$$

so that

$$
\begin{aligned}
&\prod_{j=2}^{D-2}\int_0^\pi \mathrm{d}\theta_j\ (\sin\theta_j)^{D-j-1} \\
&= \frac{\pi^{(D-3)/2}\,\Gamma(1)\Gamma(3/2)\Gamma(2)\cdots\Gamma(D/2-1)}{\Gamma(3/2)\Gamma(2)\cdots\Gamma(D/2-1)\Gamma(D/2-1/2)} \\
&= \frac{\pi^{(D-3)/2}}{\Gamma(D/2-1/2)}
\end{aligned}
$$

and thereby

$$
\begin{aligned}
I_n^D(z) &= \frac{2\,(-1)^n\,(4\pi)^{-D/2}}{\Gamma(D/2-1/2)\sqrt{\pi}}\int_0^\infty \mathrm{d}k\ k^{D-1}\,(k^2+m^2)^{-n} \\
&\times \int_0^\pi \mathrm{d}\theta\ (\sin\theta)^{D-2}\ \exp\{i\,kz_E\cos\theta\}
\end{aligned}
$$

Next we find

$$
\begin{aligned}
&\int_0^\pi \mathrm{d}\theta\ (\sin\theta)^{D-2}\ \exp\{i\,kz_E\cos\theta\} \\
&= 2\int_0^{\pi/2} \mathrm{d}\theta\ (\sin\theta)^{D-2}\ \cos(kz_E\cos\theta) \\
&= 2\int_0^1 (1-t^2)^{(D-3)/2}\ \cos(tkz_E)\ \mathrm{d}t
\end{aligned}
$$

The value of the latter integral is reported in [12] eq. **3.7717**. p. 464 and turns out to be

$$\sqrt{\pi}\ \left(\frac{2}{kz_E}\right)^{D/2-1}\Gamma\left(\frac{D-1}{2}\right)\,J_{D/2-1}(kz_E)\qquad(\Re\mathrm{e}\,D>1)$$

so that we further obtain

$$\begin{aligned} I_n^D(z) &= (-1)^n\,(4\pi)^{-\,D/2}\,2^{D/2}\,z_E^{-D/2+1} \\ &\times\ \int_0^\infty k^{D/2}\,(k^2+m^2)^{-\,n}J_{D/2-1}(kz_E)\;\mathrm{d}k \end{aligned}$$

and from [12] eq. **6.5654**. p. 710 we come to the expression

$$\begin{aligned} I_n^D(z) &= (-1)^n\,(4\pi)^{-\,D/2}\,2^{\,D/2}\,z_E^{-D/2+1} \\ &\times\ \frac{m^{\,D/2-n}\,z_E^{n-1}}{2^{n-1}\Gamma(n)}\ K_{D/2-n}(mz_E) \\ &= \frac{2\,(-1)^n}{(4\pi)^{\,D/2}\,\Gamma(n)}\left(\frac{2m}{z_E}\right)^{D/2-n}K_{D/2-n}(mz_E) \end{aligned}$$

with $0<\Re\mathrm{e}\,D<4n-1$.

Another much more quick method to get the same result is in terms of the Mellin's transform

$$\begin{aligned} I_n^D(z) &= \frac{(-1)^n}{\Gamma(n)\,(2\pi)^D}\int\mathrm{d}^Dk_E\ \exp\{\,i\,k_{E\mu}\,z_{E\mu}\} \\ &\times\ \int_0^\infty\mathrm{d}t\ t^{\,n-1}\exp\{-t\,k_E^2-t\,m^2\} \\ &= \frac{(-1)^n}{\Gamma(n)}\int_0^\infty\mathrm{d}t\ t^{\,n-1}\ \exp\{-\,t\,m^2\} \\ &\times\ \frac{1}{(2\pi)^D}\int\mathrm{d}^Dk_E\ \exp\left\{-\,t\left(k_E-i\,\frac{z_E}{2t}\right)^2-\frac{z_E^2}{4t}\right\} \\ &= \frac{(-1)^{\,n}}{(4\pi)^{D/2}\,\Gamma(n)}\int_0^\infty\mathrm{d}t\ t^{\,n-1-D/2}\ \exp\{-\,t\,m^2-z_E^2/4t\} \\ &= \frac{2\,(-1)^{\,n}}{(4\pi)^{D/2}\,\Gamma(n)}\left(\frac{2m}{\sqrt{-\,\lambda}}\right)^{D/2-n}K_{\,D/2-n}\left(m\sqrt{-\,\lambda}\right) \end{aligned}$$

where $z_E=(\mathbf{z}^2+z_4^2)^{1/2}=(\mathbf{z}^2-z_0^2)^{1/2}=(-\,\lambda)^{1/2}$, $z^2<0$. In the case $n=1$, $D=4$ we recover the Feynman propagator outside the light-cone

$$D_F(z)=-\,I_1^4(z_E)=\frac{m}{4\pi^2\sqrt{-\lambda}}\,K_1\left(m\sqrt{-\,\lambda}\right)\qquad(z^2<0)$$

and from the series representation of the Basset function of order one

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left[\ln\frac{z}{2} + \frac{1}{2}\mathbf{C} - \frac{1}{2}\psi(2)\right] + O(z^3\ln z)$$

we obtain the leading behavior of the causal Green's function in the four dimensional Minkowski space near the outer surface of the light-cone

$$D_F(z) \approx \frac{-1}{4\pi^2\lambda} + \frac{m^2}{8\pi^2}\ln\left(m\,|\,\lambda\,|^{1/2}\right) \qquad (\lambda = z^2 < 0)$$

On the other side of the light-cone, *i.e.* for $z^2 > 0$, we have to use the integral representation

$$\left(\frac{1}{k^2 - m^2 + i\varepsilon}\right)^n = \frac{(-i)^n}{\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{\,n-1}\;\exp\left\{i\,t\,(k^2 - m^2 + i\varepsilon)\right\}$$

so that

$$\begin{aligned}
I_n^D(z) &= \frac{1}{(2\pi)^D}\int \mathrm{d}^D k\;\exp\{-i\,k_\mu\,z^\mu\} \\
&\times \frac{(-i)^n}{\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{\,n-1}\;\exp\left\{i\,t\,(k^2 - m^2 + i0)\right\} \\
&= \frac{(-i)^n}{\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{\,n-1}\;\exp\left\{-\,i\,m^2\left(t + \frac{z^2}{4m^2\,t}\right)\right\} \\
&\times \frac{1}{(2\pi)^D}\int \mathrm{d}^D k\;\exp\left\{i\,t\left(k - \frac{z}{2t}\right)^2\right\} \\
&= \frac{(-\,i\,)^{\,n}}{(4\pi)^{D/2}\,\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{\,n-D/2-1}\;\exp\left\{-\,i\,m^2\left(t + \frac{\lambda}{4m^2\,t}\right)\right\}
\end{aligned}$$

with $\lambda = z^2 > 0$. Now we have [12] formula **3.471**11. p. 384

$$\int_0^\infty \mathrm{d}\,t\; t^{\,\nu\,-1}\;\exp\left\{\frac{1}{2}\,i\mu\left(t + \frac{\beta^{\,2}}{t}\right)\right\} = \pi\,i\;\exp\left\{-\,\frac{1}{2}\,\pi\,i\nu\right\}\;\beta^{\,\nu}\,H_{-\nu}^{(1)}(\beta\mu)$$

with $\Im\mathrm{m}\,\mu > 0\,,\ \Im\mathrm{m}\,(\beta^{\,2}\mu) \geq 0$. Hence we obtain

$$I_n^D(z) \;=\; \frac{\pi\,(-\,1)^{\,n+1}}{(4\pi)^{D/2}\,\Gamma(n)}\;\exp\left\{\frac{1}{4}\,\pi i D\right\}\left(\frac{2m}{\sqrt{\lambda}}\right)^{D/2-n} H_{D/2-n}^{(2)}(m\sqrt{\lambda})$$

For $n = 1\,,\ D = 4$ we recover the Feynman propagator with a time-like argument

$$D_F(z) = \frac{m}{8\pi\sqrt{\lambda}}\left[\,N_1(m\sqrt{\lambda}) + iJ_1(m\sqrt{\lambda})\right] \qquad (z^2 = \lambda > 0)$$

in agreement with the very last expression of Problem 2. From the series representations of the Bessel functions we obtain the leading behaviors

$$J_1(z) = \frac{z}{2}\left[1 + \frac{z^2}{8} + O(z^4)\right]$$
$$N_1(z) = -\frac{2}{\pi z} + \frac{z}{\pi}\left(\ln\frac{z}{2} + \mathbf{C}\right) + O(z) \tag{3.127}$$

whence

$$D_F(z) \approx -\frac{1}{4\pi^2\lambda} + \frac{m^2}{8\pi^2}\ln(m\,|\,\lambda\,|^{1/2}) \qquad (\lambda = z^2 > 0)$$

Finally, consider the causal Green's function in the four dimensional Minkowski space, that means $n = 1$, $D = 4$. To this concern it is convenient to set

$$z \equiv |\,\mathbf{z}\,| \qquad k = |\,\mathbf{k}\,| \qquad \mathbf{k}\cdot\mathbf{z} = kz\cos\theta$$

and thereby

$$D_F(z^0, z) = \frac{i}{z\,(2\pi)^3}\int_{-\infty}^{\infty}\mathrm{d}k\ k\ \sin(kz)\int_{-\infty}^{\infty}\mathrm{d}k_0\ \frac{\exp\{-\,iz^0k_0\}}{k_0^2 - k^2 - m^2 + i\varepsilon}$$

The last integral has two simple poles in the complex energy plane

$$k_0 = \omega(k) - i\varepsilon \qquad k_0 = -\,\omega(k) + i\varepsilon \qquad \omega(k) = \sqrt{k^2 + m^2}$$

For $z^0 > 0$ we have to close the contour in the lower half-plane of the complex energy, that yields

$$\begin{aligned} D_F(z^0, z) &= \frac{\theta(z^0)}{iz\,(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}k\ k\ \exp\{ikz\}\ \frac{\exp\{-\,i(z^0 - i0)\omega(k)\}}{2\omega(k)} \\ &= \frac{\theta(z^0)}{8iz\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}k\ k}{\sqrt{k^2+m^2}}\ \exp\left\{ikz -\ i(z_0 - i0)\sqrt{k^2+m^2}\,\right\} \end{aligned}$$

Conversely, for $z^0 < 0$ we have to close the contour in the upper half-plane $\Im\mathrm{m}\,(k_0) > 0$ that gives

$$\begin{aligned} D_F(z^0, z) &= \frac{\theta(-z^0)}{8iz\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}k\ k}{\sqrt{k^2+m^2}} \\ &\times\ \exp\left\{-ikz +\ i(z_0 + i0)\sqrt{k^2+m^2}\,\right\} \end{aligned}$$

As a consequence, for $z^0 = 0$ we obtain

$$\begin{aligned} D_F(0, z) &= \frac{1}{8z\pi^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k\ k\ \sin(kz)}{\sqrt{k^2 + m^2}} \\ &= \frac{1}{4z\pi^2} \left(-\frac{\mathrm{d}}{\mathrm{d}z}\right) \int_0^{\infty} \frac{\mathrm{d}k\ \cos(kz)}{\sqrt{k^2 + m^2}} \\ &= \frac{m}{4z\pi^2}\, K_1(mz) \end{aligned}$$

and thanks to the Lorentz invariance

$$D_F(z) = \frac{m}{4\pi^2\sqrt{-z^2}}\ K_1\left(m\sqrt{-z^2}\right) \qquad (z^2 < 0)$$

in accordance with the previously obtained result. Finally, when $m = 0$ we find

$$\begin{aligned} &\lim_{m\to 0}\ D_F(z^0, z) \ = \ \frac{1}{8iz\pi^2}\ \times \\ &\int_{-\infty}^{\infty} \mathrm{d}k\ \left[\,\theta(z^0)\exp\{ik(z - z^0)\} + \theta(-z^0)\exp\{-ik(z - z^0)\}\,\right] \\ &= \ \frac{1}{4\pi i}\,\delta(z_0^2 - z^2) \ = \ \frac{1}{4\pi i}\,\delta(\lambda) \end{aligned}$$

and consequently we eventually come to the singular behavior in the neighborhood of the light-cone

$$D_F(z) \approx \frac{1}{4\pi i}\,\delta(\lambda) - \ \frac{1}{4\pi^2\lambda} + \frac{m^2}{8\pi^2}\ \ln\left(m\,|\,\lambda\,|^{1/2}\right) \qquad (z^2 = \lambda\ \sim\ 0)$$

Chapter 4

The Spinor Field

4.1 The Dirac Equation

We have already obtained the Poincaré invariant and parity-even kinetic term (2.59) for the Dirac wave field as well as the other parity-even local invariant (2.58) quadratic in the Dirac spinor fields. Then, it is easy to set up the most general Lagrange density for the free Dirac field, which satisfies the general requirements listed in Sect. 2.2 : namely,

$$\mathcal{L}_D = \overline{\psi}\,(x)\left(\tfrac{1}{2}\,\gamma^{\,\mu}\,i\,\overleftrightarrow{\partial}_{\mu} - \frac{Mc}{\hbar}\right)\psi\,(x) \tag{4.1}$$

Beside this form of the Dirac Lagrangian, in which the kinetic term contains the left-right derivative operator $\overleftrightarrow{\partial}$, we can also use the equivalent form, up to a tetra-divergence term,

$$\begin{aligned}\bar{\mathcal{L}}_D &= \overline{\psi}\,(x)\,\gamma^{\,\mu}\,i\,\partial_{\mu}\psi\,(x) - (Mc/\hbar)\,\overline{\psi}\,(x)\,\psi\,(x) \\ &\doteq \mathcal{L}_D + \tfrac{1}{2}\,i\,\partial_{\mu}\left(\,\overline{\psi}\,(x)\,\gamma^{\,\mu}\,\psi\,(x)\,\right)\end{aligned} \tag{4.2}$$

Notice that the spinor fields in the four dimensional Minkowski space have canonical dimensions $[\,\psi\,] = \sqrt{\mathrm{eV}}\,\mathrm{cm}^{-1}$ in physical units, that is $[\,\psi\,] = \mathrm{cm}^{-3/2} = \mathrm{eV}^{\,3/2}$ in natural units. The free spinor wave equation can be obtained as the Euler-Lagrange field equation from the above Lagrangian by treating $\psi(x)$ and $\overline{\psi}\,(x)$ as independent fields. This actually corresponds to take independent variations with respect to $\Re\mathrm{e}\,\psi_{\alpha}(x)$ and $\Im\mathrm{m}\,\psi_{\beta}(x)$, in which the spinor component indexes run over the values $\alpha, \beta = 1L, 2L, 1R, 2R$. Then we obtain the celebrated Dirac equation

$$(i\partial\!\!\!/ - M)\,\psi(x) = 0 \tag{4.3}$$

where I have employed the customary notation

$$i\partial\!\!\!/ \equiv \gamma^{\mu}\, i\partial_{\mu}$$

Paul Adrian Maurice Dirac (1928)
The Quantum Theory of the Electron
Proceedings of the Royal Society, Vol. A**117**, p. 610

Taking the Hermitean conjugate of the Dirac equation

$$0 \;=\; i\partial_{\mu}\,\psi^{\dagger}(x)\,\gamma^{\mu\dagger} + M\,\psi^{\dagger}(x) \;=\; i\partial_{\mu}\,\psi^{\dagger}(x)\,\gamma^{0}\,\gamma^{\mu}\,\gamma^{0} + M\,\psi^{\dagger}(x) \tag{4.4}$$

and after multiplication by γ^0 from the right we come to the adjoint Dirac equation

$$i\partial_{\mu}\,\bar{\psi}\,(x)\,\gamma^{\mu} + M\,\bar{\psi}\,(x) \;\equiv\; \bar{\psi}\,(x)\,(\,i\overleftarrow{\partial\!\!\!/} + M\,) \;=\; 0 \tag{4.5}$$

The Dirac equation (4.3) can also be written *à la* Schrödinger in the form

$$i\hbar\,\frac{\partial\,\psi}{\partial t} = H\psi \qquad\qquad H = \alpha^{k}p^{k} + \beta M \equiv \boldsymbol{\alpha}\cdot\mathbf{p}\,c + \beta M c^{2} \tag{4.6}$$

where H denotes the 1-particle Hamiltonian self-adjoint operator with $\alpha^k = \gamma^0\gamma^k\,,\ p^k = -\,i\hbar\partial_k\,.$ Owing to the transformation rule (2.78) it is immediate to verify the Lorentz covariance of the Dirac equation, that means

$$\begin{aligned}
(i\partial\!\!\!/' - M)\,\psi'(x'\,) \;&=\; \big(\,\gamma^{\mu}\,\Lambda_{\mu}^{\ \nu}\,i\partial_{\nu} - M\,\big)\,\Lambda_{\frac12}\,(\omega)\,\psi(x) \\
(i\partial\!\!\!/' - M)\,\psi'(x'\,) \;&=\; \Big(\,\Lambda^{\mu}{}_{\kappa}\,\Lambda_{\frac12}(\omega)\gamma^{\kappa}\Lambda_{\frac12}^{-1}\,\Lambda_{\mu}{}^{\nu}\,i\partial_{\nu} - M\,\Big)\,\Lambda_{\frac12}\,(\omega)\,\psi(x) \\
&=\; \Big(\,\delta^{\nu}_{\kappa}\,\Lambda_{\frac12}(\omega)\gamma^{\kappa}\Lambda_{\frac12}^{-1}\;i\partial_{\nu} - M\,\Big)\,\Lambda_{\frac12}\,(\omega)\,\psi(x) \\
&=\; \Lambda_{\frac12}\,(\omega)\,(i\partial\!\!\!/ - M)\,\psi(x) = 0
\end{aligned} \tag{4.7}$$

4.1.1 Normal Modes Expansion

To solve the Dirac equation, let us first consider the plane wave stationary solutions

$$\psi_p(x) = \Gamma(p)\,\exp\{-\,i\,p\cdot x\} \tag{4.8}$$

where the spinor $\Gamma(p)$ fulfills the algebraic equation

$$(p\!\!\!/ - M)\,\Gamma(p) = 0 \quad\Leftrightarrow\quad (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta M - p_0)\,\Gamma(p) = 0 \tag{4.9}$$

H being the Hermitean matrix (4.6), which admits nontrivial solutions iff

$$\det \parallel \not p - M \parallel = 0 \tag{4.10}$$

This determinant, which is truly independent from the specific representation of the Clifford algebra, can be most easily computed in the so called ordinary or standard or even Dirac representation, that is

$$\gamma_D^0 \equiv \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad \gamma_D^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{4.11}$$

If we choose the momentum rest frame $\mathbf{p} = 0$ we find solutions iff

$$\det \parallel \not p - M \parallel = (p_0 - M)^2 (p_0 + M)^2 = 0 \tag{4.12}$$

and using Lorentz covariance

$$\det \parallel \not p - M \parallel = (p^2 - M^2)^2 = 0$$

which drives to the two pairs of degenerate solutions with frequencies

$$p_\pm^0 \;=\; \pm \left(\mathbf{p}^2 + M^2\right)^{\frac{1}{2}} \;\equiv\; \pm\,\omega_{\mathbf{p}} \tag{4.13}$$

As a consequence, it follows that we have two couples of plane wave stationary solutions (4.8) with **two possible polarization states** with $r = 1, 2$:

$$\psi_{p\,,\,r}(x) = \begin{cases} \Gamma_{-\,,\,r}(p)\,\mathrm{e}^{-\,ipx} \\ \Gamma_{+\,,\,r}(-p)\,\mathrm{e}^{ipx} \end{cases} \qquad [\,p^\mu = (\omega_{\mathbf{p}}, \mathbf{p})\,,\; r = 1, 2\,] \tag{4.14}$$

with

$$(\not p - M)\,\Gamma_{-\,,\,r}(p) = 0 \qquad (\not p + M)\,\Gamma_{+\,,\,r}(-p) = 0 \qquad (\,p_0 = \omega_{\mathbf{p}}\,)$$

Actually, it is a well established convention to set

$$\Gamma_{-\,,\,r}(p) \equiv [\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\, u_r(p) = [\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\, u_r(\mathbf{p})$$

$$\Gamma_{+\,,\,r}(-p) \equiv [\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\, v_r(p) = [\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\, v_r(\mathbf{p})$$

together with

$$\begin{aligned} u_{\mathbf{p}\,,\,r}(x) &= \sqrt{c}\,[\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-1/2}\, u_r(\mathbf{p})\, \exp\{-\,it\omega_{\mathbf{p}} + i\,\mathbf{p}\cdot\mathbf{x})\} \\ v_{\mathbf{p}\,,\,r}(x) &= \sqrt{c}\,[\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-1/2}\, v_r(\mathbf{p})\, \exp\{+\,it\omega_{\mathbf{p}} - i\,\mathbf{p}\cdot\mathbf{x})\} \end{aligned}$$

The so called spin-amplitudes or spin-states do fulfill

$$(\omega_{\mathbf{p}}\gamma^0 - \gamma^k p^k - M)\, u_r(\mathbf{p}) = 0 \qquad (\, r = 1,2\,) \tag{4.15}$$
$$(\omega_{\mathbf{p}}\gamma^0 - \gamma^k p^k + M)\, v_r(\mathbf{p}) = 0 \qquad (\, r = 1,2\,) \tag{4.16}$$

which is nothing but that the degenerate solution of the eigenvalue problem

$$H\, u_r(\mathbf{p}) = \omega_{\mathbf{p}}\, u_r(\mathbf{p}) \qquad H\, v_r(-\mathbf{p}) = -\,\omega_{-\mathbf{p}}\, v_r(-\mathbf{p}) \qquad (\, r = 1,2\,) \tag{4.17}$$

together with the orthogonality and closure relations

$$c\, u_r^{\dagger}(\mathbf{p})\, u_s(\mathbf{p}) = \; 2\omega_{\mathbf{p}}\, \delta_{rs} = \; c\, v_r^{\dagger}(\mathbf{p})\, v_s(\mathbf{p}) \tag{4.18}$$
$$\sum_{r=1,2} \left[\, u_r(\mathbf{p}) \otimes u_r^{\dagger}(\mathbf{p}) + v_r(-\mathbf{p}) \otimes v_r^{\dagger}(-\mathbf{p})\,\right] = 2\omega_{\mathbf{p}}/c \tag{4.19}$$

whence it follows that the spin states have engineering dimensions $[\, u_r\,] = [\, v_r\,] = \mathrm{cm}^{-1/2}$ in physical units. Instead of using the tensor product notation, the closure relation can also be written by means of the spinor indexes $\alpha, \beta = 1L, 2L, 1R, 2R$: namely,

$$\sum_{r=1,2} \left[\, u_{r\,,\,\alpha}(\mathbf{p})\, u^{*}_{r\,,\,\beta}(\mathbf{p}) + v_{r\,,\,\alpha}(-\mathbf{p})\, v^{*}_{r\,,\,\beta}(-\mathbf{p})\,\right] = 2\omega_{\mathbf{p}}\, \delta_{\alpha\beta} \tag{4.20}$$

As an internal consistency check, notice *en passant* that by setting $\beta = \alpha$ and summing over the spinor index $\alpha = 1L, 2L, 1R, 2R$ we get

$$\begin{aligned}
& \sum_{\alpha} \sum_{r=1,2} \left[\, u_{r\,,\,\alpha}(\mathbf{p})\, u^{*}_{r\,,\,\alpha}(\mathbf{p}) + v_{r\,,\,\alpha}(-\mathbf{p})\, v^{*}_{r\,,\,\alpha}(-\mathbf{p})\,\right] \\
= \; & \sum_{r=1,2} \sum_{\alpha} u^{*}_{r\,,\,\alpha}(\mathbf{p})\, u_{r\,,\,\alpha}(\mathbf{p}) + \sum_{r=1,2} \sum_{\alpha} v^{*}_{r\,,\,\alpha}(-\mathbf{p})\, v_{r\,,\,\alpha}(-\mathbf{p}) \\
= \; & 2 \cdot 2\omega_{\mathbf{p}} + 2 \cdot 2\omega_{-\mathbf{p}} = 2\omega_{\mathbf{p}} \cdot 4
\end{aligned}$$

where use has been made of the orthonormality relations (4.18). As a further remark, notice that the very same relations (4.18) are such to recover

$$\int \mathrm{d}\mathbf{x}\; u^{\dagger}_{\mathbf{q}\,,\,s}(x)\, u_{\mathbf{p}\,,\,r}(x) = \delta_{rs}\, \delta\,(\mathbf{p} - \mathbf{q}) = \int \mathrm{d}\mathbf{x}\; v^{\dagger}_{\mathbf{q}\,,\,s}(x)\, v_{\mathbf{p}\,,\,r}(x) \tag{4.21}$$

while the closure relations (4.19) and (4.20) indeed ensure the completeness relation

$$\sum_{\mathbf{p}\,,\,r} \Big(u_{\mathbf{p}\,,\,r}(x) \otimes u^{\dagger}_{\mathbf{p}\,,\,r}(y) + v_{\mathbf{p}\,,\,r}(x) \otimes v^{\dagger}_{\mathbf{p}\,,\,r}(y) \Big)_{x_0 \,=\, y_0} \; = \; \delta\,(\mathbf{x} - \mathbf{y}) \tag{4.22}$$

in which the notation has been introduced for brevity

$$\sum_{\mathbf{p}\,,\,r} \stackrel{\mathrm{def}}{=} \sum_{r=1,2} \int \mathrm{d}\,\mathbf{p}$$

`Proof.` From the explicit form of the spinor plane wave functions we obtain

$$\begin{aligned}
& \sum_{r=1,2} \int \mathrm{d}\,\mathbf{p}\;[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1}\,u_{\,r\,,\,\alpha}\,(\mathbf{p})\,u^{\top\,*}_{\,r\,,\,\beta}\,(\mathbf{p})\;\exp\{i\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\} \\
+ & \sum_{r=1,2} \int \mathrm{d}\,\mathbf{p}\;[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1}\,v_{\,r\,,\,\alpha}\,(\mathbf{p})\,v^{\top\,*}_{\,r\,,\,\beta}\,(\mathbf{p})\;\exp\{-\,i\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\} \\
= & \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3\,2\omega_{\mathbf{p}}}\;\exp\{\,i\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\} \\
\times & \sum_{r=1,2} \left[\,u_{\,r\,,\,\alpha}\,(\mathbf{p})\,u^{\top\,*}_{\,r\,,\,\beta}\,(\mathbf{p})\;+v_{\,r\,,\,\alpha}\,(-\mathbf{p})\,v^{\top\,*}_{\,r\,,\,\beta}\,(-\mathbf{p})\;\right] \\
= & \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\;\exp\{i\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\}\;\delta_{\alpha\beta} = \delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y})
\end{aligned}$$

□

It is worthwhile to notice that, from the covariance (4.7) of the Dirac equation, the transformation property of the spin-states readily follows. For instance, from eq. (2.78) we find

$$\begin{aligned}
(\,\not{p}^{\,\prime} - M\,)\,u_r^{\prime}\,(p^{\,\prime}\,) &= \Big(\Lambda_{\mu}^{\;\;\nu}\,p_\nu\,\gamma^{\,\mu} - M\Big)\Lambda_{\frac12}\,u_r\,(p) \\
&= \Big(\Lambda_{\mu}^{\;\;\nu}\,p_\nu\,\Lambda^{\mu}_{\;\;\rho}\,\Lambda_{\frac12}\,\gamma^{\,\rho}\,\Lambda_{\frac12}^{-1} - M\Big)\,\Lambda_{\frac12}\,u_r\,(p) \\
&= \Lambda_{\frac12}\,(\not{p} - M)\,u_r\,(p) \;=\; 0
\end{aligned} \tag{4.23}$$

Hence, the Lorentz covariance of the Dirac equation actually occurs, provided the following relationships hold true for the spin-states

$$u_r^{\prime}\,(p^{\,\prime}\,) = \Lambda_{\frac12}\,u_r\,(p) \;\Leftrightarrow\; u_r\,(p) = \Lambda_{\frac12}^{-1}\,u_r^{\prime}\,(\Lambda\,p) \tag{4.24}$$

$$v_r^{\prime}\,(p^{\,\prime}\,) = \Lambda_{\frac12}\,v_r\,(p) \;\Leftrightarrow\; v_r\,(p) = \Lambda_{\frac12}^{-1}\,v_r^{\prime}\,(\Lambda\,p) \tag{4.25}$$

It is not difficult to prove the further equality: namely,

$$\bar{u}_r(\mathbf{p})\,u_s(\mathbf{p}) = \;(2Mc/\hbar)\,\delta_{rs} \;=-\,\bar{v}_r(\mathbf{p})\,v_s(\mathbf{p}) \tag{4.26}$$

$$\begin{aligned}
& u_r^{\dagger}(\mathbf{p})\,v_s(-\mathbf{p}) = 0 \qquad \bar{u}_r(\mathbf{p})\,v_s(\mathbf{p}) = 0 \\
& v_r^{\dagger}(\mathbf{p})\,u_s(-\mathbf{p}) = 0 \qquad \bar{v}_r(\mathbf{p})\,u_s(\mathbf{p}) = 0 \qquad (r,s=1,2)
\end{aligned} \tag{4.27}$$

`Proof.` From eq. (4.15) we obviously get

$$u_s^\dagger(\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0 - \gamma^k p^k - M)\,u_r(\mathbf{p}) = 0 \qquad (\forall\, r, s = 1, 2)$$

and taking the adjoint equation

$$u_s^\dagger(\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0 + \gamma^k p^k - M)\,u_r(\mathbf{p}) = 0 \qquad (\forall\, r, s = 1, 2)$$

so that adding together

$$\bar{u}_s(\mathbf{p})\,u_r(\mathbf{p}) = \frac{M}{\omega_{\mathbf{p}}}\,u_s^\dagger(\mathbf{p})\,u_r(\mathbf{p}) = 2M\,\delta_{rs}$$

In a quite analogous way one can readily prove that

$$\bar{v}_s(\mathbf{p})\,v_r(\mathbf{p}) \;=\; -\,\frac{M}{\omega_{\mathbf{p}}}\,v_s^\dagger(\mathbf{p})\,v_r(\mathbf{p}) = -\,2M\,\delta_{rs}$$

Moreover, from eq.s (4.15) and (4.16) we obtain

$$v_s^\dagger(-\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0 - \gamma^k p^k - M)\,u_r(\mathbf{p}) = 0 \qquad (\forall\, r, s = 1, 2)$$
$$u_s^\dagger(\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0 + \gamma^k p^k + M)\,v_r(-\mathbf{p}) = 0 \qquad (\forall\, r, s = 1, 2)$$

and taking the Hermitean conjugated of the very last equality

$$v_s^\dagger(-\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0 - \gamma^k p^k + M)\,u_r(\mathbf{p}) = 0 \qquad (\forall\, r, s = 1, 2)$$

so that, by subtracting up the first equality, we get

$$v_s^\dagger(-\mathbf{p})\,u_r(\mathbf{p}) = 0 \qquad (\forall\, r, s = 1, 2)$$

and analogously

$$u_s^\dagger(\mathbf{p})\,v_r(-\mathbf{p}) = 0 \qquad (\forall\, r, s = 1, 2)$$

which completes the proof. □

Notice that from the above orthogonality properties of the spin-states the following orthogonality relations hold true between the positive and negative frequency of the spinor wave functions

$$\int \mathrm{d}\mathbf{x}\; u^\dagger_{\mathbf{q}\,,\,s}(x)\,v_{\mathbf{p}\,,\,r}(x) = 0 = \int \mathrm{d}\mathbf{x}\; v^\dagger_{\mathbf{q}\,,\,s}(x)\,u_{\mathbf{p}\,,\,r}(x) \tag{4.28}$$

It follows therefrom that the most general solution of the Dirac equation can be written in the form

$$\psi(x) \;=\; \sum_{\mathbf{p}\,,\,r}\,\Big[\,c_{\mathbf{p}\,,\,r}\,u_{\mathbf{p}\,,\,r}(x) + d^*_{\mathbf{p}\,,\,r}\,v_{\mathbf{p}\,,\,r}(x)\Big] \tag{4.29}$$

$$\bar{\psi}(x) \;=\; \sum_{\mathbf{p}\,,\,r}\,\Big[\,c^*_{\mathbf{p}\,,\,r}\,\bar{u}_{\mathbf{p}\,,\,r}(x) + d_{\mathbf{p}\,,\,r}\,\bar{v}_{\mathbf{p}\,,\,r}(x)\Big] \tag{4.30}$$

which is nothing but the normal modes expansion of the free Dirac spinor classical wave field, where $c_{\mathbf{p}\,,\,r}$ and $d_{\mathbf{p}\,,\,r}$ are arbitrary complex coefficients.

4.1.2 Spin States

In the chiral representation (2.64) for the gamma matrices we can build up a very convenient set of spin-states as follows. Consider the matrices

$$\not{p} \pm M = \begin{pmatrix} \pm M & 0 & \omega_{\mathbf{p}} - p_z & -p_x + ip_y \\ 0 & \pm M & -p_x - ip_y & \omega_{\mathbf{p}} + p_z \\ \omega_{\mathbf{p}} + p_z & p_x - ip_y & \pm M & 0 \\ p_x + ip_y & \omega_{\mathbf{p}} - p_z & 0 & \pm M \end{pmatrix} \tag{4.31}$$

where we have set $\mathbf{p} = (p^1, p^2, p^2) = (p_x, p_y, p_z)$. Notice that we can define the two projectors on the 2-dimensional spaces spanned by the positive energy and negative energy spin-states respectively: namely,

$$\mathcal{E}_\pm(p) \;\equiv\; (M \pm \not{p})\,/\,2M \qquad\qquad (\,p_0 = \omega_{\mathbf{p}}\,) \tag{4.32}$$

which satisfy by definition

$$\begin{aligned} \mathcal{E}_\pm^2 = \mathcal{E}_\pm \qquad & \mathcal{E}_+\,\mathcal{E}_- = 0 = \mathcal{E}_-\,\mathcal{E}_+ \\ \operatorname{tr}\mathcal{E}_\pm = 2 \qquad & \mathcal{E}_+ + \mathcal{E}_- = \mathbf{I} \end{aligned}$$

Moreover we have

$$\mathcal{E}_\pm^\dagger(p) \;=\; \mathcal{E}_\pm(\tilde{p}) \qquad\qquad \tilde{p}^{\,\mu} = p_\mu \tag{4.33}$$

It is worthwhile to realize that, owing to the basic equations (4.15) and (4.16), we readily get

$$\mathcal{E}_+\, u_r\,(\mathbf{p}) = \frac{2M + \not{p} - M}{2M}\; u_r\,(\mathbf{p}) = u_r\,(\mathbf{p}) \qquad\qquad (\,r = 1, 2)$$

which tell us that $u_r\,(\mathbf{p})$, $r = 1, 2$, are the pair of orthogonal eigenstates with unit eigenvalue of the projector onto the positive energy space of states with given momentum $\mathbf{p}$. A quite analogous relationship holds true, of course, for the negative energy space of states, *i.e.*

$$\mathcal{E}_-\, v_r\,(\mathbf{p}) = \frac{2M - \not{p} - M}{2M}\; v_r\,(\mathbf{p}) = v_r\,(\mathbf{p}) \qquad\qquad (\,r = 1, 2)$$

Now, if we introduce the constant bispinors

$$\xi_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xi_2 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \eta_1 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad \eta_2 \equiv \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

which are the common eigenvectors of the γ^0 matrix

$$\gamma^0\,\xi_r = \xi_r \qquad \gamma^0\,\eta_r = -\,\eta_r \qquad (\,r = 1, 2\,)$$

and of the spin matrix $\frac{1}{2}\,\Sigma_3 = \frac{i}{4}\,[\,\gamma^1\,,\,\gamma^2\,]$

$$(\Sigma_3 - 1)\,\xi_1 = (\Sigma_3 - 1)\,\eta_2 = 0 \qquad\qquad (\Sigma_3 + 1)\,\xi_2 = (\Sigma_3 + 1)\,\eta_1 = 0$$

and do indeed satisfy by direct inspection

$$\xi_r^\top\,\gamma^k\,\xi_s = 0 = \eta_r^\top\,\gamma^k\,\eta_s \qquad\quad \forall\, r, s = 1, 2\ \vee\ k = 1, 2, 3$$

then we can suitably define

$$\begin{cases} u_r\,(\mathbf{p}) \equiv 2M\,(2\omega_{\mathbf{p}} + 2M)^{-1/2}\ \mathcal{E}_+\,\xi_r \\ v_r\,(\mathbf{p}) \equiv 2M\,(2\omega_{\mathbf{p}} + 2M)^{-1/2}\ \mathcal{E}_-\,\eta_r \end{cases} \qquad\qquad (\,r = 1, 2\,) \qquad\qquad (4.34)$$

the explicit form of which is given by

$$u_1(\mathbf{p}) = (2\omega_{\mathbf{p}} + 2M)^{-1/2} \begin{pmatrix} M + \omega_{\mathbf{p}} - p_z \\ -p_x - ip_y \\ M + \omega_{\mathbf{p}} + p_z \\ p_x + ip_y \end{pmatrix}$$

$$u_2(\mathbf{p}) = (2\omega_{\mathbf{p}} + 2M)^{-1/2} \begin{pmatrix} -p_x + ip_y \\ M + \omega_{\mathbf{p}} + p_z \\ p_x - ip_y \\ M + \omega_{\mathbf{p}} - p_z \end{pmatrix}$$

$$v_1(\mathbf{p}) = (2\omega_{\mathbf{p}} + 2M)^{-1/2} \begin{pmatrix} -p_x + ip_y \\ M + \omega_{\mathbf{p}} + p_z \\ -p_x + ip_y \\ -M + p_z - \omega_{\mathbf{p}} \end{pmatrix}$$

$$v_2(\mathbf{p}) = (2\omega_{\mathbf{p}} + 2M)^{-1/2} \begin{pmatrix} -M - \omega_{\mathbf{p}} + p_z \\ p_x + ip_y \\ \omega_{\mathbf{p}} + M + p_z \\ p_x + ip_y \end{pmatrix} \qquad\qquad (4.35)$$

their orthonormality and completeness relations being in full accordance with formulæ (4.18), (4.19) and (4.27). In fact we have for instance

$$\begin{aligned} v_r^\dagger(\mathbf{p})\,v_s(\mathbf{p}) &= (2\omega_{\mathbf{p}} + 2M)^{-1}\ \eta_r^\top (M - \tilde{\not p})(M - \not p)\eta_s \\ &= (2\omega_{\mathbf{p}} + 2M)^{-1}\ \eta_r^\top \left(M^2 - 2M\gamma^0\omega_{\mathbf{p}} + \omega_{\mathbf{p}}^2 + \mathbf{p}^2\right)\eta_s \\ &= (2\omega_{\mathbf{p}} + 2M)^{-1}\ \eta_r^\top \left(2\omega_{\mathbf{p}}^2 + 2M\omega_{\mathbf{p}}\right)\eta_s \\ &= 2\omega_{\mathbf{p}}\ \tfrac{1}{2}\ \eta_r^\top\eta_s = 2\omega_{\mathbf{p}}\ \delta_{rs} \end{aligned}$$

in which I have made use of the property

$$\eta_r^\top \gamma^k \gamma^0 \eta_s = -\,\eta_r^\top \gamma^k \eta_s = 0 \qquad\qquad \forall\, r, s = 1, 2 \,\vee\, k = 1, 2, 3$$

Finally, taking the normalization (4.26) into account together with

$$\mathcal{E}_+ \; u_r\,(\mathbf{p}) = u_r\,(\mathbf{p}) \qquad \mathcal{E}_- \; v_r\,(\mathbf{p}) = v_r\,(\mathbf{p}) \qquad (\,r = 1, 2\,)$$

it is immediate to obtain the so called sums over the spin-states, that is

$$\frac{1}{2M} \sum_{r=1,\,2} \left\{ \begin{array}{c} u_r\,(\mathbf{p}) \otimes \bar{u}_r\,(\mathbf{p}) = \mathcal{E}_+(p) \\ v_r\,(\mathbf{p}) \otimes \bar{v}_r\,(\mathbf{p}) = \,-\,\mathcal{E}_-(p) \end{array} \right. \tag{4.36}$$

or even for $\alpha, \beta = 1L, 2L, 1R, 2R\,$,

$$\sum_{r=1,\,2} \left\{ \begin{array}{l} u_{r\,,\,\alpha}\,(\mathbf{p})\,\bar{u}_{r\,,\,\beta}\,(\mathbf{p}) = (\,\not{p} + M\,)_{\alpha\beta} \\ v_{r\,,\,\alpha}\,(\mathbf{p})\,\bar{v}_{r\,,\,\beta}\,(\mathbf{p}) = (\,\not{p} - M\,)_{\alpha\beta} \end{array} \right. \qquad (\,p_0 = \omega_{\mathbf{p}}\,) \tag{4.37}$$

Hence, from the orthonormality relations (4.26) we can immediately verify that we have

$$\sum_{r=1,\,2} u_r\,(\mathbf{p}) \otimes \bar{u}_r\,(\mathbf{p})\, u_s\,(\mathbf{p}) = 2M\, u_s\,(\mathbf{p}) = 2M\mathcal{E}_+\, u_s\,(\mathbf{p})$$

$$\sum_{r=1,\,2} v_r\,(\mathbf{p}) \otimes \bar{v}_r\,(\mathbf{p})\, v_s\,(\mathbf{p}) = -\,2M\, v_s\,(\mathbf{p}) = -\,2M\mathcal{E}_-\, v_s\,(\mathbf{p})$$

4.2 Nöther Currents

From Nöther theorem and from the Lagrange density (4.1) we obtain the canonical energy-momentum tensor of the free Dirac spinor wave field which turns out to be real though not symmetric

$$\begin{aligned} T^{\mu}{}_{\nu}(x) &\equiv (\delta\,\mathcal{L}_D\,/\,\delta\,\partial_\mu\,\psi)\,\partial_\nu\,\psi + \partial_\nu\,\bar\psi\,(\delta\,\mathcal{L}_D\,/\,\delta\,\partial_\mu\,\bar\psi) - \delta^{\mu}_{\nu}\,\mathcal{L}_D \\ &= \frac{i}{2}\Big(\bar\psi(x)\,\gamma^{\mu}\,\partial_\nu\,\psi(x) - \partial_\nu\,\bar\psi(x)\,\gamma^{\mu}\,\psi(x)\Big) \end{aligned} \tag{4.38}$$

$$T_{\mu\nu}(x) \neq T_{\nu\mu}(x)$$

where we have taken into account that the Dirac Lagrangian vanishes if the equations of motion hold true as it occurs in the Nöther theorem. The corresponding canonical total angular momentum density tensor for the Dirac field can be obtained from the general expression (2.98) and reads

$$\begin{aligned} M^{\lambda\mu\nu}(x) &\stackrel{\text{def}}{=} x^{\mu}\,T^{\lambda\nu}(x) - x^{\nu}\,T^{\lambda\mu}(x) + S^{\lambda\mu\nu}(x) \\ &= x^{\mu}\,T^{\lambda\nu}(x) - x^{\nu}\,T^{\lambda\mu}(x) + \tfrac{1}{2}\,\bar\psi(x)\,\{\sigma^{\mu\nu}\,,\,\gamma^{\lambda}\}\,\psi(x) \end{aligned}$$

As a matter of fact we have

$$\begin{aligned} \delta\,\mathcal{L}\,/\,\delta\,\partial_\mu\,\psi(x) &= \tfrac{1}{2}\,\bar\psi(x)\,i\,\gamma^{\mu} & \qquad \delta\,\mathcal{L}\,/\,\delta\,\partial_\mu\,\bar\psi(x) &= -\,\tfrac{1}{2}\,i\,\gamma^{\mu}\,\psi(x) \\ (-\,i)(\,S^{\lambda\kappa})\,\psi(x) &= -\,i\,\sigma^{\lambda\kappa}\,\psi(x) & \qquad i\,\bar\psi(x)\,(\,S^{\lambda\kappa}\,) &= i\,\bar\psi(x)\,\sigma^{\lambda\kappa} \end{aligned}$$

where

$$\sigma^{\lambda\kappa} \equiv \frac{i}{4}\,\left[\,\gamma^{\lambda}\,,\,\gamma^{\kappa}\,\right] \qquad\qquad \left(\,\sigma^{\lambda\kappa}\,\right)^{\dagger} = \gamma^{0}\,\sigma^{\lambda\kappa}\,\gamma^{0}$$

is the spin tensor for the Dirac field. Hence from the general expression (2.98) we get

$$\begin{aligned} S^{\mu\lambda\kappa}(x) &\stackrel{\text{def}}{=} \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\,(-\,i)\,\left(S^{\lambda\kappa}\right)_{AB}\,u_B(x) \\ &= \delta\mathcal{L}/\delta\,\partial_\mu\,\psi(x)\,\,(-\,i)\,\left(S^{\lambda\kappa}\right)\,\psi(x) \\ &+ \; i\,\bar\psi(x)\,\left(S^{\lambda\kappa}\right)\,\delta\mathcal{L}/\delta\,\partial_\mu\,\bar\psi(x) \\ &= \tfrac{1}{2}\,\bar\psi(x)\,\{\,\gamma^{\mu}\,,\,\sigma^{\lambda\kappa}\,\}\,\psi(x) \end{aligned}$$

Notice that

$$M^{0jk}(x) = x^{j}\,T^{0k}(x) - x^{k}\,T^{0j}(x) + \psi^{\dagger}(x)\,\sigma^{jk}\,\psi(x) \tag{4.39}$$

$$M^{0k0}(x) = x^{k}\,T^{00}(x) - x^{0}\,T^{0k}(x) \tag{4.40}$$

It is rather easy to check, using the Dirac equation, that the continuity equations actually hold true

$$\partial_\mu T^{\mu\nu} = 0 \tag{4.41}$$

$$\partial_\mu M^{\mu\lambda\kappa} = 0 \quad \Longleftrightarrow \quad T^{\lambda\kappa} - T^{\kappa\lambda} = \partial_\mu S^{\mu\kappa\lambda} \tag{4.42}$$

which lead to the four conserved energy-momentum Noether charges

$$\begin{aligned} P_\mu &= \int \mathrm{d}\mathbf{x}\, T^0{}_\mu(x^0,\mathbf{x}) = \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\ \psi^\dagger(x)\, i\overleftrightarrow{\partial}_\mu\, \psi(x) \\ &= \int \mathrm{d}\mathbf{x}\ \psi^\dagger(x)\, i\,\partial_\mu\, \psi(x) \end{aligned} \tag{4.43}$$

while from the spatial integration of eq. (4.39) we obtain the three conserved Nöther charges corresponding to the spatial components of the relativistic total angular momentum

$$\begin{aligned} M_{jk} &= \int \mathrm{d}\mathbf{x}\, M^0{}_{jk}(t,\mathbf{x}) \doteq \\ &\int \mathrm{d}\mathbf{x}\left[\, x_j\, \psi^\dagger(x)\, i\,\partial_k\, \psi(x) - \{\, j \leftrightarrow k\,\} + \psi^\dagger(x)\, \sigma_{jk}\, \psi(x)\,\right] \end{aligned} \tag{4.44}$$

in which we have discarded, as customary, the boundary term

$$\frac{i}{2}\int \mathrm{d}\mathbf{x}\, \partial_k\Big(\, x_j\, \psi^\dagger(t,\mathbf{x})\, \psi(t,\mathbf{x})\Big) - \{\, j \leftrightarrow k\,\} \;=\; 0$$

Furthermore, from the spatial integration of eq. (4.40) we find

$$M^{k0} \;=\; \int \mathrm{d}\mathbf{x}\, M^{0k0}(t,\mathbf{x}) = \int \mathrm{d}\mathbf{x}\, x^k\, T^{00}(t,\mathbf{x}) - x^0\, P^k \tag{4.45}$$

in such a way that the constancy in time of the latter space-time components of the relativistic total angular momentum leads to the definition of the velocity for the center of the energy, *viz.,*

$$X_t^k \;\stackrel{\mathrm{def}}{=}\; \int \mathrm{d}\mathbf{x}\ \frac{x^k}{cP_0}\, T^{00}(t,\mathbf{x})$$

which corresponds to the relativistic generalization of the center of mass and travels at a constant speed

$$\dot{M}^{0k} = 0 \;\Leftrightarrow\; \dot{\mathbf{X}}_t = c\,\frac{\mathbf{P}}{P_0} \tag{4.46}$$

It follows that the so called center of momentum frame $\mathbf{P} = 0$ just coincides with the inertial reference frame in which the center of the energy is at rest.

It is very important to realize that owing to the lack of symmetry for the canonical energy-momentum tensor of the Dirac field [1] its spin angular momentum tensor is not constant in time. Actually we find, for instance,

$$\partial_\mu \, S^{\,\mu jk}(x) = T^{\,kj} - T^{\,jk} \neq 0 \tag{4.47}$$

and consequently the corresponding Noether charges are not conserved in time so that

$$S_{ij}(t) = \int \mathrm{d}\mathbf{x} \; S^{\,0}{}_{ij}\,(t,\mathbf{x}) = \tfrac{1}{2}\,\varepsilon_{\,ijk} \int \mathrm{d}\mathbf{x} \; \psi^{\,\dagger}\,(t,\mathbf{x})\,\Sigma_{\,k}\,\psi\,(t,\mathbf{x}) \tag{4.48}$$

where

$$\Sigma_{\,k} \;\equiv\; \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{4.49}$$

However, if the spinor wave field ψ does not depend upon some of the spatial coordinates $(x^1, x^2, x^3) = (x, y, z)$ it is possible to achieve the continuity equation for some of the components of the spin angular momentum density tensor so that the corresponding Nöther charges – *i.e.* its spatial integrals – keep constant in time. For example, if $\psi(t,z) = \psi\,(t,0,0,z)$ then we get

$$2i\,T^{\,12} = \bar{\psi}\,(t,z)\,\gamma^1\,\overset{\leftrightarrow}{\partial}_y\,\psi\,(t,z) = 0 = \bar{\psi}\,(t,z)\,\gamma^2\,\overset{\leftrightarrow}{\partial}_x\,\psi\,(t,z) = 2i\,T^{\,21}$$

and consequently

$$\partial_\mu\,M^{\,\mu}_{\,12} = \partial_\mu\,S^{\,\mu}_{\,12} = 0 \tag{4.50}$$

Hence it follows that the component of the spin vector along the direction of propagation, which is named the helicity, is conserved in time: namely,

$$\frac{\mathrm{dh}}{\mathrm{d}t} \;=\; 0 \qquad\qquad \mathrm{h} \;\overset{\mathrm{def}}{=}\; \int_{-\infty}^{\infty} \mathrm{d}z\;\tfrac{1}{2}\,\psi^{\,\dagger}\,(t,z)\,\Sigma_3\,\psi\,(t,z) \tag{4.51}$$

After insertion of the normal modes expansions (4.30) one gets

$$\mathrm{h} = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p\,\left[\,c^*_{\,p\,,1}\,c_{\,p\,,1} - c^*_{\,p\,,2}\,c_{\,p\,,2} - d_{\,p\,,1}\,d^*_{\,p\,,1} + d_{\,p\,,2}\,d^*_{\,p\,,2}\right] \tag{4.52}$$

[1] The lack of symmetry in the canonical energy-momentum tensor of the Dirac field can be removed using a trick due to Frederik J. Belifante, *On the spin angular momentum of mesons*, Physica **6** (1939) 887-898, and L. Rosenfeld, *Sur le tenseur d'impulsion-énergie*, Mémoires de l'Academie Roy. Belgique **18** (1940) 1-30, see **Problem** 1.

`Proof.` By passing in (4.49) to the momentum representations (4.30) and carrying out the integration over the Oz real line we obtain

$$\begin{aligned} \mathrm{h} &= \int_{-\infty}^{\infty} \mathrm{d}z\, \tfrac{1}{2}\, \psi^{\dagger}(t,z)\, \Sigma_3\, \psi(t,z) \\ &= \int_{-\infty}^{\infty} \mathrm{d}z \sum_{p\,,\,r} \left[c^*_{p\,,\,r}\, u^{\dagger}_{p\,,\,r}(t,z) + d_{\,p\,,\,r}\, v^{\dagger}_{p\,,\,r}(t,z) \right] \\ &\times \;\; \tfrac{1}{2}\, \Sigma_3 \sum_{q\,,\,s} \left[c_{\,q\,,\,s}\, u_{\,q\,,\,s}(t,z) + d^*_{q\,,\,s}\, v_{\,q\,,\,s}(t,z) \right] \end{aligned}$$

in which we have set $\mathbf{p} = (0,0,p)\,, \mathbf{q} = (0,0,q)\,,\ \omega_{\,p} = \sqrt{(p^2 + M^2)}$ together with

$$\begin{aligned} u_{\,p\,,\,r}(t,z) &= [\,4\pi\omega_{\,p}\,]^{-1/2}\, u_r(p)\, \exp\{ipz - it\omega_{\,p}\} \qquad (r = 1,2) \\ v_{\,q\,,\,s}(t,z) &= [\,4\pi\omega_{\,q}\,]^{-1/2}\, v_s(q)\, \exp\{it\omega_{\,q} - iqz\} \qquad (s = 1,2) \end{aligned}$$

the normalization being now consistent with the plane waves independent of the transverse $x^{\top} = (x^1\,,\,x^2)$ spatial coordinates. From (4.27) and the commutation relation

$$[\,\omega_{\,p}\gamma^0 - p\,\gamma^3\,,\,\Sigma_3\,] = 0 \tag{4.53}$$

together with the definition (4.34)

$$\left\{ \begin{array}{l} u_{\,r}(p) \equiv (2\omega_{\,p} + 2M)^{-1/2}\ \left(M + \omega_{\,p}\gamma^0 - p\,\gamma^3\right) \xi_{\,r} \\ v_{\,r}(p) \equiv (2\omega_{\,p} + 2M)^{-1/2}\ \left(M - \omega_{\,p}\gamma^0 + p\,\gamma^3\right) \eta_{\,r} \end{array} \right. \qquad (r = 1,2)$$

it can be readily derived that

$$\begin{gathered} (\,\Sigma_3 - 1\,)\, u_1(p) = (\,\Sigma_3 - 1\,)\, v_2(p) = 0 \\ (\,\Sigma_3 + 1\,)\, u_2(p) = (\,\Sigma_3 + 1\,)\, v_1(p) = 0 \\ u^{\dagger}_r(p)\, \Sigma_3\, v_s(-\,p) = 0 \qquad v^{\dagger}_r(-\,p)\, \Sigma_3\, u_s(p) = 0 \qquad (r,s = 1,2) \end{gathered} \tag{4.54}$$

Hence the spin component along the direction of propagation, which is named **helicity** of the Dirac spinor wave field, turns out to be time independent and takes the form

$$\begin{aligned} \mathrm{h} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{4\omega_{\,p}} \sum_{r=1,\,2} \left[c^*_{p\,,\,r}\, c_{\,p\,,\,r}\, u^{\dagger}_r(p)\, \Sigma_3\, u_r(p) + d_{\,p\,,\,r}\, d^*_{p\,,\,r}\, v^{\dagger}_r(p)\, \Sigma_3\, v_r(p) \right] \\ &= \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}p \left[c^*_{p\,,\,1}\, c_{\,p\,,\,1} - c^*_{p\,,\,2}\, c_{\,p\,,\,2} - d_{\,p\,,\,1}\, d^*_{p\,,\,1} + d_{\,p\,,\,2}\, d^*_{p\,,\,2} \right] \end{aligned}$$

which proves equation (4.52) and shows that particles and antiparticles exhibit opposite helicity. □

Finally, the Dirac Lagrangian is manifestly invariant under the internal symmetry group $U(1)$ of the phase transformations

$$\psi^{\,\prime}(x) = e^{\,iq\theta}\, \psi(x) \qquad \bar{\psi}^{\,\prime}(x) = \bar{\psi}(x)\, e^{-\,iq\theta} \qquad 0 \le \theta < 2\pi$$

where q denotes the particle electric charge, *e.g.* $q = -\,e$ $(e > 0)$ for the electron. Then from eq. (2.105) the corresponding Nöther current becomes

$$J^{\,\mu}(x) = q\, \bar{\psi}(x)\, \gamma^{\,\mu}\, \psi(x) \tag{4.55}$$

and will be identified with the electric current carried on by the spinor field, which transforms as a true four vector under the space inversion (2.55) or, more generally, under the improper orthochronus Lorentz transformations of $L_{-}^{\uparrow}$, that means

$$J_0'(x^0, -\mathbf{x}) = J_0(x) \qquad \mathbf{J}'(x^0, -\mathbf{x}) = -\mathbf{J}(x)$$

If instead we consider the internal symmetry group $U(1)$ of the chiral phase transformations

$$\psi'(x) = \exp\{-i\theta\,\gamma_5\}\,\psi(x) \qquad \bar{\psi}'(x) = \bar{\psi}(x)\,\exp\{-i\theta\,\gamma_5\} \qquad 0 \le \theta < 2\pi$$

then the free Dirac Lagrange density is not invariant, owing to the presence of the mass term, so that the Nöther theorem (2.89) yields in this case

$$\partial_\mu\,J_5^\mu(x) = \partial_\mu\Big(\bar{\psi}(x)\,\gamma^\mu\,\gamma_5\,\psi(x)\Big) = 2iM\,\bar{\psi}(x)\,\gamma_5\,\psi(x) \tag{4.56}$$

Notice that the axial vector current $J_5^\mu(x)$ is a pseudo-vector since we have the transformation law under parity or spatial inversion (2.55)

$$J_5^{0\,\prime}(x^0, -\mathbf{x}) = -J_5^0(x) \qquad J_5^{k\,\prime}(x^0, -\mathbf{x}) = J_5^k(x)$$

It follows therefrom that the chiral phase transformations are a symmetry group of the classical free Dirac theory only in the massless limit.

4.3 Dirac Quantum Field

The above discussion eventually leads to the energy-momentum vector of the free Dirac wave field that reads

$$P_\mu = \int \mathrm{d}\mathbf{x}\; T^0{}_\mu(x^0, \mathbf{x}) = \int \mathrm{d}\mathbf{x}\; \psi^\dagger(x)\, i\partial_\mu\, \psi(x) \tag{4.57}$$

and inserting the normal mode expansions (4.30) we obtain

$$\begin{aligned} P_\mu \;&=\; \int \mathrm{d}\mathbf{x} \sum_{\mathbf{q},s} \Big[c^*_{\mathbf{q},s}\, u^\dagger_{\mathbf{q},s}(x) \;+\; d_{\mathbf{q},s}\, v^\dagger_{\mathbf{q},s}(x) \Big] \\ &\times\; \sum_{\mathbf{p},r} p_\mu \Big[c_{\mathbf{p},r}\, u_{\mathbf{p},r}(x) \;-\; d^*_{\mathbf{p},r}\, v_{\mathbf{p},r}(x) \Big] \end{aligned} \tag{4.58}$$

where we have set $p_\mu \equiv (\omega_{\mathbf{p}}\, ,\, -\mathbf{p})$. Taking the orthogonality relations (4.21) and (4.28) into account we come to the expression

$$P^\mu \;=\; \sum_{\mathbf{p},r} p^\mu \left(c^*_{\mathbf{p},r}\, c_{\mathbf{p},r} \;-\; d_{\mathbf{p},r}\, d^*_{\mathbf{p},r} \right) \tag{4.59}$$

Now the key point: as we shall see in a while, in order to quantize the relativistic spinor wave field in such a manner to obtain a positive semi-definite energy operator, then we must impose canonical anti-commutation relations. As a matter of fact, once the normal mode expansion coefficients turn into creation and destruction operators, that means

$$c_{\mathbf{p},r}\, ,\, c^*_{\mathbf{p},r} \;\mapsto\; c_{\mathbf{p},r}\, ,\, c^\dagger_{\mathbf{p},r} \qquad\qquad d_{\mathbf{p},r}\, ,\, d^*_{\mathbf{p},r} \;\mapsto\; d_{\mathbf{p},r}\, ,\, d^\dagger_{\mathbf{p},r}$$

had we assumed the canonical commutation relations

$$[\, d_{\mathbf{p},r}\, ,\, d^\dagger_{\mathbf{q},s}\,] = \delta_{rs}\, \delta(\mathbf{p} - \mathbf{q})$$

as in the scalar field case, then we would find

$$P^\mu \;=\; \sum_{\mathbf{p},r} p^\mu \left(c^\dagger_{\mathbf{p},r}\, c_{\mathbf{p},r} \;-\; d^\dagger_{\mathbf{p},r}\, d_{\mathbf{p},r} \right) - 4\, U_0\, g^{\mu 0}$$

where U_0 is again the divergent zero-point energy (3.56)

$$\begin{aligned} cU_0 \;&=\; \delta(\mathbf{0}) \sum_{\mathbf{p}} \tfrac{1}{2}\hbar\, \omega_{\mathbf{p}} = V\hbar \int \frac{\omega_{\mathbf{p}}\, \mathrm{d}\mathbf{p}}{2(2\pi)^3} \\ &=\; \frac{V\hbar c}{4\pi^2} \int_0^K \mathrm{d}p\, p^2 \sqrt{p^2 + M^2c^2/\hbar^2} \end{aligned} \tag{4.60}$$

whereas V is the volume of a very large box and $\hbar K \gg Mc$ is a very large wave-number, the factor two being due to spin. This means, however, that even assuming normal ordering prescription to discard U_0, still the spinor energy operator P_0 is no longer positive semi-definite.

Turning back to the classical spinor wave field, it turns out that it is not convenient to understand the normal mode expansion coefficients

$$d_{\mathbf{p},r} \quad \left(r = 1,2\,,\ \mathbf{p} \in \mathbb{R}^3\right)$$

as ordinary complex numbers. On the contrary, we can assume all those coefficients to be anti-commuting numbers, also named Graßmann numbers

Hermann Günther Graßmann (Stettino, 15.04.1809 – 26.09.1877)
Die Lineale Ausdehnungslehre, ein neuer Zweig del Mathematik
(Linear Extension Theory, a New Branch of Mathematics (1844)

$$\{d_{\mathbf{p},r}\,,\,d_{\mathbf{q},s}\} = 0 = \{d_{\mathbf{p},r}\,,\,d^{*}_{\mathbf{q},s}\} \qquad (\forall\, r,s = 1,2 \quad \mathbf{p},\mathbf{q} \in \mathbb{R}^3\,)$$

in which $\{a,b\} = ab + ba$. Notice *en passant* that in the particular case $r = s$ and $\mathbf{p} = \mathbf{q}$ we find $d^{2}_{\mathbf{q},s} = 0 = d^{*2}_{\mathbf{q},s}$. Internal consistency then requires that also the normal mode expansion coefficients $c_{\mathbf{p},r}$ $(r = 1,2\,,\ \mathbf{p} \in \mathbb{R}^3)$ must be taken Graßmann numbers, in such a manner that the whole classical Dirac spinor relativistic wave field becomes Graßmann valued so that

$$\{\psi(x)\,,\,\psi(y)\} = \{\psi(x)\,,\,\bar{\psi}(y)\} = \{\bar{\psi}(x)\,,\,\bar{\psi}(y)\} = 0 \tag{4.61}$$

Under this assumption, the canonical quantization of such a system is then achieved by replacing the Graßmann numbers valued coefficients of the normal mode expansion by creation annihilation operators acting on a Fock space and postulating the canonical anti-commutation relations, that means

$$\begin{aligned}&\{c_{\mathbf{p},r}\,,\,c^{\dagger}_{\mathbf{q},s}\} = \delta_{rs}\,\delta(\mathbf{p}-\mathbf{q}) = \{d_{\mathbf{p},r}\,,\,d^{\dagger}_{\mathbf{q},s}\}\\ &(\,\forall\, r,s = 1,2 \quad \mathbf{p},\mathbf{q} \in \mathbb{R}^3\,)\\ &\text{all the other anti} - \text{commutators vanishing}\end{aligned} \tag{4.62}$$

As a consequence, the Dirac spinor quantum wave field becomes an operator valued tempered distribution acting on a Fock space and reads

$$\psi(x) = \sum_{\mathbf{p},r} \left[c_{\mathbf{p},r}\,u_{\mathbf{p},r}(x) + d^{\dagger}_{\mathbf{p},r}\,v_{\mathbf{p},r}(x)\right]\sqrt{\hbar c} \tag{4.63}$$

$$\psi^{\dagger}(x) = \sum_{\mathbf{p},r} \left[c^{\dagger}_{\mathbf{p},r}\,u^{\top *}_{\mathbf{p},r}(x) + d_{\mathbf{p},r}\,v^{\top *}_{\mathbf{p},r}(x)\right]\sqrt{\hbar c} \tag{4.64}$$

where it has been suitably stressed that the symbol † is indeed referred to the Hermitean conjugation of operators acting on a Fock space. The canonical anti-commutation relations (4.62) actually imply

$$\{\psi_\alpha(t,\mathbf{x})\,,\,\psi_\beta(t,\mathbf{y})\} = 0 = \{\psi_\alpha^\dagger(t,\mathbf{x})\,,\,\psi_\beta^\dagger(t,\mathbf{y})\} \tag{4.65}$$

$$\{\psi_\alpha(t,\mathbf{x})\,,\,\psi_\beta^\dagger(t,\mathbf{y})\} = \hbar c\,\delta\,(\mathbf{x}-\mathbf{y})\,\delta_{\alpha\beta} \tag{4.66}$$

$$(\,\alpha,\beta = 1L, 2L, 1R, 2R\,)$$

`Proof.` Taking the normal modes expansions into account we actually get

$$\begin{aligned}
\{\psi_\alpha(t,\mathbf{x})\,,\,\psi_\beta^\dagger(t,\mathbf{y})\} \;&=\; \Big\{\sum_{\mathbf{p}\,,\,r}\Big[c_{\mathbf{p}\,,\,r}\,u_{\mathbf{p}\,,\,r\,,\,\alpha}\,(t,\mathbf{x}) + d_{\mathbf{p}\,,\,r}^\dagger\,v_{\mathbf{p}\,,\,r\,,\,\alpha}\,(t,\mathbf{x})\Big]\,,\\
&\sum_{\mathbf{q}\,,\,s}\Big[c_{\mathbf{q}\,,\,s}^\dagger\,u_{\mathbf{q}\,,\,s\,,\,\beta}^{\top\,*}\,(t,\mathbf{y}) + d_{\mathbf{q}\,,\,s}\,v_{\mathbf{q}\,,\,s\,,\,\beta}^{\top\,*}\,(t,\mathbf{y})\Big]\Big\}\\
&=\; \sum_{\mathbf{p}\,,\,r}\Big[u_{\mathbf{p}\,,\,r\,,\,\alpha}\,(t,\mathbf{x})\,u_{\mathbf{p}\,,\,r\,,\,\beta}^\dagger\,(t,\mathbf{y}) + v_{\mathbf{p}\,,\,r\,,\,\alpha}\,(t,\mathbf{x})\,v_{\mathbf{p}\,,\,r\,,\,\beta}^\dagger\,(t,\mathbf{y})\Big]
\end{aligned}$$

where use has been made of the canonical anti-commutation relations (4.62) among the creation and destruction operators of the spinor operator field. Now, by taking into account the closure relation (4.22) we immediately get the result

$$\{\psi_\alpha(t,\mathbf{x})\,,\,\psi_\beta^\dagger(t,\mathbf{y})\} = \delta_{\alpha\beta}\;\delta(\mathbf{x}-\mathbf{y})$$

and the related anti-commutator

$$\{\psi\,(t,\mathbf{x})\,,\,\overline{\psi}\,(t,\mathbf{y})\} = \gamma^0\,\delta(\mathbf{x}-\mathbf{y})$$

□

Then, if we adopt once again the normal product to remove the divergent and negative zero-point energy contribution to the cosmological constant, we come to the operator expression for the energy-momentum of the Dirac spinor quantum free field: namely,

$$\begin{aligned}
E \;&=\; \int \mathrm{d}\mathbf{x}\;:\;\psi^\dagger\,(x)\,i\partial_0\,\psi\,(x)\;:\\
&=\; \frac{1}{\hbar c}\int \mathrm{d}\mathbf{x}\;\;:\;\psi^\dagger\,(x)\,H\,\psi\,(x)\;:\\
&=\; \sum_{\mathbf{p}\,,\,r}\hbar\omega_{\mathbf{p}}\Big[c_{\mathbf{p}\,,\,r}^\dagger\,c_{\mathbf{p}\,,\,r}\;+\;d_{\mathbf{p}\,,\,r}^\dagger\,d_{\mathbf{p}\,,\,r}\Big] \qquad (4.67)\\
\mathbf{P} \;&=\; \frac{1}{c}\int \mathrm{d}\mathbf{x}\;\;:\;\psi^\dagger\,(x)\,(-\,i\boldsymbol{\nabla}\,)\,\psi\,(x)\;:\\
&=\; \sum_{\mathbf{p}\,,\,r}\hbar\mathbf{p}\Big[c_{\mathbf{p}\,,\,r}^\dagger\,c_{\mathbf{p}\,,\,r}\;+\;d_{\mathbf{p}\,,\,r}^\dagger\,d_{\mathbf{p}\,,\,r}\Big] \qquad (4.68)
\end{aligned}$$

`Normal Product of Anti-commuting Fields.` As a matter of fact we have, for instance,

$$\begin{aligned}
\psi_\alpha^\dagger(y)\, i\partial_0\, \psi_\beta(x) \;&=\; \sum_{\mathbf{p}\,,\,r}\Big[c^\dagger_{\mathbf{p}\,,\,r}\, u^\dagger_{\mathbf{p}\,,\,r\,,\,\alpha}(y) + d_{\mathbf{p}\,,\,r}\, v^\dagger_{\mathbf{p}\,,\,r\,,\,\alpha}(y)\Big] \\
&\times\; \sum_{\mathbf{q}\,,\,s}\omega_{\mathbf{q}}\Big[c_{\mathbf{q}},\, u_{\mathbf{q}\,,\,s\,,\,\beta}(x) - d^\dagger_{\mathbf{q}\,,\,s}\, v_{\mathbf{q}\,,\,s\,,\,\beta}(x)\Big] \\
&=\; \sum_{\mathbf{p}\,,\,r}\sum_{\mathbf{q}\,,\,s}\omega_{\mathbf{q}}\Big[c^\dagger_{\mathbf{p}\,,\,r}\, c_{\mathbf{q}},\, u^\dagger_{\mathbf{p}\,,\,r\,,\,\alpha}(y)\, u_{\mathbf{q}\,,\,s\,,\,\beta}(x) \\
&-\; c^\dagger_{\mathbf{p}\,,\,r}\, d^\dagger_{\mathbf{q}},\, u^\dagger_{\mathbf{p}\,,\,r\,,\,\alpha}(y)\, v_{\mathbf{q}\,,\,s\,,\,\beta}(x) \\
&+\; d_{\mathbf{p}\,,\,r}\, c_{\mathbf{q}\,,\,s}\, v^\dagger_{\mathbf{p}\,,\,r\,,\,\alpha}(y)\, u_{\mathbf{q}\,,\,s\,,\,\beta}(x) \\
&+\; d^\dagger_{\mathbf{q}\,,\,s}\, d_{\mathbf{p}\,,\,r}\, v^\dagger_{\mathbf{p}\,,\,r\,,\,\alpha}(y)\, v_{\mathbf{q}\,,\,s\,,\,\beta}(x)\Big] \\
&-\; \sum_{\mathbf{q}\,,\,s}\omega_{\mathbf{q}}\, v^\dagger_{\mathbf{q}\,,\,s\,,\,\alpha}(y)\, v_{\mathbf{q}\,,\,s\,,\,\beta}(x) \\
&\equiv\; :\psi_\alpha^\dagger(y)\, i\dot\psi_\beta(x): - \sum_{\mathbf{q}\,,\,s}\;\omega_{\mathbf{q}}\, v^\dagger_{\mathbf{q}\,,\,s\,,\,\alpha}(y)\, v_{\mathbf{q}\,,\,s\,,\,\beta}(x)
\end{aligned}$$

Thus we can eventually write

$$:\psi_\alpha^\dagger(y)\,\dot\psi_\beta(x) := \psi_\alpha^\dagger(y)\,\dot\psi_\beta(x) - \sum_{\mathbf{q}\,,\,s}\; i\omega_{\mathbf{q}}\, v^\dagger_{\mathbf{q}\,,\,s\,,\,\alpha}(y)\, v_{\mathbf{q}\,,\,s\,,\,\beta}(x) = -\; :\dot\psi_\beta(x)\,\psi_\alpha^\dagger(y):$$

that can be easily generalized to any local bilinear in the anti-commuting spinor fields. □

It is important to remark that the canonical anti-commutation relations

$$\{d_{\mathbf{p}\,,\,r}\,,\, d^\dagger_{\mathbf{q}\,,\,s}\} = \delta_{rs}\,\delta(\mathbf{p}-\mathbf{q}) \qquad \{d_{\mathbf{p}\,,\,r}\,,\, d_{\mathbf{q}\,,\,s}\} = 0$$

$$(\,\forall\, r,s = 1,2 \quad \mathbf{p},\mathbf{q}\in\mathbb{R}^3\,)$$

do guarantee the positive semi-definiteness of the energy operator P_0, while the remaining anti-commutators (4.62) can be derived from the requirement that the energy-momentum operators P_μ realize the self-adjoint generators of the space-time translations of a unitary representation of the Poincaré group. As a matter of fact, from a repeated use of the algebraic identity $[ab,c] = a\,\{b,c\} - \{a,c\}\,b$ we obtain

$$\begin{aligned}
&[\,P_\mu\,,\,\psi(x)\,] \;= \\
&\sum_{\mathbf{p}\,,\,r} p_\mu\Big[\Big(c^\dagger_{\mathbf{p}\,,\,r}\, c_{\mathbf{p}\,,\,r} \;+\; d^\dagger_{\mathbf{p}\,,\,r}\, d_{\mathbf{p}\,,\,r}\Big)\,,\, \sum_{\mathbf{q}\,,\,s}\; c_{\mathbf{q}\,,\,s}\, u_{\mathbf{q}\,,\,s}(x)\Big] \;+ \\
&\sum_{\mathbf{p}\,,\,r} p_\mu\Big[\Big(c^\dagger_{\mathbf{p}\,,\,r}\, c_{\mathbf{p}\,,\,r} \;+\; d^\dagger_{\mathbf{p}\,,\,r}\, d_{\mathbf{p}\,,\,r}\Big)\,,\, \sum_{\mathbf{q}\,,\,s}\; d^\dagger_{\mathbf{q}\,,\,s}\, v_{\mathbf{q}\,,\,s}(x)\Big)\Big] \;= \\
&\sum_{\mathbf{p}\,,\,r} p_\mu \sum_{\mathbf{q}\,,\,s}\; u_{\mathbf{q}\,,\,s}(x)\Big(c^\dagger_{\mathbf{p}\,,\,r}\,\{c_{\mathbf{p}\,,\,r}\,,\, c_{\mathbf{q}\,,\,s}\} - \{c^\dagger_{\mathbf{p}\,,\,r}\,,\, c_{\mathbf{q}\,,\,s}\}\, c_{\mathbf{p}\,,\,r}\Big) \;+
\end{aligned}$$

$$\sum_{\mathbf{p},r} p_\mu \sum_{\mathbf{q},s} u_{\mathbf{q},s}(x) \Big(d^\dagger_{\mathbf{p},r} \{ d_{\mathbf{p},r} \,,\, c_{\mathbf{q},s} \} - \{ d^\dagger_{\mathbf{p},r} \,,\, c_{\mathbf{q},s} \} \, d_{\mathbf{p},r} \Big) \; +$$
$$\sum_{\mathbf{p},r} p_\mu \sum_{\mathbf{q},s} v_{\mathbf{q},s}(x) \Big(c^\dagger_{\mathbf{p},r} \{ c_{\mathbf{p},r} \,,\, d^\dagger_{\mathbf{q},s} \} - \{ c^\dagger_{\mathbf{p},r} \,,\, d^\dagger_{\mathbf{q},s} \} \, c_{\mathbf{p},r} \Big) \; +$$
$$\sum_{\mathbf{p},r} p_\mu \sum_{\mathbf{q},s} v_{\mathbf{q},s}(x) \Big(d^\dagger_{\mathbf{p},r} \{ d_{\mathbf{p},r} \,,\, d^\dagger_{\mathbf{q},s} \} - \{ d^\dagger_{\mathbf{p},r} \,,\, d^\dagger_{\mathbf{q},s} \} \, d_{\mathbf{p},r} \Big)$$
$$= \; - i\partial_\mu \psi(x) = \sum_{\mathbf{p},r} p_\mu \Big[- c_{\mathbf{p},r} \, u_{\mathbf{p},r}(x) + d^\dagger_{\mathbf{p},r} \, v_{\mathbf{p},r}(x) \Big] \tag{4.69}$$

if and only if the canonical anti-commutation relations (4.62) hold true. A quite close calculation for the total angular momentum operator eventually yields

$$[\, \psi(x) \,,\, P_\mu \,] \;=\; i\hbar\partial_\mu \psi(x) \tag{4.70}$$
$$\hbar^{-1}[\, \psi(x) \,,\, M_{\mu\nu} \,] \;=\; \big(\, x_\mu \, i\partial_\nu - x_\nu \, i\partial_\mu + g_{j\mu} \, g_{k\nu} \, \sigma^{\,jk} \, \big) \, \psi(x) \tag{4.71}$$

From Nöther theorem and canonical anti-commutation relations (4.62) it follows that the classical vector current (4.55) is turned into the quantum operator

$$J^{\,\mu}(x) \;\equiv \, : q \, \bar\psi(x) \, \gamma^{\,\mu} \, \psi(x) : \tag{4.72}$$

which satisfies the continuity operator equation

$$\partial_\mu J^{\,\mu}(x) = \frac{1}{i\hbar} \, [\, P_\mu \,,\, J^{\,\mu}(x) \,] = 0 \tag{4.73}$$

As a consequence we have the conserved charge operator

$$Q \;\equiv\; \int \mathrm{d}\mathbf{x} \;\; : q \, \psi^\dagger(x) \, \psi(x) : \;=\; q \sum_{\mathbf{p},r} \Big(c^\dagger_{\mathbf{p},r} \, c_{\mathbf{p},r} \;-\; d^\dagger_{\mathbf{p},r} \, d_{\mathbf{p},r} \Big) \tag{4.74}$$

whence it is manifest that the two types of quanta of the Dirac field do carry opposite charges [2]. According to the customary convention for the Dirac spinor describing the electron positron field, we shall associate to particles the creation annihilation operators of the c −type and the negative electric charge $q = -e$ ($e > 0$), whilst the creation-annihilation operators of the d-type and the positive electric charge $+e$ will be associated to the antiparticles, so that the electric charge operator becomes

$$Q = \sum_{\mathbf{p},r} (-e) \Big(c^\dagger_{\mathbf{p},r} \, c_{\mathbf{p},r} \;-\; d^\dagger_{\mathbf{p},r} \, d_{\mathbf{p},r} \Big) \tag{4.75}$$

[2] On the contrary, the quanta of the charge self-conjugated Majorana bispinor field are neutral in such a manner that particles and antiparticles actually coincide, see Problem 3.

Finally, it turns out that also the helicity (4.52) of the Dirac field, which is a constant of motion, will be turned by the quantization procedure into the normal ordered operator expression

$$\begin{aligned} \mathrm{h} &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}z\ :\psi^{\dagger}(t,z)\,\Sigma_3\,\psi(t,z): \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}p \sum_{r=1,2}\Big[c_{p,r}^{\dagger}\,c_{p,r}\,u_r^{\dagger}(p)\,\Sigma_3\,u_r(p) \\ &- d_{p,r}^{\dagger}\,d_{p,r}\,v_r^{\dagger}(p)\,\Sigma_3\,v_r(p)\Big]\,(2\omega_p)^{-1} \end{aligned} \tag{4.76}$$

where

$$\{c_{p,r}\,,\,c_{q,s}^{\dagger}\} = \delta_{rs}\,\delta(p-q) = \{d_{p,r}\,,\,d_{q,s}^{\dagger}\}$$
$$(\,\forall\, r,s = 1,2 \quad p,q\in\mathbb{R}\,)$$

and all other anti-commutators vanish.

It is convenient to choose our standard spin-states, *i.e.* the orthogonal and normalized solutions of eq.s (4.15) and (4.16), in which we have to put $\mathbf{p} = (0,0,p)$. Then the helicity operator eventually becomes

$$\mathrm{h} = \tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}p\,[\,c_{p,1}^{\dagger}\,c_{p,1} - c_{p,2}^{\dagger}\,c_{p,2} + d_{p,1}^{\dagger}\,d_{p,1} - d_{p,2}^{\dagger}\,d_{p,2}\,] \tag{4.77}$$

The above expression actually clarifies the meaning of the polarization indexes $r,s = 1,2$. Hence, from the expressions (4.67), (4.68), (4.75) and (4.77), it follows that the operators $c_{\mathbf{p},r}^{\dagger}$ and $c_{\mathbf{p},r}$ do correspond respectively to the creation and annihilation operators for the particles of momentum $\mathbf{p}$, mass M with $p^2 = M^2$, electric charge $-\,e$, and positive helicity equal to $\frac{1}{2}$ $(r=1)$ or negative helicity equal to $-\,\frac{1}{2}$ $(r=2)$. Conversely, the operators $d_{\mathbf{p},r}^{\dagger}$ and $d_{\mathbf{p},r}$ will correspond respectively to the creation and annihilation operators for the antiparticles of momentum $\mathbf{p}$, mass M with $p^2 = M^2$, electric charge $+\,e$, and positive helicity equal to $\frac{1}{2}$ $(r=1)$ or negative helicity equal to $-\,\frac{1}{2}$ $(r=2)$.

The cyclic Fock vacuum is defined by

$$c_{\mathbf{p},r}\,|\,0\,\rangle = 0 \qquad d_{\mathbf{p},r}\,|\,0\,\rangle = 0 \qquad (\,\forall\, r=1,2 \quad \mathbf{p}\in\mathbb{R}^3\,) \tag{4.78}$$

while the 1-particle energy-momentum, helicity and charge eigenstates will correspond to

$$|\,\mathbf{p}\,r\,-\rangle \equiv c_{\mathbf{p},r}^{\dagger}\,|\,0\,\rangle \qquad (\,r=1,2 \quad \mathbf{p}\in\mathbb{R}^3\,) \tag{4.79}$$

whereas the 1-antiparticle energy-momentum, helicity and charge eigenstates will be

$$|\,\mathbf{p}\,r+\,\rangle \equiv d^{\dagger}_{\mathbf{p}\,,r}\,|\,0\,\rangle \qquad (\,r=1,2 \quad \mathbf{p}\in\mathbb{R}^3\,) \tag{4.80}$$

Owing to the canonical anti-commutation relations (4.62), it is impossible to put two particles or two antiparticles in the very same quantum state as *e.g.*

$$\begin{aligned} c^{\dagger}_{\mathbf{p}\,,r}\,c^{\dagger}_{\mathbf{p}\,,r}\,|\,0\,\rangle &= -\,c^{\dagger}_{\mathbf{p}\,,r}\,c^{\dagger}_{\mathbf{p}\,,r}\,|\,0\,\rangle = 0 \\ d^{\dagger}_{\mathbf{p}\,,r}\,d^{\dagger}_{\mathbf{p}\,,r}\,|\,0\,\rangle &= -\,d^{\dagger}_{\mathbf{p}\,,r}\,d^{\dagger}_{\mathbf{p}\,,r}\,|\,0\,\rangle = 0 \end{aligned}$$

As a consequence the many-particle states do obey Fermi-Dirac statistics and the occupation numbers solely take the two possible values

$$N_{\mathbf{p}\,,r\,,\pm} = 0,1 \qquad\qquad (\,r=1,2 \quad \mathbf{p}\in\mathbb{R}^3\,)$$

which drives to the Pauli exclusion principle valid for all identical particles with half-integer spin. In other words, it turns out that the exclusion principle is no longer an assumption in Relativistic Quantum Field Theory albeit a direct necessary consequence of the canonical anti-commutation relations. This statement, which goes under the name of the Spin-Statistics theorem, does actually guarantee the stability of Matter Systems. The generic and properly normalized quantum state with $N-$particles and $\overline{N}-$antiparticles of definite momentum and polarization, that corresponds to an element of the basis of the Fock space, will be written in the form

$$\big(N!\,\overline{N}!\big)^{-\frac{1}{2}}\prod_{a=1}^{N}\prod_{b=1}^{\overline{N}} c^{\dagger}(\mathbf{p}_a\,,\,r_a\,)\,d^{\dagger}(\mathbf{p}_b\,,\,r_b\,)\,|\,0\,\rangle \tag{4.81}$$

By virtue of (4.62) those states are completely anti-symmetric with regard to the exchange of any pairs $(\mathbf{p}_a\,,\,r_a\,)$ and $(\mathbf{p}_b\,,\,r_b\,)$ and correspond to the presence of N particles and $\overline{N}$ antiparticles.

As a last but not least development, we can obtain the wave functions for particles and antiparticles in the form of some suitable matrix elements. From the general normal mode expansion of the quantized Dirac field (4.64)

$$\begin{aligned} \psi\,(x) &= \sum_{\mathbf{p}\,,r}\Big[\,c_{\mathbf{p}\,,r}\,u_{\mathbf{p}\,,r}\,(x) + d^{\dagger}_{\mathbf{p}\,,r}\,v_{\mathbf{p}\,,r}\,(x)\,\Big] \\ \bar{\psi}\,(x) &= \sum_{\mathbf{p}\,,r}\Big[\,c^{\dagger}_{\mathbf{p}\,,r}\,\bar{u}_{\mathbf{p}\,,r}\,(x) + d_{\mathbf{p}\,,r}\,\bar{v}_{\mathbf{p}\,,r}\,(x)\,\Big] \end{aligned}$$

we readily find for instance

$$\begin{aligned}
\langle\, 0\,|\,\psi\,(x)|\,\mathbf{q}\; s\,-\,\rangle &= \langle\, 0\,|\,\psi\,(x)\, c^{\dagger}_{\mathbf{q}\,,\,s}\,|\,0\,\rangle \\
&= \langle\, 0\,|\sum_{\mathbf{p}\,,\,r}\,\Big[c_{\mathbf{p}\,,\,r}\,u_{\mathbf{p}\,,\,r}\,(x)+d^{\dagger}_{\mathbf{p}\,,\,r}\,v_{\mathbf{p}\,,\,r}\,(x)\Big]\,c^{\dagger}_{\mathbf{q}\,,\,s}\,|\,0\,\rangle
\end{aligned}$$

and from the canonical anti-commutation relations (4.62) we eventually get

$$\langle\, 0\,|\,\psi\,(x)\,|\,\mathbf{q}\; s\,-\,\rangle = \sum_{\mathbf{p}\,,\,r}\langle\, 0\,|\,\{c_{\mathbf{p}\,,\,r}\,,\, c^{\dagger}_{\mathbf{q}\,,\,s}\}\,|\,0\,\rangle\, u_{\mathbf{p}\,,\,r}\,(x) = u_{\mathbf{q}\,,\,s}\,(x)$$

which represents the wave function for an incoming particle of spin $\frac{1}{2}$ and negative charge $-\,e$. In a quite analogous way we find

$$\langle\, 0\,|\,\bar{\psi}\,(x)|\,\mathbf{q}\; s\,+\,\rangle = \sum_{\mathbf{p}\,,\,r}\langle\, 0\,|\,\{d_{\mathbf{p}\,,\,r}\,,\, d^{\dagger}_{\mathbf{q}\,,\,s}\}\,|\,0\,\rangle\, \bar{v}_{\mathbf{p}\,,\,r}\,(x) = \bar{v}_{\mathbf{q}\,,\,s}\,(x)$$

which represents the wave function for an incoming antiparticle of spin $\frac{1}{2}$ and positive charge e . To find out the wave functions for the outgoing quanta we shall consider the matrix elements

$$\langle\, -\; s\;\mathbf{q}\,|\,\bar{\psi}\,(x)\,|\,0\,\rangle = \langle\, 0\,|\; c_{\mathbf{q}\,,\,s}\,\bar{\psi}\,(x)\,|\,0\,\rangle = \bar{u}_{\mathbf{q}\,,\,s}\,(x)$$

that represents the wave function for an outgoing particle of spin $\frac{1}{2}$ and negative charge $-\,e$, while

$$\langle\, +\; s\;\mathbf{q}\,|\,\psi\,(x)\,|\,0\,\rangle = \langle\, 0\,|\; d_{\mathbf{q}\,,\,s}\,\psi\,(x)\,|\,0\,\rangle = v_{\mathbf{q}\,,\,s}\,(x)$$

does correspond to the wave function for an outgoing antiparticle of spin $\frac{1}{2}$ and negative charge $+\,e$.

4.4 Covariance of the Quantum Dirac Field

Consider the structure of the generators of the Poincaré transformations, that appears in the form of normal ordered bi-linear operator expressions, which can be readily obtained from the Nöther theorem using *e.g.* the classical Lagrange density (4.2): namely,

$$\begin{aligned}
P_{\mu} &= \int \mathrm{d}\mathbf{x}\;\; :\psi^{\,\dagger}\,(x)\, i\partial_{\mu}\,\psi\,(x): \\
M^{\,\lambda\nu} &= \int \mathrm{d}\mathbf{x}\;\; :\psi^{\,\dagger}\,(x)[\,x^{\,\lambda} i\partial^{\,\nu} - x^{\,\nu} i\partial^{\,\lambda} + \sigma^{\,\lambda\nu}\,]\psi\,(x): \qquad (4.82)
\end{aligned}$$

It can be verified by direct inspection that, owing to the anti-commutation relations (4.62) and (4.66), those operator expressions indeed generate the local infinitesimal (passive) Poincaré transformations for the operator valued quantum Dirac spinor field

$$\begin{aligned}\delta\,\psi(x) &= i\,[\,P_\mu\,,\,\psi\,(x)\,]\,\epsilon^{\,\mu} - \tfrac{1}{2}\,i\,[\,M_{\rho\sigma}\,,\,\psi\,(x)\,]\,\epsilon^{\,\rho\sigma} \\ &= \big[\,\epsilon^{\,\mu}\,\partial_\mu + \tfrac{1}{2}\,\epsilon_{\,\mu\nu}\,(\,x^{\,\nu}\,\partial^{\,\mu} - x^{\,\mu}\,\partial^{\,\nu} + i\sigma^{\,\mu\nu}\,)\,\big]\,\psi\,(x)\end{aligned} \tag{4.83}$$

whilst for a finite transformation we can write

$$\psi^{\,\prime}(x) - \psi(x) = U(\mathrm{a},\omega)\,\psi(x)\,U^{\,\dagger}(\mathrm{a},\omega) - \psi(x) \approx \delta\,\psi(x)$$

which precisely corresponds to the local change of the Dirac field operator under a change of inertial reference frame, where

$$U(\mathrm{a},\omega)\;\equiv\;\exp\left\{\frac{i}{\hbar}\,\mathrm{a}^{\,\mu}\,P_\mu - \frac{i}{2\hbar}\,\omega^{\,\rho\sigma}M_{\rho\sigma}\right\} \tag{4.84}$$

Moreover it is straightforward albeit tedious to verify that, by virtue of the canonical anti-commutation relations, the operators (4.82) do indeed fulfill the Lie algebra (1.44) of the Poincaré group – for the explicit check see [21] **8.6**, pp. 163 - 165. Alternatively, had we started from the symmetric classical Lagrangian (4.1), we come to the infinitesimal local variation of the spinorial operator valued tempered distribution

$$\begin{aligned}\delta\psi(x) &= i\,[\,P_\mu\,,\,\psi(x)\,]\,\epsilon^{\,\mu} - \tfrac{1}{2}\,i\,[\,M_{\rho\sigma}\,,\,\psi(x)\,]\,\epsilon^{\,\rho\sigma} \\ &= \big[\,\epsilon^{\,\mu}\,\partial_\mu + \tfrac{1}{2}\,\epsilon_{\mu\nu}\,\big(\,x^{\,\nu}\,\partial^{\,\mu} - x^{\,\mu}\,\partial^{\,\nu} + i\,\delta^{\,\mu}_{\,\jmath}\,\delta^{\,\nu}_{\,\ell}\sigma^{\,\jmath\ell}\,\big)\,\big]\,\psi(x)\end{aligned} \tag{4.85}$$

which is an equivalent though not identical form exhibiting the unitary nature of the quantum Poincaré transformation.

The covariant 1-particle states and creation annihilation operators can be defined in analogy with the construction (3.90) and take the form

$$|\,p\;r\;-\,\rangle = [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{\frac{1}{2}}\,c^{\,\dagger}_{\mathbf{p}\,,\,r}\,|\,0\,\rangle \equiv c^{\,\dagger}_r(p)\,|\,0\,\rangle \tag{4.86}$$
$$\forall\;\mathbf{p}\in\mathbb{R}^3 \qquad \forall\;r=1,2$$
$$|\,q\;s\;+\,\rangle = [\,(2\pi)^3\,2\omega_{\mathbf{q}}\,]^{\frac{1}{2}}\,d^{\,\dagger}_{\mathbf{q}\,,\,s}\,|\,0\,\rangle \equiv d^{\,\dagger}_s(p)\,|\,0\,\rangle \tag{4.87}$$
$$\forall\;\mathbf{q}\in\mathbb{R}^3 \qquad \forall\;s=1,2$$

which satisfy

$$\langle\,\pm\;s\;q\,|\,p\;r\;\pm\,\rangle \;=\; \delta_{\,rs}\,(2\pi)^3\,2\omega_{\,\mathbf{p}}\,\delta(\mathbf{p}-\mathbf{q})$$

The normal mode decomposition of the Dirac field with respect to the new set of covariant creation-annihilation operators becomes

$$\psi(x) = \sum_{r=1,2} \int \mathrm{D}p \left[c_r(p)\, u_r(p)\, \mathrm{e}^{-\,ipx} + d_r^\dagger(p)\, v_r(p)\, \mathrm{e}^{ipx} \right]_{p_0=\omega_{\mathbf{p}}} \tag{4.88}$$

$$\bar{\psi}(x) = \sum_{r=1,2} \int \mathrm{D}p \left[c_r^\dagger(p)\, \bar{u}_r(p)\, \mathrm{e}^{ipx} + d_r(p)\, \bar{v}_r(p)\, \mathrm{e}^{-\,ipx} \right]_{p_0=\omega_{\mathbf{p}}} \tag{4.89}$$

where $p_\mu = (\omega_{\mathbf{p}}\,,\,-\,\mathbf{p})$ together with

$$\int \mathrm{D}\,p = \int \mathrm{d}\mathbf{p}/(2\pi)^3\, 2\omega_{\mathbf{p}} = (2\pi)^{-3} \int \mathrm{d}^4p\; \theta(p_0)\, \delta(\,p^2 - M^2)$$

while the spin-states satisfy the algebraic equations

$$(\not p - M)\, u_r(p) = 0 = (\not p + M)\, v_r(p) \quad r = 1,2 \vee p_0 = \omega_{\mathbf{p}}$$

If we define

$$\begin{cases} \langle\, 0\,|\, \psi(x)\,|\, p\ r\ -\rangle \equiv u_{p,r}(x) = u_r(p)\, \mathrm{e}^{-\,ipx} \\ \langle +\ r\ p\,|\, \psi(x)\,|\, 0\,\rangle \equiv v_{p,r}(x) = v_r(p)\, \mathrm{e}^{ipx} \end{cases} \quad r = 1,2 \quad p_0 = \omega_{\mathbf{p}} \tag{4.90}$$

then we can write the inversion formulæ

$$(u_{p,r}\,,\,\psi)\ \equiv\ \int \mathrm{d}\mathbf{x}\ \bar{u}_{p,r}(t,\mathbf{x})\, \gamma^0\, \psi(t,\mathbf{x}) = c_r(p) \tag{4.91}$$

$$(\psi\,,\, v_{p,r})\ \equiv\ \int \mathrm{d}\mathbf{x}\ \bar{\psi}(t,\mathbf{x})\, \gamma^0\, v_{p,r}(t,\mathbf{x}) = d_r(p) \tag{4.92}$$

As a matter of fact we have for instance

$$\begin{aligned} (u_{p,r}\,,\,\psi) &= \int \mathrm{d}\mathbf{x}\ u_{p,r}^{\top\,*}(t,\mathbf{x})\, \psi(t,\mathbf{x}) \\ &= \sum_{s=1,2} \int \mathrm{D}\,q \left[c_s(q)\ (u_{p,r}\,,\, u_{s,q}) + d_s^\dagger(q)\ (u_{p,r}\,,\, v_{s,q}) \right] \\ &= \sum_{s=1,2} \int \mathrm{d}\mathbf{q}\ c_s(q)\, \delta(\mathbf{p}-\mathbf{q})\, \delta_{rs} = c_r(p) \qquad r = 1,2 \quad p_0 = \omega_{\mathbf{p}} \end{aligned}$$

The inversion formulas are based upon the existence of an inner product which turns out to be time-independent and Poincaré invariant for, otherwise, the notion of creation and destruction operators would be time and frame dependent and thereby meaningless. As a matter of fact, for a pair of classical complex spinor fields $\psi, \Psi \in \boldsymbol{\tau}_{\frac{1}{2}\,0} \bigoplus \boldsymbol{\tau}_{0\,\frac{1}{2}}$ that satisfy the Dirac equation, the quantity

$$(\psi \, , \, \Psi \,) = \int \mathrm{d}\mathbf{x} \; \psi_\alpha^*(t, \mathbf{x}) \, \Psi_\alpha(t, \mathbf{x})$$

turns out to be time independent and Poincaré invariant, owing to Nöther theorem and the equality (2.107), as well as positive semi-definite when $\psi = \Psi$. Hence, due to the IO(1,3) invariance of the measure $\mathrm{D}q$, from the previous relationship

$$(u_{\,p,r} \, , \, \psi) = \sum_{s=1,2} \int \mathrm{D}q \, \Big[\, c_{\,s}(q) \; (u_{\,p,r} \, , \, u_{\,s,q}) + d_{\,s}^{\dagger}(q) \; (u_{\,p,r} \, , \, v_{\,s,q}) \, \Big] = c_{\,r}(p)$$

it is evident that consistency requires the complex or Graßmann valued Fourier coefficients $c_{\,r}(p), d_{\,r}(p)$ to be inertial frame independent entities, for any classical Dirac spinor ψ. Of course, the very same Relativity argument holds true after transition to the Quantum Theory, because the canonical commutation and anti-commutation relations must be Poincaré invariant as it does. As a consequence we can suitably set the infinitesimal local variation

$$\delta c_{\,r}(\,p\,) = \left[\, - \, i\epsilon^{\,\mu} \, p_\mu + \tfrac{1}{2} \, \epsilon_{\,\mu\nu} \left(p^{\,\nu} \, \frac{\partial}{\partial \, p_\mu} - p^{\,\mu} \, \frac{\partial}{\partial \, p_\nu} \right) \right] c_{\,r}(p) \qquad (4.93)$$

which exactly corresponds to the infinitesimal form of the passive Poincaré transformation (3.104) of the destruction operators of the Klein-Gordon scalar quantum field. Thus, turning to the finite transformation, we are definitely allowed to write

$$\begin{aligned} & U(\omega, \mathrm{a}) \, c_{\,r}(\,p\,) \, U^{\,-1}(\omega, \mathrm{a}) \;=\; \exp\{ - \, i \, p \cdot \mathrm{a} \} \, c_{\,r}(\Lambda \, p) \\ =\; & (1 - i \, p_\mu \, \mathrm{a}^{\,\mu} + \cdots) \left(1 - \omega^{\mu\nu} \, p_\nu \, \frac{\partial}{\partial p^\mu} + \cdots \right) c_{\,r}(\,p\,) \\ =\; & \left\{ 1 - i \, p_\mu \, \mathrm{a}^{\,\mu} + \tfrac{1}{2} \, \omega^{\mu\nu} \left(p_\nu \, \frac{\partial}{\partial p^\mu} - p_\mu \, \frac{\partial}{\partial p^\nu} \right) \right\} c_{\,r}(\,p\,) + \cdots \end{aligned}$$

and thereby

$$\left\{ \begin{array}{lcl} c_{\,r}^{\prime}(\,p\,) \;\equiv\; U(\omega, \mathrm{a}) \, c_{\,r}(\,p\,) \, U^{\,-1}(\omega, \mathrm{a}) & = & c_{\,r}(\Lambda \, p) \exp\{ - \, i \, p \cdot \mathrm{a} \} \\ d_{\,r}^{\prime}(\,p\,) \;\equiv\; U(\omega, \mathrm{a}) \, d_{\,r}(\,p\,) \, U^{\,-1}(\omega, \mathrm{a}) & = & d_{\,r}(\Lambda \, p) \exp\{ - \, i \, p \cdot \mathrm{a} \} \end{array} \right.$$

with $r = 1, 2$ and $p_0 = \omega_{\mathbf{p}}$, that endorse the invariance of the creation and destruction operators of a quantum Dirac field under homogeneous Lorentz transformations. Of course, for the inverse Poincaré unitary transformation we get instead

$$\left\{ \begin{array}{lcl} c_{\,r}^{\prime}(\,p^{\prime}\,) \;\equiv\; U^{\,-1}(\omega, \mathrm{a}) \, c_{\,r}(\,\Lambda \, p\,) \, U(\omega, \mathrm{a}) & = & c_{\,r}(p) \exp\{ \, i \, p \cdot \mathrm{a} \} \\ d_{\,r}^{\prime}(\,p^{\prime}\,) \;\equiv\; U^{\,-1}(\omega, \mathrm{a}) \, d_{\,r}(\,\Lambda \, p\,) \, U(\omega, \mathrm{a}) & = & d_{\,r}(p) \exp\{ \, i \, p \cdot \mathrm{a} \} \end{array} \right. \qquad (4.94)$$

$$\forall\, r = 1,2 \,\vee\, \mathbf{p} \in \mathbb{R}^3 \quad \text{with} \quad p_0 = \hbar\omega_{\mathbf{p}}/c$$

It is clear that the transformation law for the covariant 1-particle states and corresponding creation annihilation operators will be determined by the unitary operators associated to the Poincaré transformations. If we denote as usual the Lorentz matrices by $\Lambda(\omega) = \Lambda(\boldsymbol{\alpha}, \boldsymbol{\eta})$ and by $U(\omega, \mathrm{a}) = U(\boldsymbol{\alpha}, \boldsymbol{\eta}, \mathrm{a}^\nu)$ the related unitary operators (4.84) of the infinite dimensional irreducible representation of Poincaré group acting on the Fock space $\mathcal{F}$, where $(\boldsymbol{\alpha}, \boldsymbol{\eta})$ are the canonical angular and rapidity coordinates of the restricted Lorentz group $L_+^\uparrow = O(1,3)_+^\uparrow$, then we evidently get

$$U(\mathrm{a}, \omega)\,|\,0\,\rangle = |\,0\,\rangle \qquad \langle\, 0\,|\,U^\dagger(\mathrm{a}, \omega) = \langle\, 0\,|\,U^{-1}(\mathrm{a}, \omega) = \langle\, 0\,| \tag{4.95}$$

owing to the normal ordering prescription, whence we get the transformation laws for the 1-particle states

$$\begin{aligned} & U(\omega, \mathrm{a})\,|\,p\ r\ -\rangle = U(\omega, \mathrm{a})\, c_r^\dagger(p)\,|\,0\,\rangle \\ = \;& U(\omega, \mathrm{a})\, c_r^\dagger(p)\, U^{-1}(\mathrm{a}, \omega)\, U(\omega, \mathrm{a})\,|\,0\,\rangle = U(\omega, \mathrm{a})\, c_r^\dagger(p)\, U^\dagger(\mathrm{a}, \omega)\,|\,0\,\rangle \\ = \;& \exp\{\, i\, p \cdot \mathrm{a}\}\, c_r^\dagger(\Lambda\, p)\,|\,0\,\rangle = \exp\{\, i\, p \cdot \mathrm{a}\}\,|\,\Lambda\, p\ r\ -\rangle \end{aligned}$$

so that

$$U(\omega, \mathrm{a})\,|\,p\ r\ \mp\rangle \;=\; \exp\{\, i\, p \cdot \mathrm{a}\}\,|\,\Lambda\, p\ r\ \mp\rangle \tag{4.96}$$

$$\langle\, \pm\ s\ \Lambda\, q\,|\,\Lambda\, p\ r\ \pm\,\rangle = \delta_{rs}\,(2\pi)^3\, 2\omega_{\mathbf{p}}\, \delta\,(\mathbf{p} - \mathbf{q}) = \langle\, \pm\ s\ q\,|\,p\ r\ \pm\,\rangle$$

$$\forall\, r = 1,2 \,\vee\, \mathbf{p} \in \mathbb{R}^3 \quad \text{with} \quad p_0 = \hbar\omega_{\mathbf{p}}/c$$

It is a fundamental assumption of the local Quantum Field Theory that if a Poincaré transformation takes the event from $x^\mu = (ct, \mathbf{x})$ to $x'^\mu = (ct', \mathbf{x}')$ with $x' = \Lambda(x + \mathrm{a})$ and any arbitrary Fock state $|\,\Upsilon\,\rangle$ into the new state $|\,\Upsilon'\,\rangle \equiv U(\omega, \mathrm{a})\,|\,\Upsilon\,\rangle$, where U is a unitary operator, then the components of $\psi(ct', \mathbf{x}')\,|\,\Upsilon'\,\rangle$ MUST BE RELATED BY A LINEAR TRANSFORMATION to the components of $U(\omega, \mathrm{a})\,\psi(ct, \mathbf{x})\,|\,\Upsilon\,\rangle$: namely,

$$\psi_\alpha(x')\,|\,\Upsilon'\,\rangle = A_{\alpha\beta}\, U(\omega, \mathrm{a})\,\psi_\beta(x)\,|\,\Upsilon\,\rangle$$

where $\alpha, \beta = L1, L2, R1, R2$ are the the spinor indexes while $A_{\alpha\beta}$ complex number to be further specified. To this concern, one important remark is in order: the Dirac spinor field is a local quantum field, *i.e.* an operator valued tempered distribution which contains both an operator part as well as a $c-$number part involving the spinor plane wave functions. Thus, when we write $\psi(ct', \mathbf{x}')$ we understand that the whole space-time dependent part – which is of a $c-$number kind – of the local quantum field is evaluated at the point $x'^{\mu} = (ct', \mathbf{x}')$: namely,

$$\psi(ct', \mathbf{x}') = \sum_{\mathbf{p}\,,\,r} \left[c_{\mathbf{p}\,,\,r}\, u'_{\mathbf{p}\,,\,r}\,(x') + d^{\dagger}_{\mathbf{p}\,,\,r}\, v'_{\mathbf{p}\,,\,r}\,(x') \right]$$

Hence, the field operator must transform according to

$$\psi(ct', \mathbf{x}')\, U(\omega, \mathrm{a})\,|\,\Upsilon\,\rangle = \mathbb{A}\, U(\omega, \mathrm{a})\, \psi(ct, \mathbf{x})\,|\,\Upsilon\,\rangle \qquad \forall\,|\,\Upsilon\,\rangle \in \mathcal{F}$$

or

$$\psi'(x') \equiv U^{\dagger}(\omega, \mathrm{a})\, \psi(x')\, U(\omega, \mathrm{a}) = \mathbb{A}\, \psi(x) \tag{4.97}$$

where $\mathbb{A}$ is a 4×4 matrix – which commutes by definition with the unitary operator $U(\omega, \mathrm{a})$ – that specifies the algebraic transformation properties of the spinor, the components of which, like those of a vector or tensor, are indeed reshuffled in the symmetry operation.

Thus, from the unitary similarity transformation (4.94) for the creation-destruction operators and the transformation rule (4.25) for the spin states, we can eventually write

$$\begin{aligned}
\psi'(x') \;&\equiv\; U^{-1}(\omega, \mathrm{a})\, \psi\,(x')\, U(\omega, \mathrm{a}) \\
&=\; \sum_{r=1,2} \int \mathrm{D}\,p' \left[U^{\dagger}(\omega, \mathrm{a})\, c_r\,(p')\, U(\omega, \mathrm{a})\, u'_r\,(p')\, \mathrm{e}^{-\,ip'\cdot x'} \right. \\
&+\; \left. U^{\dagger}(\omega, \mathrm{a})\, d^{\dagger}_r\,(p')\, U(\omega, \mathrm{a})\, v'_r\,(p')\, \mathrm{e}^{ip'\cdot x'} \right]_{p'_0\,=\,\omega_{\mathbf{p}'}} \\
&=\; \sum_{r=1,2} \int \mathrm{D}\,(\Lambda p) \left[c_r\,(p)\, \Lambda_{\frac{1}{2}}(\omega)\, u_r\,(p)\, \mathrm{e}^{-\,ip\cdot x} \right. \\
&+\; \left. d^{\dagger}_r\,(p)\, \Lambda_{\frac{1}{2}}(\omega)\, v_r\,(p)\, \mathrm{e}^{ip\cdot x} \right]_{p_0\,=\,\omega_{\mathbf{p}}} \\
&=\; \Lambda_{\frac{1}{2}} \sum_{r=1,2} \int \mathrm{D}\,p \left[c_r\,(p)\, u_r\,(p)\, \mathrm{e}^{-\,ip\cdot x} + d^{\dagger}_r\,(p)\, v_r\,(p)\, \mathrm{e}^{ip\cdot x} \right]_{p_0\,=\,\omega_{\mathbf{p}}} \\
&=\; \Lambda_{\frac{1}{2}}(\omega)\, \psi\,(x)
\end{aligned} \tag{4.98}$$

which means that the requirement of the covariance for the Dirac equation satisfied by the operator valued tempered distribution $\psi(x)$ does actually implies

$$\mathbb{A} = \Lambda_{\frac{1}{2}}(\omega)$$

which holds true for any Poincaré transformations working contextually upon space-time events and quantum states of the system in some given inertial reference frame. This is the way for the Poincaré covariance is implemented in relativistic quantum field theory by UNITARY OPERATORS acting upon the Fock space of quantum states, in spite of the presence of the 4×4 matrix $\Lambda_{\frac{1}{2}}(\omega)$ belonging the representation $\boldsymbol{\tau}_{0\,\frac{1}{2}} \oplus \boldsymbol{\tau}_{\frac{1}{2}\,0}$ of the Lorentz group, which is not unitary. In particular, from the finite transformation rule it is simple to check *e.g.* that the mass operator is Lorentz invariant, *i.e.*

$$\overline{\psi'}(x')\psi'(x') = \overline{\psi}(x)\psi(x)$$

As a final remark, notice that the density

$$\varrho\,(x) = \psi^{\dagger}(t, \mathbf{x})\,\psi(t, \mathbf{x})$$

does represent the positive semi-definite probability density in the old Dirac theory, *i.e.* the relativistic quantum mechanics of the electron. Conversely, the corresponding operator valued local density

$$\widehat{\varrho}\,(x) = q : \psi^{\dagger}(t, \mathbf{x})\,\psi(t, \mathbf{x}) :$$

has not a definite sign for it represents the charge density of the quantized spinor field, see equation (4.74).

Beside the transformation laws of the quantized Dirac field under the continuous Poincaré group, a very important role in the Standard Model of the fundamental interactions for Particle Physics is played by the charge conjugation, parity and time reversal discrete transformations, the so called CPT symmetries, on the quantized spinor matter fields. We shall analyze in details the latter ones at the end of the present chapter.

4.5 Special Distributions

From the canonical anti-commutation relations (4.62) and the normal mode expansion (4.30) of the Dirac field, the so called canonical anti-commutator at arbitrary points between two free Dirac spinor field operators can be readily shown to be equal to zero

$$\{\psi_\alpha(x)\,,\,\psi_\beta(y)\} = 0 = \{\bar\psi_\alpha(x)\,,\,\bar\psi_\beta(y)\} \qquad (\,\alpha,\beta = 1L, 2L, 1R, 2R\,)$$

On the contrary, the canonical anti-commutator at arbitrary points between the free Dirac field and its adjoint does not vanish: it can be easily calculated to be

$$\begin{aligned}\{\psi_\alpha(x)\,,\,\bar\psi_\beta(y)\} &\equiv S_{\alpha\beta}(x-y)\\ &= \sum_{\mathbf{p}\,,\,r}\left[\,u_{\,\alpha\,,\,\mathbf{p}\,,\,r}(x)\,\bar u_{\,\beta\,,\,\mathbf{p}\,,\,r}(y) + v_{\,\alpha\,,\,\mathbf{p}\,,\,r}(x)\,\bar v_{\,\beta\,,\,\mathbf{p}\,,\,r}(y)\,\right]\\ &= -\,i\,(\,i\partial\!\!\!/_x + M\,)_{\alpha\beta}\,D(x-y) \qquad (4.99)\end{aligned}$$

where $D(x-y)$ is the Pauli-Jordan distribution (3.112).

`Proof.` From the normal modes expansions (4.64) of the spinor fields

$$\begin{aligned}\psi_\alpha(x) &= \sum_{\mathbf{p}\,,\,r}\left[\,(2\pi)^3 2\omega_{\mathbf{p}}\,\right]^{-\,\frac12}\\ &\times\ \left[c_{\mathbf{p}\,,\,r}\,u_{\,\alpha\,,\,r}(\mathbf{p})\,\mathrm{e}^{-\,i\,p\,x} + d^{\,\dagger}_{\mathbf{p}\,,\,r}\,v_{\,\alpha\,,\,r}(\mathbf{p})\,\mathrm{e}^{\,i\,p\,x}\right]_{p_0\,=\,\omega_{\mathbf{p}}}\\ \bar\psi_\beta(y) &= \sum_{\mathbf{q}\,,\,s}\left[\,(2\pi)^3 2\omega_{\mathbf{q}}\,\right]^{-\,\frac12}\\ &\times\ \left[c^{\,\dagger}_{\mathbf{q}\,,\,s}\,\bar u_{\,\beta\,,\,s}(\mathbf{q})\,\mathrm{e}^{\,i\,q\,y} + d_{\,\mathbf{q}\,,\,s}\,\bar v_{\,\beta\,,\,s}(\mathbf{q})\,\mathrm{e}^{-\,i\,q\,y}\right]_{q_0\,=\,\omega_{\mathbf{q}}}\end{aligned}$$

and the canonical anti-commutation relations (4.62) one finds

$$\begin{aligned}\{\psi_\alpha(x)\,,\,\bar\psi_\beta(y)\} &= \int \mathrm{D}p\sum_{r\,=\,1,2}\Big[\,u_{\,\alpha\,,\,r}(\mathbf{p})\,\bar u_{\,\beta\,,\,r}(\mathbf{p})\,\mathrm{e}^{-\,i\,p\,(x-y)}\\ &+\ v_{\,\alpha\,,\,r}(\mathbf{p})\,\bar v_{\,\beta\,,\,r}(\mathbf{p})\,\mathrm{e}^{\,i\,p\,(x-y)}\Big]_{p_0\,=\,\omega_{\mathbf{p}}}\end{aligned}$$

Taking into account the sums over the spin states (4.37)

$$\sum_{r\,=\,1,2}\left\{\begin{array}{l} u_{\,\beta\,,\,r}(\mathbf{p})\,\bar u_{\,\alpha\,,\,r}(\mathbf{p}) = (\,p\!\!\!/ + M\,)_{\,\alpha\beta}\\ v_{\,\beta\,,\,r}(\mathbf{p})\,\bar v_{\,\alpha\,,\,r}(\mathbf{p}) = (\,p\!\!\!/ - M\,)_{\,\alpha\beta}\end{array}\right. \qquad (\,p_0 = \omega_{\mathbf{p}}\,)$$

we readily come to the expression, by omitting the spinor indexes for the sake of brevity,

$$\{\psi(x)\,,\,\bar\psi(y)\} \ \equiv\ S(x-y)$$

$$
\begin{aligned}
&= \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\left(\frac{\not p + M}{2\omega_{\mathbf{p}}}\,\mathrm{e}^{-\,ip\,(x-y)} - \mathrm{e}^{\,ip\,(x-y)}\,\frac{M - \not p}{2\omega_{\mathbf{p}}}\right)_{p_0\,=\,\omega_{\mathbf{p}}} \\
&= (i\not\partial_x + M)\int \mathrm{D}p\,\left(\mathrm{e}^{-\,ip\,(x-y)} - \mathrm{e}^{\,ip\,(x-y)}\right)_{p_0\,=\,\omega_{\mathbf{p}}} \\
&= (i\not\partial_x + M)\int \frac{\mathrm{d}^4 p}{(2\pi)^3}\,\mathrm{e}^{-\,ip\,(x-y)}\,\delta\,(p^2 - M^2)\,\mathrm{sgn}\,(p_0) \\
&= -\,i\,(i\not\partial_x + M)\,D\,(x-y) \qquad (4.100)
\end{aligned}
$$

where use has been made of the formulæ (4.36). □

The canonical anti-commutator at arbitrary points (4.100) is a solution of the Dirac equation which does not vanish when $(x-y)$ is a space-like interval with $x^0 \neq y^0$. In fact, at variance with the real scalar field case in which

$$
D\,(x-y) \equiv 0 \qquad \forall\,(x_0 - y_0)^2 < (\mathbf{x}-\mathbf{y})^2 \qquad (x^0 \neq y^0) \tag{4.101}
$$

we find instead the non-vanishing equal time anti-commutator

$$
S\,(0, \mathbf{x}-\mathbf{y}) = \gamma^0\,\delta\,(\mathbf{x}-\mathbf{y}) \tag{4.102}
$$

and more generally the spinor micro-causality property

$$
\gamma^0\,S\,(x-y) = \dot{D}(x-y) \qquad (x-y)^2 < 0 \tag{4.103}
$$

in agreement with (4.66).

The causal Green's function or Feynman propagator for the Dirac field is

$$
\begin{aligned}
\langle\,0\,|\,T\,\psi_\alpha\,(x)\,\bar\psi_\beta\,(y)\,|\,0\,\rangle &= \begin{cases} \langle\,0\,|\,\psi_\alpha\,(x)\,\bar\psi_\beta\,(y)\,|\,0\,\rangle & \text{for } x^0 > y^0 \\ -\,\langle\,0\,|\,\bar\psi_\beta\,(y)\,\psi_\alpha\,(x)\,|\,0\,\rangle & \text{for } x^0 < y^0 \end{cases} \\
&\equiv i\,S^c_{\alpha\beta}\,(x-y) = S^F_{\alpha\beta}\,(x-y)
\end{aligned}
$$

Actually we have

$$
S^F\,(x-y) = (i\not\partial_x + M)\;D_F\,(x-y) \tag{4.104}
$$

where $D_F\,(x-y)$ is the Feynman propagator of the real scalar field.

`Proof.` From the very definition as the vacuum expectation value of the chronological product of spinor fields we can write

$$
\begin{aligned}
S^F\,(x-y) &= \theta\,(x^0-y^0)\,\langle\,\psi\,(x)\,\bar\psi\,(y)\,\rangle_0 - \theta\,(y^0-x^0)\,\langle\,\bar\psi\,(y)\,\psi\,(x)\,\rangle_0 \\
&= \theta\,(x^0-y^0)\sum_{\mathbf{p}\,,\,r}[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1}\,u_r\,(\mathbf{p})\otimes\bar u_r\,(\mathbf{p}) \\
&\times\ \exp\{-\,i\omega_{\mathbf{p}}\,(x^0-y^0) + i\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\} \\
&-\ \theta\,(y^0-x^0)\sum_{\mathbf{p}\,,\,r}[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1}\,v_r\,(\mathbf{p})\otimes\bar v_r\,(\mathbf{p}) \\
&\times\ \exp\{+\,i\omega_{\mathbf{p}}\,(x^0-y^0) -\,i\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\} \qquad (4.105)
\end{aligned}
$$

Hence, from the sums over the spin-states (4.36) we obtain

$$\begin{aligned}
S^{F}(x-y) &= \theta(x^0-y^0)\int Dp\ \Big[(M+\not p)\,\mathrm{e}^{-\,i\,p\,(x-y)}\Big]_{p_0\,=\,\omega_{\mathbf p}}\\
&+\ \theta(y^0-x^0)\int Dp\ \Big[(M-\not p)\,\mathrm{e}^{\,i\,p\,(x-y)}\Big]_{p_0\,=\,\omega_{\mathbf p}}\\
&= \theta(x^0-y^0)\ (i\not\partial_x+M)\int Dp\ \ \mathrm{e}^{-\,i\,p\,(x-y)}\Big|_{p_0\,=\,\omega_{\mathbf p}}\\
&+\ \theta(y^0-x^0)\ (i\not\partial_x+M)\int Dp\ \ \mathrm{e}^{\,i\,p\,(x-y)}\Big|_{p_0\,=\,\omega_{\mathbf p}}
\end{aligned}$$

and if we recall the definitions

$$\begin{aligned}
\pm\,iD^{(\pm)}(x-y) &= \int Dp\ \ \mathrm{e}^{\pm\,i\,p\,(x-y)}\Big|_{p_0\,=\,\omega_{\mathbf p}}\\
&= \int\frac{\mathrm{d}\mathbf p}{(2\pi)^3\,2\omega_{\mathbf p}}\ \exp\{\pm i\omega_{\mathbf p}(x^0-y^0)\mp i\,\mathbf p\cdot(\mathbf x-\mathbf y)\}
\end{aligned}$$

we can readily establish the equality

$$\begin{aligned}
& S^{F}(x-y)=\theta(x^0-y^0)\ (i\not\partial_x+M)\ \frac{1}{i}\,D^{(-)}(x-y)\\
+&\ \ i\,\theta(y^0-x^0)\ (i\not\partial_x+M)\ D^{(+)}(x-y)\\
=&\ \ (i\not\partial_x+M)\ i\,\theta\big(y^0-x^0\big)\ D^{(+)}(x-y)-(i\not\partial_x+M)\ i\,\theta\big(x^0-y^0\big)\ D^{(-)}(x-y)\\
-&\ \ \gamma^0\,\delta\big(x^0-y^0\big)\ \Big[D^{(-)}(x-y)+D^{(+)}(x-y)\Big]\ =\ (i\not\partial_x+M)\ D_F(x-y)
\end{aligned}$$

where I did make use of the Pauli-Jordan distribution property

$$D(x)\ =\ D^{(+)}(x)+D^{(-)}(x)\,,\qquad\qquad D(0,\mathbf x)=0$$

and of the relation (3.119)

$$D_F(x-y)=i\,\theta\big(y^0-x^0\big)\ D^{(+)}(x-y)-i\,\theta\big(x^0-y^0\big)\ D^{(-)}(x-y)$$

that was what we had to prove. □

The Fourier representation of the Feynman's spinor propagator reads

$$\begin{aligned}
S^{F}(x-y;\,M) &= (i\not\partial_x+M)\,D_F(x-y;\,M)\\
&= \frac{i}{(2\pi)^4}\int\mathrm{d}^4p\ \frac{\not p+M}{p^2-M^2+i\varepsilon}\ \exp\{-\,ip\cdot(x-y)\}\\
&= \frac{1}{(2\pi)^4}\int\mathrm{d}^4p\ \left(\frac{i}{\not p-M}\right)\ \exp\{-\,ip\cdot(x-y)\}
\end{aligned}$$

with

$$\frac{i\,(\,\not p+M\,)_{\alpha\beta}}{p^2-M^2+i\varepsilon}\ \stackrel{\mathrm{def}}{=}\ \left(\frac{i}{\not p-M}\right)_{\alpha\beta}\tag{4.106}$$

and consequently

$$(i\partial\!\!\!/_x - M)\, S^F(x-y) \;=\; i\hbar c\, \delta(x-y) \tag{4.107}$$

It is convenient to write the adjoint form of the non-homogeneous equation for the Feynman propagator of the spinor field. To this concern, let us first obtain the Hermitean conjugate of equation (4.107) *viz.,*

$$\begin{aligned} i\,\delta(x-y) &= i\,(\partial/\partial x^\mu)\, S_F^\dagger(x-y)\,\gamma^{\mu\,\dagger} + M\, S_F^\dagger(x-y) \\ &= i\,(\partial/\partial x^\mu)\, S_F^\dagger(x-y)\,\gamma^0\,\gamma^\mu\,\gamma^0 + M\, S_F^\dagger(x-y) \end{aligned} \tag{4.108}$$

Multiplication by γ^0 from left and right yields

$$\begin{aligned} i\,\delta(x-y) &= i\,\gamma^0\,(\partial/\partial x^\mu)\, S_F^\dagger(x-y)\,\gamma^0\,\gamma^\mu + \gamma^0\, M\, S_F^\dagger(x-y)\,\gamma^0 \\ &\stackrel{\rm def}{=} \bar{S}^F(y-x)\,(\,i\overleftarrow{\partial\!\!\!/}_x + M\,) \end{aligned} \tag{4.109}$$

where

$$\begin{aligned} \bar{S}^F(y-x) &= \gamma^0\, S_F^\dagger(x-y)\,\gamma^0 \\ = \frac{-\,i}{(2\pi)^4} &\int \mathrm{d}p\; \frac{p\!\!\!/ + M}{p^2 - M^2 - i\varepsilon}\; \exp\{-\,i\,p\cdot(y-x)\} \end{aligned} \tag{4.110}$$

4.6 Euclidean Fermions

Also for the above fermion propagator we can safely perform the very same replacements

$$i\,p^0 = p_4 \qquad\qquad i x^0 = x_4 \tag{4.111}$$

as in the scalar field case, in such a manner to obtain a positive semi-definite denominator in the Fourier transform, that is

$$\begin{aligned} S^F_{\alpha\beta}(\,-i\,x_4, \mathbf{x}) \;&= i\,(2\pi)^{-4} \int \mathrm{d}\,\mathbf{p} \int_{-\infty}^{\infty} \mathrm{d}\,p_4 \\ \frac{\exp\{i\,p_4\,x_4 + i\,p_\jmath\,x_\jmath\}}{p_4^2 + \mathbf{p}^2 + M^2}\; &(\,\gamma^0\,p_4 - i\,\gamma^k\,p^k + i\,M\,)_{\alpha\beta} \end{aligned}$$

It follows that if we define the Hermitean Euclidean Dirac Matrices

$$\overline{\gamma}_4 \;\equiv\; \gamma^0 \qquad \overline{\gamma}_k \;\equiv\; i\,\gamma_k = -\,i\,\gamma^k \qquad\qquad (\,k = 1,2,3\,) \tag{4.112}$$

$$\overline{\gamma}_\mu = (\overline{\gamma}_k\,,\,\overline{\gamma}_4) \;=\; \overline{\gamma}_\mu^\dagger \qquad\qquad \left\{\overline{\gamma}_\mu\,,\,\overline{\gamma}_\nu\right\} = 2\,\delta_{\mu\nu} \tag{4.113}$$

or even more explicitly

$$\overline{\gamma}_1 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{4.114}$$

$$\overline{\gamma}_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.115}$$

$$\overline{\gamma}_3 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{4.116}$$

$$\overline{\gamma}_4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.117}$$

Then we can write

$$\begin{aligned} S^F_{\alpha\beta}(-ix_4, \mathbf{x}) &= \frac{i}{(2\pi)^4} \int \mathrm{d}p_E \, \frac{\exp\{ip_E \cdot x_E\}}{p_E^2 + M^2} \, (\overline{\gamma}_\mu p_{E\mu} + iM)_{\alpha\beta} \\ &= \frac{i}{(2\pi)^4} \int \mathrm{d}p_E \, \exp\{ip_E \cdot x_E\} \left(\frac{1}{\not{p}_E - iM} \right)_{\alpha\beta} \\ &= (\not{\partial}_E - M)_{\alpha\beta} \, D_E(x_E) \\ &\equiv -S^E_{\alpha\beta}(x_E) \end{aligned} \tag{4.118}$$

where we understand

$$\mathbf{p} = (p^1, p^2, p^3) = (p_x, p_y, p_z) = \vec{p}_E = (\bar{p}_1, \bar{p}_2, \bar{p}_3)$$

together with

$$\begin{aligned} \overline{\gamma}_4 p_4 + \overline{\gamma}_k \bar{p}_k &= \overline{\gamma}_\mu p_{E\mu} \equiv \not{p}_E \\ i\not{\partial}_E \equiv \overline{\gamma}_\mu i\partial_\mu &= \overline{\gamma}_k i\partial_k + \overline{\gamma}_4 i\partial_4 \end{aligned}$$

This suggests that the proper classical variables for setting up of an Euclidean formulation of the Dirac spinor field theory, in analogy with what we have already seen in the scalar field case, should be two Euclidean bispinors ψ_E and $\bar{\psi}_E$ obeying

$$\{\psi_E(x), \psi_E(y)\} = \{\bar{\psi}_E(x), \bar{\psi}_E(y)\} = \{\psi_E(x), \bar{\psi}_E(y)\} = 0 \tag{4.119}$$

for all points x and y of the four-dimensional Euclidean space $\mathbb{R}^4$.

The last of these relations is crucial, for it implies that $\bar{\psi}_E$ does not necessarily coincide with the adjoint of ψ_E times some matrix γ_4 . Thus, if we want to set up a meaningful Euclidean formulation for the Dirac spinor field theory, then we can treat ψ_E and $\bar{\psi}_E$ as **totally independent classical Graßmann valued fields** . This independence is the main novelty of the Euclidean fermion field theory; the rest of the construction is straightforward.

For instance, we use the definition of the Hermitean Euclidean gamma matrices to derive the $O(4)$ transformation law for ψ_E in the usual way[3], while define $\bar{\psi}_E$ to transform like the transposed of ψ_E . Next we define

$$\overline{\gamma}_5 \;=\; \overline{\gamma}_1\overline{\gamma}_2\overline{\gamma}_3\overline{\gamma}_4 \;=\; \overline{\gamma}_5^{\,\dagger} \;=\; -\,\gamma_5$$

Thus $\bar{\psi}_E\,\psi_E$ is a scalar, $\bar{\psi}_E\overline{\gamma}_5\psi_E$ a pseudo-scalar, $\bar{\psi}_E\overline{\gamma}_\mu\psi_E$ a vector *etc.*

The Euclidean Action for the free Dirac field is given by

$$S_E\,[\,\psi_E,\bar{\psi}_E\,] \;=\; \int \mathrm{d}^4x_E\;\bar{\psi}_E(x_E)\,(\bar{\partial\!\!\!/}_E + M)\;\psi_E(x_E) \tag{4.120}$$

Here the overall sign is purely conventional: we could always absorb it into ψ_E if we wanted to – remember that we are free to change ψ_E without touching $\bar{\psi}_E$. Conversely, the lack of the factor i in front of the derivative term is not at all conventional: it is there just to ensure that the Euclidean fermion propagator, which is named 2-point *Schwinger's function*, is proportional to $(i\not{p}_E - M)/(p_E^2 + M^2)$; if it were not for this i , then we would have tachyon poles after transition back to the momentum space.

It is worthwhile to notice that the above Dirac Euclidean Action can be obtained from the corresponding one in the Minkowski space, after the customary standard replacements (4.111) and (4.112)

$$x_4 = i\,x_0 \qquad \overline{\gamma}_k \;\equiv\; -\,i\gamma^k \quad (\,k = 1,2,3\,) \qquad \overline{\gamma}_4 \;\equiv\; \gamma^0$$

$$\psi\,(x) \;\mapsto\; \psi_E\,(x_E) \qquad\qquad \bar{\psi}\,(x) \;\mapsto\; \bar{\psi}_E\,(x_E)$$

As a matter of fact we readily obtain

$$\begin{aligned} i\,S\,[\,\psi,\bar{\psi}\,] \;&\mapsto\; -\,S_E\,[\,\psi_E,\bar{\psi}_E\,] \\ &=\; -\int \mathrm{d}^4x_E\;\bar{\psi}_E\,(x_E)\;(\bar{\partial\!\!\!/}_E + M\,)\;\psi_E\,(x_E) \end{aligned} \tag{4.121}$$

[3]Remember that the orthogonal group $O(4)$ of the rotations in the Euclidean space $\mathbb{R}^4$ is a semi-simple Lie group $O(4) = O(3)_L \times O(3)_R$.

Furthermore, the Euclidean Dirac operator $(\not{\partial}_E + M)$ is precisely that one which gives, according to the definition (4.118), the 2-point Schwinger function inversion formula[4]

$$(\not{\partial}_E + M)_{\alpha\beta}\, S^E_{\beta\eta}(x_E) \;=\; \delta(x_E)\,\delta_{\alpha\eta}$$

Finally, it is worthwhile to remark that the Euclidean Action for the free Dirac field is not a real quantity. As we shall see, this fact will not cause any troubles in the analytic continuation to the Minkowski space.

To end up we have

$$\begin{aligned} S^0_E[\phi_E] &= \int \mathrm{d}x_E\;\left[\tfrac{1}{2}\,\partial_\mu\phi_E\,\partial_\mu\phi_E + \tfrac{1}{2}\,m^2\,\phi_E^2\right] \\ &\doteq \int \mathrm{d}x_E\;\tfrac{1}{2}\,\phi_E(x_E)\left(-\,\partial^2 + m^2\right)\phi_E(x_E) \qquad (4.122) \\ S^0_E[\bar\psi\,,\,\psi_E] &= \int \mathrm{d}x_E\;\bar\psi(x_E)\,(\not{\partial}_E + M)\,\psi_E(x_E) \qquad (4.123) \end{aligned}$$

in such a manner that we can summarize the useful relationships

$$D_F(-\,i\,x_4\,,\,\mathbf{x}) \;\to\; -\,D_E(x_E) \qquad (4.124)$$

$$D_E(x_E) \;=\; \frac{1}{(2\pi)^4}\int \mathrm{d}k_E\;\frac{\exp\{\,i\,k_E\cdot x_E\,\}}{k_E^2 + m^2} \qquad (4.125)$$

$$\left(-\,\partial_E^2 + m^2\right)\,D_E(x_E) \;=\; \delta(x_E) \qquad (4.126)$$

$$S^F_{\alpha\beta}(-i\,x_4,\mathbf{x}) \;\to\; -\,S^E_{\alpha\beta}(x_E) \qquad (4.127)$$

$$S^E_{\alpha\beta}(x_E) \;=\; \int \frac{\mathrm{d}p_E}{(2\pi)^4}\;\exp\{\,ip_E\cdot x_E\,\}\;\left(\frac{i}{-\,\not{p}_E + iM}\right)_{\alpha\beta} \qquad (4.128)$$

$$(\not{\partial}_E + M)_{\alpha\beta}\,S^E_{\beta\eta}(x_E) \;=\; \delta(x_E)\,\delta_{\alpha\eta} \qquad (4.129)$$

[4]The $n-$point Green functions in the Euclidean space are usually named the $n-$point Schwinger functions.

4.7 The $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ Transformations

The charge conjugation $\mathcal{C}$ is the discrete internal symmetry transformation under which the particles and antiparticles are interchanged. The parity transformation $\mathcal{P}$ or spatial inversion is the discrete space-time symmetry transformation such that

$$x^{\mu} = (t, \mathbf{r}) \mapsto x'^{\mu} = (t, -\mathbf{r}) = \mathcal{P}\, x^{\mu}$$

Under $\mathcal{P}$ the handedness of the particles motion is reversed so that, for example, a left handed electron e_L^- is transformed into a right handed positron e_R^+ under the combined $\mathcal{CP}$ symmetry transformation. Thus, if $\mathcal{CP}$ were an exact symmetry, the laws of Nature would be the same for matter and for antimatter.

The experimental evidence actually shows that most phenomena in the particles Physics are $\mathcal{C}$ and $\mathcal{P}$ symmetric and thereby also $\mathcal{CP}$ symmetric. In particular, these symmetries are respected by the electromagnetic and strong interactions as well as by classical gravity. On the other hand, the weak interactions violate $\mathcal{C}$ and $\mathcal{P}$ in the strongest possible way. Hence, while weak interactions do violate $\mathcal{C}$ and $\mathcal{P}$ symmetries separately, the combined $\mathcal{CP}$ symmetry is still preserved. The $\mathcal{CP}$ symmetry is, however, violated in certain rare processes, as discovered long ago in neutral K mesons decays and recently observed in neutral B decays. Thus, only the combined discrete $\mathcal{CPT}$ symmetry transformation, where $\mathcal{T}$ denotes the time inversion

$$x^{\mu} = (t, \mathbf{r}) \mapsto x'^{\mu} = (-\,t, \mathbf{r}) = \mathcal{T}\, x^{\mu}$$

is an exact symmetry for all laws of Nature, just like for the invariance under the restricted Poincaré continuous group. In the sequel we will examine in some detail the $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations for the quantized Dirac field.

4.7.1 The Charge Conjugation

We have seen before that the quantization of the free Dirac wave field leads to the appearance of particle and antiparticles, which are characterized by the very same mass and spin although opposite charge. The latter one may have a different interpretation depending upon the physical context: namely, it could be electric, barionic, leptonic *et cetera*. In any case, the existence of the charge conjugation invariance just implies

1. the existence of the antiparticles
2. the equality of all the quantum numbers, but charge, for a particle antiparticle pair

We are looking for a unitary charge conjugation operator in the theory of the quantized Dirac field, the action of which will be given by

$$\psi^{\,c}(x) = \mathcal{C}\,\psi\,(x)\,\mathcal{C}^{\,\dagger} \qquad\qquad \mathcal{C}^{\,\dagger} = \mathcal{C}^{\,-1} \tag{4.130}$$

Charge conjugation is conventionally defined as the operation in which particles and antiparticles are interchanged, up to an arbitrary overall phase factor. It follows thereby that if we set

$$\mathcal{C}\,c_{\mathbf{p}\,,\,r}\,\mathcal{C}^{\,-1} = \mathrm{e}^{i\eta}\,d_{\mathbf{p}\,,\,r} \qquad \mathcal{C}\,d_{\mathbf{p}\,,\,s}\,\mathcal{C}^{\,-1} = \mathrm{e}^{-\,i\eta}\,c_{\mathbf{p}\,,\,s} \tag{4.131}$$

$$\forall\, r,s = 1,2 \quad \mathbf{p} \in \mathbb{R}^3$$

then the repeated application of the charge conjugation operation yields

$$\mathcal{C}\left(\mathcal{C}\,c_{\mathbf{p}\,,\,r}\,\mathcal{C}^{\,-1}\right)\mathcal{C}^{\,-1} = \mathrm{e}^{i\eta}\,\mathcal{C}\,d_{\mathbf{p}\,,\,r}\,\mathcal{C}^{\,-1} = c_{\mathbf{p}\,,\,r}$$

so that we can always assume the following properties: namely,

$$\mathcal{C}^{\,2} \;=\; \mathbb{I} \qquad\qquad \Longrightarrow \qquad\qquad \mathcal{C} = \mathcal{C}^{\,\dagger} = \mathcal{C}^{\,-1} \tag{4.132}$$

It turns out that the standard spin-states (4.35) do fulfill the remarkable relationship

$$\begin{aligned} u_r(\mathbf{p}) &= -\,i\,\gamma^2\,v_r^*(\mathbf{p}) \\ v_r(\mathbf{p}) &= -\,i\,\gamma^2\,u_r^*(\mathbf{p}) \end{aligned}$$

Hence, the transformation law (4.130) can be rewritten as

$$\begin{aligned} \psi^{\,c}(x) &= \sum_{\mathbf{p}\,,\,r}\ \mathrm{e}^{i\eta}\left[\,d_{\mathbf{p}\,,\,r}\,u_{\mathbf{p}\,,\,r}\,(x) + c^{\dagger}_{\mathbf{p}\,,\,r}\,v_{\mathbf{p}\,,\,r}\,(x)\,\right] \\ &= \mathrm{e}^{\,i(\eta-\pi/2)}\sum_{\mathbf{p}\,,\,r}\ \left[\,d_{\mathbf{p}\,,\,r}\,\gamma^2\,v^*_{\mathbf{p}\,,\,r}\,(x) + c^{\dagger}_{\mathbf{p}\,,\,r}\,\gamma^2\,u^*_{\mathbf{p}\,,\,r}\,(x)\,\right] \\ &= \mathrm{e}^{\,i(\eta-\pi/2)}\,\gamma^2\,\left(\psi^{\,\dagger}\,(x)\right)^{\top} \end{aligned} \tag{4.133}$$

Hence, if we choose $\eta = \pi/2$ then we obtain the transformation rule

$$\psi^{\,c}(x) = \gamma^2\,\left(\psi^{\,\dagger}\,(x)\right)^{\top} = -\left(\psi^{\,\dagger}\,(x)\,\gamma^2\right)^{\top} = -\left(\bar{\psi}\,(x)\,\gamma^0\,\gamma^2\right)^{\top} \tag{4.134}$$

which perfectly corresponds to the quantum mechanical counterpart of the classical transformation (2.79). Moreover we readily find

$$\bar{\psi}^{\,c}(x) = \psi^{\,c\,\dagger}\,(x)\,\gamma^0 = \left(-\,\gamma^2\,\psi\,(x)\right)^{\top}\,\gamma^0 = \left(-\,\gamma^0\,\gamma^2\,\psi\,(x)\right)^{\top} \tag{4.135}$$

Working out the transformations of bilinears is a little bit tricky and it helps to write the spinor indices explicitly. For instance, the mass scalar operator becomes

$$\begin{aligned}
:\bar{\psi}^{\,c}(x)\,\psi^{\,c}(x): \;&=\; :\big(-\gamma^0\gamma^2\,\psi(x)\big)^\top\;\big(\bar{\psi}(x)\,\gamma^0\gamma^2\big)^\top: \\
&=\; -\;:\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta(x)\;\bar{\psi}_\eta(x)\,\gamma^0_{\eta\omega}\,\gamma^2_{\omega\alpha}: \\
&=\; +\;:\bar{\psi}_\eta(x)\,\gamma^0_{\eta\omega}\,\gamma^2_{\omega\alpha}\;\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta(x): \\
&=\; +\;:\bar{\psi}(x)\,\gamma^0\,\gamma^2\,\gamma^0\,\gamma^2\,\psi(x): \\
&=\; -\;:\bar{\psi}(x)\,(\gamma^0)^2(\gamma^2)^2\,\psi(x): \\
&=\; +\;:\bar{\psi}(x)\,\psi(x): \qquad (4.136)
\end{aligned}$$

where the change of sign in the third line is just owing to the spinor field canonical anti-commuation relations. Hence the femion mass scalar operator is invariant under charge conjugation, *i.e.*

$$:\bar{\psi}^{\,c}(x)\,\psi^{\,c}(x): \;=\; :\bar{\psi}(x)\,\psi(x):$$

while, in a quite analogous way, one can show that the electric current density changes its sign : namely,

$$:\bar{\psi}^{\,c}(x)\,\gamma^\mu\,\psi^{\,c}(x): \;=\; -\;:\bar{\psi}(x)\,\gamma^\mu\,\psi(x):$$

4.7.2 The Parity Transformation

Suppose it is possible, with a suitable modification of some experimental apparatus, to realize the space inversion and to obtain the parity transformed state of a Dirac particle or antiparticle. This means that, from the point of view of the Dirac quantum field, we look for a unitary operator $\mathcal{P}$ satisfying

$$\mathcal{P}\,c_{\mathbf{p},r}\,\mathcal{P}^\dagger = \mathrm{e}^{i\eta}\,c_{-\mathbf{p},r} \qquad \mathcal{P}\,d_{\mathbf{p},r}\,\mathcal{P}^\dagger = \mathrm{e}^{i\theta}\,d_{-\mathbf{p},r} \qquad (4.137)$$

$$0 \le \eta \le 2\pi \qquad 0 \le \theta \le 2\pi$$

where η, θ are free arbitrary phases, in such a manner that, for example, the quantum state $d^\dagger_{\mathbf{p},r}|0\rangle$ is turned into $\mathrm{e}^{\mathrm{i}\theta}\,d^\dagger_{-\mathbf{p},r}|0\rangle$. Inserting the normal mode expansion (4.30) we come to the relations

$$\begin{aligned}
\psi'(x') \;&=\; \mathcal{P}\,\psi(t,-\mathbf{x})\,\mathcal{P}^\dagger \\
&=\; \sum_{\mathbf{p},r}\Big[\mathcal{P}\,c_{\mathbf{p},r}\,\mathcal{P}^\dagger\,u_{\mathbf{p},r}(t,-\mathbf{x}) + \mathcal{P}\,d^\dagger_{\mathbf{p},r}\,\mathcal{P}^\dagger\,v_{\mathbf{p},r}(t,-\mathbf{x})\Big] \\
&=\; \sum_{\mathbf{p},r}\Big[\mathrm{e}^{i\eta}\,c_{-\mathbf{p},r}\,u_{\mathbf{p},r}(t,-\mathbf{x}) + \mathrm{e}^{-i\theta}\,d^\dagger_{-\mathbf{p},r}\,v_{\mathbf{p},r}(t,-\mathbf{x})\Big]
\end{aligned}$$

Now change variables according to $\mathbf{p} \longmapsto -\mathbf{p}$, by taking into account that $\omega_{\mathbf{p}} = \omega_{-\mathbf{p}}$ and that the previously introduced standard spin-states (4.35) do satisfy

$$u_r(-\mathbf{p}) = \gamma^0 u_r(\mathbf{p}) \qquad v_r(-\mathbf{p}) = -\gamma^0 v_r(\mathbf{p})$$

Then we obtain

$$\mathcal{P}\,\psi(t,-\mathbf{x})\,\mathcal{P}^{\dagger} = \gamma^0 \sum_{\mathbf{p},r} \left[\mathrm{e}^{i\eta}\, c_{\mathbf{p},r}\, u_{\mathbf{p},r}(t,\mathbf{x}) - \mathrm{e}^{-i\theta}\, d^{\dagger}_{\mathbf{p},r}\, v_{\mathbf{p},r}(t,\mathbf{x}) \right]$$

and if we set $\theta = -\eta \pm \pi$ we come to the parity transformation law of the Dirac field operator, viz.,

$$\psi'(x') \;=\; \mathcal{P}\,\psi(t,-\mathbf{x})\,\mathcal{P}^{\dagger} = \mathrm{e}^{i\eta}\gamma^0\,\psi(x) \qquad (0 \le \eta < 2\pi) \tag{4.138}$$

which corresponds to the transformation law (2.55) of the classical Dirac wave field under spatial inversion. Furthermore, it can be readily verified from the expressions (4.67) and (4.68) that we have

$$\mathcal{P}\,P^{\mu}\,\mathcal{P}^{\dagger} \;=\; P_{\mu} \tag{4.139}$$

If we choose $\eta = 2\pi k \quad (k \in \mathbb{Z})$ then the relative parity of the particle-antiparticle system is equal to minus one and, contextually, the square of the parity operator (or space inversion operator) $\mathcal{P}$ is equal to the identity operator, that is

$$\mathcal{P}^2 = \mathbf{I} \qquad \mathcal{P}^{\dagger} = \mathcal{P} = \mathcal{P}^{-1} \qquad (\eta = 2\pi k\,,\; k \in \mathbb{Z})$$

As a consequence we can write for instance

$$\mathcal{P}\,\psi(t,-\mathbf{x})\,\mathcal{P} = \gamma^0\,\psi(t,\mathbf{x}) \qquad \mathcal{P}\,\psi^{\dagger}(t,-\mathbf{x})\,\mathcal{P} = \bar{\psi}(t,\mathbf{x}) \tag{4.140}$$

$$\mathcal{P}\,\bar{\psi}(t,-\mathbf{x})\,\mathcal{P} = \mathcal{P}\,\psi^{\dagger}(t,-\mathbf{x})\,\gamma^0\,\mathcal{P} = \mathcal{P}\,\psi^{\dagger}(t,-\mathbf{x})\,\mathcal{P}\,\gamma^0 = \bar{\psi}(t,\mathbf{x})\,\gamma^0$$

$$\mathcal{P}\,(\bar{\psi}\,\psi)(t,-\mathbf{x})\,\mathcal{P} = +\,(\bar{\psi}\,\psi)(t,\mathbf{x}) \tag{4.141}$$

$$\mathcal{P}\,(\bar{\psi}\,\gamma_5\,\psi)(t,-\mathbf{x})\,\mathcal{P} = -\,(\bar{\psi}\,\gamma_5\,\psi)(t,\mathbf{x}) \qquad \text{et cetera} \tag{4.142}$$

4.7.3 The Time Reversal

Let us turn now to the implementation of time reversal transformation. It is known[5] that the time reversal transformation in quantum mechanics is

[5]See E.P. Wigner, *Group Theory and Its Application to Quantum Mechanics of Atomic Spectra*, translated by J.J. Griffin, Academic Press, New York, 1959, Appendix to Chapter 20, p. 233. See also V. Bargmann, J. Math. Phys. **5** (1964) 862.

achieved by means of anti-linear and anti-unitary operators $\mathcal{T} : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, which satisfy

$$\mathcal{T}\,(a\,|\,\alpha\,\rangle + b\,|\,\beta\,\rangle) = a^*\,\mathcal{T}\,|\,\alpha\,\rangle + b^*\,\mathcal{T}\,|\,\beta\,\rangle \qquad a, b \in \mathbb{C} \quad \alpha, \beta \in \mathcal{H}$$

$$\langle\,\alpha\,|\,\mathcal{T}^\dagger\mathcal{T}\,\beta\,\rangle \;=\; \langle\,\mathcal{T}\,\alpha\,|\,\mathcal{T}\,\beta\,\rangle \;=\; \langle\,\beta\,|\,\alpha\,\rangle \qquad\quad \forall\,\alpha, \beta \in \mathcal{H}$$

so that invariance under time reversal requires that

$$[\,\mathcal{T}\,,\,cP_0\,] \;=\; [\,\mathcal{T}\,,\,H\,] \;=\; 0$$

where $cP_0 = H$ denotes the infinitesimal generator of the time translations of the system.

`Time Reversal Symmetry.` A mechanical system is said to exhibit symmetry under time reversal if, at least in principle, its time evolution may be reversed in such a manner that all physical processes run backwards in time, with the initial and final states interchanged. In Classical Mechanics, for an isolated system with time independent constraints, the time reversal symmetry follows in a straightforward way from the second order degree in time of the classical Lagrange-Newton equations. Things are considerably more tricky and complicated in non-relativistic Quantum Mechanics, because the symmetry between the two directions of the time development actually implies that to every state $|\Upsilon\rangle$ there corresponds a time-reversed state $\mathcal{T}|\Upsilon\rangle$ and that the transformation $\mathcal{T}$ does indeed preserve the values of all the probabilities, thus leaving invariant the absolute value of any scalar product between any pair of states. If the time development of the system is provided by

$$|\Upsilon(t)\rangle = \mathrm{e}^{-(i/\hbar)Ht}\,|\Upsilon(0)\rangle \qquad\quad H = H^\dagger \qquad\quad \forall\,|\Upsilon(0)\rangle \in \mathcal{H}$$

time reversal symmetry demands that the time-reversed initial state $\mathcal{T}|\Upsilon(0)\rangle$ evolves into

$$\mathcal{T}\,|\Upsilon(-\,t)\rangle = \mathcal{T}\,\mathrm{e}^{(i/\hbar)Ht}\,|\Upsilon(0)\rangle = \mathrm{e}^{-(i/\hbar)Ht}\,\mathcal{T}\,|\Upsilon(0)\rangle$$

If the theory must be invariant under time reversal, from the last two equations we get the condition

$$\mathcal{T}\,\mathrm{e}^{(i/\hbar)Ht} = \mathrm{e}^{-(i/\hbar)Ht}\,\mathcal{T}$$

For a unitary $\mathcal{T}$ the above condition would be equivalent to $\{\mathcal{T}\,,\,H\} = 0$. If such an operator $\mathcal{T}$ existed, every stationary state $|\Upsilon_E\rangle$ of the system with energy E would be accompanied with another time-reversed stationary state $\mathcal{T}|\Upsilon_E\rangle$ of energy $-E$ (the so called Kramers degeneracy). This doubling of the energy spectrum is in a manifest conflict with the existence of a lower bound to the energy and thereby with the stability of the mechanical system itself: hence, $\mathcal{T}$ can not be unitary but necessarily anti-linear and anti-unitary. Then we get

$$\begin{aligned} \mathcal{T}\,|\Upsilon_E(t)\rangle &= \mathcal{T}\,\mathrm{e}^{-(i/\hbar)Ht}\,|\Upsilon_E\rangle = \mathcal{T}\,\mathrm{e}^{-(i/\hbar)Et}\,|\Upsilon_E\rangle \\ &= \mathrm{e}^{(i/\hbar)Et}\,\mathcal{T}\,|\Upsilon_E\rangle = \mathrm{e}^{(i/\hbar)Ht}\,\mathcal{T}\,|\Upsilon_E\rangle = |\Upsilon_E(-t)\rangle \qquad \forall\,|\Upsilon_E\rangle \in \mathcal{H} \end{aligned}$$

which entails in turn

$$\mathcal{T}\,\mathrm{e}^{(i/\hbar)Ht} = \mathrm{e}^{(i/\hbar)Ht}\,\mathcal{T} \quad\Longrightarrow\quad [\,H\,,\,\mathcal{T}\,] = 0$$

The physical significance of $\mathcal{T}$ as the time reversal operator requires that, while spatial relations must be unchanged, all the momenta and angular momenta must be reversed. Hence, we shall postulate the conditions

$$\mathcal{T}\,\mathbf{P}\,\mathcal{T}^{\dagger} = -\,\mathbf{P} \qquad \{\mathcal{T}\,,\,\mathbf{P}\} = 0 \tag{4.143}$$
$$\mathcal{T}\,L^{\rho\sigma}\,\mathcal{T}^{\dagger} = -\,L^{\rho\sigma} \qquad \{\mathcal{T}\,,\,L^{\rho\sigma}\} = 0 \tag{4.144}$$
$$\mathcal{T}\,S^{\rho\sigma}\,\mathcal{T}^{\dagger} = -\,S^{\rho\sigma} \qquad \{\mathcal{T}\,,\,S^{\rho\sigma}\} = 0 \tag{4.145}$$

Since the anti-unitary time reversal operator $\mathcal{T}$ is defined to reverse the sign of all momenta and spins we therefore require

$$\mathcal{T}\,c_{\mathbf{p}\,,\,r}\,\mathcal{T}^{\dagger} = \exp\{\,i\eta_{\mathbf{p}\,,\,r}\}\,c_{-\mathbf{p}\,,\,-r} \tag{4.146}$$
$$\mathcal{T}\,d_{\mathbf{p}\,,\,r}\,\mathcal{T}^{\dagger} = \exp\{\,i\zeta_{\mathbf{p}\,,\,r}\}\,d_{-\mathbf{p}\,,\,-r} \tag{4.147}$$
$$\mathcal{T}\,c^{\dagger}_{\mathbf{p}\,,\,r}\,\mathcal{T}^{\dagger} = \exp\{-\,i\eta_{\mathbf{p}\,,\,r}\}\,c^{\dagger}_{-\mathbf{p}\,,\,-r} \tag{4.148}$$
$$\mathcal{T}\,d^{\dagger}_{\mathbf{p}\,,\,r}\,\mathcal{T}^{\dagger} = \exp\{-\,i\zeta_{\mathbf{p}\,,\,r}\}\,d^{\dagger}_{-\mathbf{p}\,,\,-r} \tag{4.149}$$

where the notation $-\,r$ refers to the change of handedness of the polarization, which implies the change of sign of the helicity eigenvalues $\pm\frac{1}{2}\hbar$: namely,

$$c_{-\mathbf{p}\,,\,-1} = c_{-\mathbf{p}\,,\,2} \qquad c_{-\mathbf{p}\,,\,-2} = c_{-\mathbf{p}\,,\,1} \quad \text{et cetera}$$

Although the phase factors $\exp\{\,i\,\eta_{\mathbf{p}\,,\,r}\}$ and $\exp\{\,i\,\zeta_{\mathbf{p}\,,\,r}\}$ are arbitrary, it is always possible to choose them in such a manner that the (Dirac spinor) fields undergo utmost simple transformation laws under time reversal. Hence we can suitably choose the arbitrary phase factors to be equal to 1, *viz.,*

$$\eta_{\mathbf{p}\,,\,r} = \zeta_{\mathbf{p}\,,\,r} = 2n\pi i \qquad (n \in \mathbb{Z}\,,\ \forall\,\mathbf{p} \in \mathbb{R}^3\,,\ r = 1,2)$$

This means that, in order to obtain the time reversed spinor field operator $\mathcal{T}\,\psi(x)\,\mathcal{T}^{\dagger}$, we have to perform on the complex conjugated c-number part (*i.e.* not operator part) of the spinor some anti-unitary operation in such a way that eventually we come to the transformation rule

$$\psi'(x') = \psi'(-\,t, \mathbf{x}) = \mathcal{T}\,\psi(-\,t, \mathbf{x})\,\mathcal{T}^{\dagger}$$

By inserting once again the the normal mode expansion (4.30) we get, up to some trivial substitutions in the integrands,

$$\begin{aligned}\mathcal{T}\,\psi(-\,t, \mathbf{x})\,\mathcal{T}^{\dagger} &= \sum_{\mathbf{p}\,,\,r} c_{-\mathbf{p}\,,\,-r}\,u^{*}_{r}(\mathbf{p}) \\ &\times\ [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1/2}\,\exp\{i\,(-t)\,\omega_{\mathbf{p}} - i\,\mathbf{p}\cdot\mathbf{x}\}\end{aligned}$$

$$
\begin{aligned}
&+ \quad \sum_{\mathbf{p}\,,\,r} d^{\dagger}_{-\mathbf{p}\,,\,-r}\, v^{*}_{r}(\mathbf{p}) \\
&\times \quad [\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-1/2}\, \exp\{it\,\omega_{\mathbf{p}} + i\,\mathbf{p}\cdot\mathbf{x}\} \\
&= \quad \sum_{\mathbf{p}\,,\,r} c_{\mathbf{p}\,,\,r}\, u^{*}_{-r}(-\mathbf{p}) \\
&\times \quad [\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-1/2}\, \exp\{\,-\,it\omega_{\mathbf{p}} + i\mathbf{p}\cdot\mathbf{x}\} \\
&+ \quad \sum_{\mathbf{p}\,,\,r} d^{\dagger}_{\mathbf{p}\,,\,r}\, v^{*}_{-r}(-\mathbf{p}) \\
&\times \quad [\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,]^{-1/2}\, \exp\{\,it\omega_{\mathbf{p}} - i\mathbf{p}\cdot\mathbf{x}\}
\end{aligned}
$$

In arriving at this equation, the anti-unitarity nature of $\mathcal{T}$ has been exploited and has resulted in complex conjugation. It is easily seen that the right-hand side of this equation becomes a local expression for the field at time t if a 4×4 matrix Θ can be found such that

$$ u^{*}_{-r}\,(-\mathbf{p}) \;=\; \Theta\, u_{r}\,(\mathbf{p}) \tag{4.150} $$

$$ v^{*}_{-r}\,(-\mathbf{p}) \;=\; \Theta\, v_{r}\,(\mathbf{p}) \tag{4.151} $$

The closure relation (4.19) and the orthonormality relation (4.18) imply that Θ must be unitary. Furthermore, the above relations (4.150) and (4.151) are consistent with the spin-states eigenvalue equations (4.17) only if Θ satisfies the conditions

$$ [\,\Theta\,,\, H\,] \;=\; 0 $$

$$ \left(\alpha^{\,k}\right)^{*}\,\Theta = -\,\Theta\,\alpha^{\,k} \qquad (k = 1,2,3) \qquad \beta^{*}\,\Theta = \Theta\,\beta $$

As a matter of fact we have, for example,

$$
\begin{aligned}
H\,u_r\,(\mathbf{p}) = \left(\alpha^{k}\,p^{k} + \beta\,M\right)\; u_r\,(\mathbf{p}) &= \omega_{\mathbf{p}}\,u_r\,(\mathbf{p}) \\
\left(-\,\alpha^{k*}\,p^{k} + \beta^{*}\,M\right)\; u^{*}_{r}\,(-\,\mathbf{p}) &= \omega_{\mathbf{p}}\,u^{*}_{r}\,(-\,\mathbf{p}) \\
\left(-\,\alpha^{k*}\,p^{k} + \beta^{*}\,M\right)\;\Theta\,u_r\,(\mathbf{p}) &= \omega_{\mathbf{p}}\,\Theta\,u_r\,(\mathbf{p}) \\
\Theta\left(\alpha^{k}\,p^{k} + \beta\,M\right)\; u_r\,(\mathbf{p}) = \Theta\,H\,u_r\,(\mathbf{p}) &= \omega_{\mathbf{p}}\,\Theta\,u_r\,(\mathbf{p})
\end{aligned}
$$

In the spinorial-chiral-Weyl representation (2.64), the matrix

$$ \Theta \;=\; -\,\gamma^{1}\,\gamma^{3} \;=\; -\,i\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \;=\; -\,i\,\Sigma_2 $$

is a solution. It has the important property

$$ \Theta\,\Theta^{*} \;=\; -\,\mathbb{I} \qquad\qquad \Theta\,\Theta^{\dagger} \;=\; \mathbb{I} $$

which can be proved to hold true independently of the representation of the Dirac matrices. Actually, it turns out that the spin states (4.35) do actually fulfill equations (4.150) and (4.151) for $\Theta = -i\Sigma_2$.

It follows that the Dirac particle-antiparticle spinor quantum wave field is transformed under time reversal according to

$$\begin{aligned} \psi'(x') = \mathcal{T}\,\psi\,(-\,t,\mathbf{x})\,\mathcal{T}^{\,\dagger} &= \Theta\,\psi\,(t,\mathbf{x}) \\ \mathcal{T}\,\psi^{\,\dagger}\,(-\,t,\mathbf{x})\,\mathcal{T}^{\,\dagger} &= \psi^{\,\dagger}\,(t,\mathbf{x})\,\Theta^{\,\dagger} \end{aligned} \tag{4.152}$$

It is now rather easy to derive the action of the time reversal operator on various bilinears. First of all we have

$$\begin{aligned} \overline{\psi}'(x') = \mathcal{T}\,\bar{\psi}\,(-\,t,\mathbf{x})\,\mathcal{T}^{\,\dagger} &= \mathcal{T}\,\psi^{\,\dagger}\,(-\,t,\mathbf{x})\,\mathcal{T}^{\,\dagger}\,\gamma^{0*} \\ &= \psi^{\,\dagger}\,(t,\mathbf{x})\,\Theta^{\,\dagger}\,\gamma^{0*} \\ &= \bar{\psi}\,(t,\mathbf{x})\,\gamma^1\,\gamma^3 \end{aligned} \tag{4.153}$$

Then the transformation law for the scalar mass bi-linear under time reversal becomes

$$\begin{aligned} \mathcal{T}\,\bar{\psi}\,(-\,t,\mathbf{x})\,\psi\,(-\,t,\mathbf{x})\,\mathcal{T}^{\,\dagger} &= \bar{\psi}\,(t,\mathbf{x})\,\gamma^1\,\gamma^3\left(-\,\gamma^1\,\gamma^3\right)\,\psi\,(t,\mathbf{x}) \\ &= +\,\bar{\psi}\,(t,\mathbf{x})\,\psi\,(t,\mathbf{x}) \end{aligned} \tag{4.154}$$

A quite analogous calculation gives for instance

$$\mathcal{T}\,\bar{\psi}\,(-\,t,\mathbf{x})\,\gamma^{\mu}\,\psi\,(-\,t,\mathbf{x})\,\mathcal{T}^{\,\dagger} = \begin{cases} \psi^{\dagger}\,(t,\mathbf{x})\,\psi\,(t,\mathbf{x}) & \text{for } \mu = 0 \\ -\,\bar{\psi}\,(t,\mathbf{x})\gamma^{k}\,\psi\,(t,\mathbf{x}) & \text{for } \mu = 1,2,3 \end{cases}$$

Summary of $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations

The transformation properties of the various Dirac field bi-linear functional expressions under charge conjugation, parity and time reversal symmetries are summarized[6] in the following table

	$\overline{\psi}\psi$	$i\overline{\psi}\gamma_5\psi$	$\overline{\psi}\gamma^{\mu}\psi$	$\overline{\psi}\gamma^{\mu}\gamma_5\psi$	$i\partial_{\mu}$
$\mathcal{C}$	$+1$	$+1$	-1	$+1$	-1
$\mathcal{P}$	$+1$	-1	$(-1)^{\mu}$	$-(-1)^{\mu}$	$(-1)^{\mu}$
$\mathcal{T}$	$+1$	-1	$(-1)^{\mu}$	$(-1)^{\mu}$	$(-1)^{\mu}$

[6] See Ref. [6] here below, Ch. 3, §**3.6**, p. 71.

which holds true both for classical and quantum Dirac spinor fields, where use has been made of the customary notation

$$(-1)^{\mu} = \begin{cases} 1 & \text{for } \mu = 0 \\ -1 & \text{for } \mu = 1, 2, 3 \end{cases}$$

From the above table it immediately follows that the Dirac Lagrangian $\mathcal{L} = \overline{\psi}\,(i\gamma^{\mu}\partial_{\mu} - M)\,\psi$ is invariant under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ separately. Actually, it has been rigorously proved, under quite general conditions, that it is not possible to build up any Lorentz invariant quantum field theory, governed by a Hermitean Hamiltonian, that violates $\mathcal{CPT}$ – see Raymond Frederick Streater & Arthur Strong Wightman (1964) *PCT, Spin, Statistics and all that*, Princeton University Press (New York, Benjamin).

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4.8 Problems

4.8.1 Belifante Energy-Momentum Tensor

In general, for a relativistic wave field with spin angular momentum, the canonical energy-momentum tensor is not symmetric. Show that one can always find a term $\mathcal{B}^{\lambda\mu\nu}$ anti-symmetric under $\lambda \to \mu$ or ν such that the Belifante tensor

$$\Theta^{\mu\nu}(x) = T^{\mu\nu}(x) - \partial_\lambda \mathcal{B}^{\lambda\mu\nu}(x)$$

is symmetric and the corresponding total angular momentum density can be written in the purely orbital form

$$M^{\mu\lambda\kappa}(x) = x^\lambda\, \Theta^{\mu\kappa}(x) - x^\kappa\, \Theta^{\mu\lambda}(x)$$

Solution. One can always decompose the canonical energy-momentum tensor of the Dirac wave field into its symmetric and anti-symmetric parts

$$\begin{aligned} T^{\mu\nu}(x) &= \tfrac{1}{4}\Big[\, \bar\psi(x)\,\gamma^\mu\, i\overleftrightarrow{\partial}^{\,\nu}\,\psi(x) + \bar\psi(x)\,\gamma^\nu\, i\overleftrightarrow{\partial}^{\,\mu}\,\psi(x)\Big] \\ &+ \tfrac{1}{4}\Big[\, \bar\psi(x)\,\gamma^\mu\, i\overleftrightarrow{\partial}^{\,\nu}\,\psi(x) - \bar\psi(x)\,\gamma^\nu\, i\overleftrightarrow{\partial}^{\,\mu}\,\psi(x)\Big] \end{aligned}$$

in which

$$\begin{aligned} T^{\mu\nu}(x) - T^{\nu\mu}(x) &= \tfrac{1}{2}\Big[\, \bar\psi(x)\,\gamma^\mu\, i\overleftrightarrow{\partial}^{\,\nu}\,\psi(x) - \bar\psi(x)\,\gamma^\nu\, i\overleftrightarrow{\partial}^{\,\mu}\,\psi(x)\Big] \\ &= \partial_\lambda\, S^{\lambda\nu\mu}(x) \end{aligned}$$

in accordance with eq. (4.42), where the third rank tensor of the spin angular momentum density of the Dirac field is defined by the Noether theorem and reads

$$S^{\lambda\mu\nu}(x) \equiv \tfrac{1}{2}\, \bar\psi(x)\,\{\gamma^\lambda\,,\,\sigma^{\mu\nu}\}\,\psi(x)$$

which enjoys by construction

$$S^{\lambda\mu\nu}(x) + S^{\lambda\nu\mu}(x) = 0 \qquad\qquad S^{\lambda\mu\nu}(x) = S^{\mu\nu\lambda}(x)$$

If we introduce the auxiliary quantity

$$\mathcal{B}^{\lambda\mu\nu}(x) \equiv \tfrac{1}{2}\,\Big[\, S^{\lambda\mu\nu}(x) - S^{\mu\lambda\nu}(x) - S^{\nu\lambda\mu}(x)\Big]$$

which is evidently related to a non-vanishing spin angular momentum density tensor, it can be readily checked that it fulfills by definition

$$\mathcal{B}^{\lambda\mu\nu} + \mathcal{B}^{\mu\lambda\nu} = 0 \qquad\qquad \partial_\lambda\, \mathcal{B}^{\lambda\mu\nu}(x) = \tfrac{1}{2}\,\partial_\lambda\, S^{\lambda\mu\nu}(x)$$

In fact we have

$$
\begin{aligned}
2\partial_\lambda \mathcal{B}^{\lambda\mu\nu}(x) &= \partial_\lambda S^{\lambda\mu\nu}(x) - \partial_\lambda S^{\mu\lambda\nu}(x) - \partial_\lambda S^{\nu\lambda\mu}(x) \\
&= \partial_\lambda S^{\lambda\mu\nu}(x) + \partial_\lambda S^{\mu\nu\lambda}(x) + \partial_\lambda S^{\nu\mu\lambda}(x) \\
&= \partial_\lambda S^{\lambda\mu\nu}(x) + \partial_\lambda S^{\lambda\mu\nu}(x) + \partial_\lambda S^{\lambda\nu\mu}(x) \\
&= 2\partial_\lambda S^{\lambda\mu\nu}(x) + \partial_\lambda S^{\lambda\nu\mu}(x) = \partial_\lambda S^{\lambda\mu\nu}(x)
\end{aligned}
$$

Then the symmetric energy-momentum tensor for the Dirac field is obtained by setting

$$
\begin{aligned}
\Theta^{\mu\nu}(x) &\stackrel{\text{def}}{=} \tfrac{1}{2}\Big[T^{\mu\nu}(x) + T^{\nu\mu}(x)\Big] \\
&= \tfrac{1}{4}\Big[\bar{\psi}(x)\,\gamma^{\mu}\,i\overleftrightarrow{\partial}^{\nu}\psi(x) + \bar{\psi}(x)\,\gamma^{\nu}\,i\overleftrightarrow{\partial}^{\mu}\psi(x)\Big] \\
&= T^{\mu\nu}(x) + \tfrac{1}{2}\,\partial_\lambda S^{\lambda\mu\nu}(x) \\
&= T^{\mu\nu}(x) + \partial_\lambda \mathcal{B}^{\lambda\mu\nu}(x)
\end{aligned}
$$

which apparently satisfies the continuity equation because

$$
\partial_\mu \Theta^{\mu\nu}(x) = \partial_\mu T^{\mu\nu}(x) + \partial_\lambda \partial_\mu \mathcal{B}^{\lambda\mu\nu}(x) = \partial_\mu T^{\mu\nu}(x) = 0
$$

The symmetric energy-momentum tensor of a non vanishing spin field is also called the Belifante-Rosenfeld form of the energy-momentum tensor. It actually enjoys the trace property

$$
\Theta^{\mu}{}_{\mu}(x) = T^{\mu}{}_{\mu}(x) = \overline{\psi}(x)\,i\partial\!\!\!/\,\psi(x) = M\,\overline{\psi}(x)\,\psi(x)
$$

which entails the trace-less property in the scale invariant mass-less case. Furthermore, the third rank tensor of THE TOTAL ANGULAR MOMENTUM DENSITY FOR THE DIRAC FIELD CAN ALWAYS BE WRITTEN IN THE PURELY ORBITAL FORM. In fact we have

$$
\begin{aligned}
M^{\mu\kappa\lambda}(x) &= x^{\kappa}\,\Theta^{\mu\lambda}(x) - x^{\lambda}\,\Theta^{\mu\kappa}(x) \\
&= x^{\kappa}\,T^{\mu\lambda}(x) + \tfrac{1}{2}\,x^{\kappa}\,\partial_\rho S^{\rho\mu\lambda}(x) \\
&- x^{\lambda}\,T^{\mu\kappa}(x) - \tfrac{1}{2}\,x^{\lambda}\,\partial_\rho S^{\rho\mu\kappa}(x) \\
&= x^{\kappa}\,T^{\mu\lambda}(x) - x^{\lambda}\,T^{\mu\kappa}(x) \\
&- \tfrac{1}{2}\Big[S^{\kappa\mu\lambda}(x) - S^{\lambda\mu\kappa}(x)\Big] \\
&- \tfrac{1}{2}\,\partial_\rho\Big[x^{\lambda}\,S^{\rho\mu\kappa}(x) - x^{\kappa}\,S^{\rho\mu\lambda}(x)\Big]
\end{aligned}
$$

and taking the symmetry properties of the spin angular momentum density tensor suitably into account

$$
M^{\mu\kappa\lambda}(x) = x^{\kappa}\,\Theta^{\mu\lambda}(x) - x^{\lambda}\,\Theta^{\mu\kappa}(x)
$$

$$
\begin{aligned}
&= x^{\kappa} T^{\mu\lambda}(x) - x^{\lambda} T^{\mu\kappa}(x) + S^{\mu\lambda\kappa}(x) \\
&+ \tfrac{1}{2}\,\partial_\rho \Big[x^{\kappa} S^{\lambda\rho\mu}(x) - x^{\lambda} S^{\kappa\rho\mu}(x) \Big]
\end{aligned}
$$

it becomes apparent that the continuity equations hold true, *viz.,*

$$
\begin{aligned}
\partial_\mu M^{\mu\kappa\lambda}(x) &= \partial_\mu \Big(x^{\kappa}\,\Theta^{\mu\lambda}(x) - x^{\lambda}\,\Theta^{\mu\kappa}(x) \Big) \\
&= \partial_\mu \Big(x^{\kappa} T^{\mu\lambda}(x) - x^{\lambda} T^{\mu\kappa}(x) + S^{\mu\lambda\kappa}(x) \Big) \\
&- \tfrac{1}{2}\,\partial_\rho\,\partial_\mu \Big(x^{\lambda} S^{\kappa\rho\mu}(x) - x^{\kappa} S^{\lambda\rho\mu}(x) \Big) \\
&= \partial_\mu \Big(x^{\kappa} T^{\mu\lambda}(x) - x^{\lambda} T^{\mu\kappa}(x) + S^{\mu\lambda\kappa}(x) \Big) = 0
\end{aligned}
$$

It follows therefrom that the generators of the Lorentz group for the quantized Dirac field can even be recast in a purely orbital form

$$
\begin{aligned}
M^{\kappa\lambda} &\equiv \int \mathrm{d}\mathbf{x}\ : M^{0\kappa\lambda}(t,\mathbf{x}) : \\
&= \int \mathrm{d}\mathbf{x}\ : x^{\kappa}\,\Theta^{0\lambda}(t,\mathbf{x}) - x^{\lambda}\,\Theta^{0\kappa}(t,\mathbf{x}) : \\
&= \tfrac{1}{4} \int \mathrm{d}\mathbf{x}\, x^{\kappa} : \psi^{\dagger}(t,\mathbf{x})\, i \overleftrightarrow{\partial}^{\lambda} \psi(t,\mathbf{x}) + \bar{\psi}(t,\mathbf{x})\,\gamma^{\lambda}\, i \overleftrightarrow{\partial}_{0} \psi(t,\mathbf{x}) : \\
&- \tfrac{1}{4} \int \mathrm{d}\mathbf{x}\, x^{\lambda} : \psi^{\dagger}(t,\mathbf{x})\, i \overleftrightarrow{\partial}^{\kappa} \psi(t,\mathbf{x}) + \bar{\psi}(t,\mathbf{x})\,\gamma^{\kappa}\, i \overleftrightarrow{\partial}_{0} \psi(t,\mathbf{x}) :
\end{aligned}
$$

4.8.2 Mass-less Neutrino and Anti-Neutrino Fields

The quantum theory of a mass-less left-handed or right-handed Weyl spinor fields leads to the description of the neutrino and anti-neutrino particles, in the limit of a negligible neutrino mass. Discuss the corresponding canonical quantum theory.

Solution. The classical Weyl spinor fields are two-component complex spinor wave fields which undergo, by definition, the following transformation rule for a change of inertial reference frame $x' = \Lambda \cdot (x + a)$: namely,

$$
\begin{aligned}
\psi'_L(x') &= \Lambda_L\,\psi_L(x) \qquad & \Lambda_L &= \exp\{\tfrac{1}{2}\, i\boldsymbol{\sigma}\cdot(\boldsymbol{\alpha} - i\,\boldsymbol{\eta})\} \\
\psi'_R(x') &= \Lambda_R\,\psi_R(x) \qquad & \Lambda_R &= \exp\{\tfrac{1}{2}\, i\boldsymbol{\sigma}\cdot(\boldsymbol{\alpha} + i\,\boldsymbol{\eta})\}
\end{aligned}
$$

The real Lagrange densities for the left-handed and right-handed mass-less Weyl spinors are very simple and read

$$
\mathcal{L}_L = \tfrac{1}{2}\,\psi_L^{\dagger}(x)\,\sigma^{\mu}\, i\,\partial_\mu\,\psi_L(x) + \text{c. c.} \tag{4.155}
$$

$$
\mathcal{L}_R = \tfrac{1}{2}\,\psi_R^{\dagger}(x)\,\bar{\sigma}^{\mu}\, i\,\partial_\mu\,\psi_R(x) + \text{c. c.} \tag{4.156}
$$

where

$$\sigma_\mu = (\,\mathbb{I}\,,\,\sigma_1,\sigma_2,\sigma_3\,) = \sigma_\mu^\dagger = \bar\sigma^{\,\mu}$$

$\mathbb{I}$ being the 2×2 identity matrix. The Euler-Lagrange field equations drive immediately to the Weyl equation for left-handed and right-handed mass-less spinors

$$\sigma^{\,\mu}\,i\partial_\mu\,\psi_L(x) = 0 \qquad\qquad i\partial_\mu\,\psi_L^\dagger(x)\,\sigma^{\,\mu} = 0 \tag{4.157}$$

$$\bar\sigma^{\,\mu}\,i\partial_\mu\,\psi_R(x) = 0 \qquad\qquad i\partial_\mu\,\psi_R^\dagger(x)\,\bar\sigma^{\,\mu} = 0 \tag{4.158}$$

or equivalently in physical units

$$i\hbar\,\frac{\partial}{\partial t}\,\psi_L(t,\mathbf{x}) = i\hbar c\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi_L(t,\mathbf{x}) \tag{4.159}$$

$$i\hbar\,\frac{\partial}{\partial t}\,\psi_R(t,\mathbf{x}) + i\hbar c\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi_R(t,\mathbf{x}) = 0 \tag{4.160}$$

that indicates how we are allowed to identify *à la* Schrödinger

$$H_L = i\hbar c\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} = -\,c\,\mathbf{p}\cdot\boldsymbol{\sigma} \qquad\qquad H_R = -\,i\hbar c\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} = c\,\mathbf{p}\cdot\boldsymbol{\sigma}$$

as the Weyl Hamiltonian. After setting

$$\psi_L(x) = (2\pi)^{-3/2}\int \mathrm{d}^4p\;\tilde\psi_L(p)\,\mathrm{e}^{-i\,p\cdot x}$$

$$\psi_R(x) = (2\pi)^{-3/2}\int \mathrm{d}^4p\;\tilde\psi_R(p)\,\mathrm{e}^{-i\,p\cdot x}$$

we come to the decoupled spinorial equations

$$(\,p_0 + \boldsymbol{\sigma}\cdot\mathbf{p}\,)\,\tilde\psi_L(p) = 0 \qquad\qquad (\,p_0 - \boldsymbol{\sigma}\cdot\mathbf{p}\,)\,\tilde\psi_R(p) = 0$$

These homogeneous algebraic equations admit nontrivial solutions iff

$$\det\,(\,p_0 \pm \boldsymbol{\sigma}\cdot\mathbf{p}\,) = p_0^2 - \mathbf{p}^2 = 0 \qquad \Leftrightarrow \qquad p_0 = \pm\,|\mathbf{p}| = \pm\,p$$

whence we readily obtain the algebraic equations for positive energy particles of momentum $\mathbf{p}$ for the left and right spinors respectively

$$\begin{pmatrix} p+p_z & p_x - ip_y \\ p_x + ip_y & p - p_z \end{pmatrix}\tilde\psi_L(p,\mathbf{p}) = 0 \tag{4.161}$$

$$\begin{pmatrix} p-p_z & -p_x + ip_y \\ -p_x - ip_y & p + p_z \end{pmatrix}\tilde\psi_R(p,\mathbf{p}) = 0 \tag{4.162}$$

which clearly drive to the relations

$$\begin{array}{l}\tilde{\psi}_R(p,\mathbf{p})=\tilde{\psi}_L(p,-\mathbf{p})=\tilde{\psi}_L(-p,\mathbf{p})=\tilde{\psi}_R(-p,-\mathbf{p})\\ \tilde{\psi}_L(p,\mathbf{p})=\tilde{\psi}_R(p,-\mathbf{p})=\tilde{\psi}_R(-p,\mathbf{p})=\tilde{\psi}_L(-p,-\mathbf{p})\end{array}$$

For example, one can easily write two linearly dependent left-handed spinor solutions with the positive energy for the Weyl equation, *e.g.* ,

$$\tilde{\psi}_L(p,\mathbf{p})=\left(\begin{array}{c}-p_x+ip_y\\ p+p_z\end{array}\right)\qquad \tilde{\psi}_L'(p,\mathbf{p})=\left(\begin{array}{c}-p+p_z\\ p_x+ip_y\end{array}\right)\tag{4.163}$$

which are proportional because

$$\tilde{\psi}_L'(p,\mathbf{p})=\frac{p_x+ip_y}{p+p_z}\,\tilde{\psi}_L(p,\mathbf{p})$$

The linearly dependent right-handed spinor solutions for the positive energy Weyl equation are obtained after sending $\mathbf{p}$ into $-\mathbf{p}$, that yields

$$\begin{array}{r}\tilde{\psi}_R(p,\mathbf{p})=\left(\begin{array}{c}p_x-ip_y\\ p-p_z\end{array}\right)\\ \tilde{\psi}_R'(p,\mathbf{p})=(-1)\left(\begin{array}{c}p+p_z\\ p_x+ip_y\end{array}\right)\end{array}\tag{4.164}$$

Furthermore, it is easy to check that the following relations hold true

$$\tilde{\psi}_R'(p,\mathbf{p})=-i\sigma_2\tilde{\psi}_L^*(p,\mathbf{p})\qquad \tilde{\psi}_L'(p,\mathbf{p})=-i\sigma_2\tilde{\psi}_R^*(p,\mathbf{p})\tag{4.165}$$

Of course the negative energy solutions of the left Weyl equation (4.157) do involve right-handed spin states and correspond to the antiparticles, the converse holding true for the right Weyl's equation (4.158). As a matter of fact, if we send p into $-p$ in eq. (4.163), *i.e.* if we turn to the antiparticle spin states, we get for instance

$$\tilde{\psi}_L(p,\mathbf{p})\quad\mapsto\{\,p\leftrightarrow -p\}\mapsto\quad \tilde{\psi}_L(-p,\mathbf{p})\propto\tilde{\psi}_R'(p,\mathbf{p})$$

Thus, to our purposes, it is utmost convenient to take by definition the following properly normalized spin states

$$\begin{array}{r}u_L(\mathbf{p})=(\,p+p_z)^{-\frac12}\left(\begin{array}{c}-p_x+ip_y\\ p+p_z\end{array}\right)\equiv u(\,p)\\ v_L(\mathbf{p})=i\,(\,p-p_z)^{-\frac12}\left(\begin{array}{c}-p_x+ip_y\\ -p+p_z\end{array}\right)\equiv u(-p)\\ u_R(\mathbf{p})=(\,p+p_z)^{-\frac12}\left(\begin{array}{c}p+p_z\\ p_x+ip_y\end{array}\right)=i\sigma_2u_L^*(\mathbf{p})\equiv v(\,p)\\ v_R(\mathbf{p})=i\,(\,p-p_z)^{-\frac12}\left(\begin{array}{c}p_z-p\\ p_x+ip_y\end{array}\right)=-i\sigma_2v_L^*(\mathbf{p})\equiv v(-p)\end{array}$$

which satisfy the orthogonality relations

$$u_L^\dagger(\mathbf{p})\, u_L(\mathbf{p}) = v_L^\dagger(\mathbf{p})\, v_L(\mathbf{p}) = 2p \qquad u_L^\dagger(\mathbf{p})\, v_L(\mathbf{p}) = 0$$

$$u_L^\dagger(\mathbf{p})\, u_R(\mathbf{p}) = 0 \qquad v_L^\dagger(\mathbf{p})\, v_R(\mathbf{p}) = 0 \tag{4.166}$$

$$u_R^\dagger(\mathbf{p})\, u_R(\mathbf{p}) = v_R^\dagger(\mathbf{p})\, v_R(\mathbf{p}) = 2p \qquad u_R^\dagger(\mathbf{p})\, v_R(\mathbf{p}) = 0$$

Then we can suitably define the mass-less Weyl spinor wave functions

$$\begin{aligned} u_{\pm p}(x) &\equiv [\,(2\pi)^3 2\,|\,\mathbf{p}\,|\,]^{-\frac{1}{2}}\, u(\pm p)\, \mathrm{e}^{\mp i\,p\cdot x} \qquad &(\,p_0 = |\,\mathbf{p}\,|\,) \\ v_{\pm p}(x) &\equiv [\,(2\pi)^3 2\,|\,\mathbf{p}\,|\,]^{-\frac{1}{2}}\, v(\pm p)\, \mathrm{e}^{\mp i\,p\cdot x} \qquad &(\,p_0 = |\,\mathbf{p}\,|\,) \end{aligned}$$

which are solutions of the Weyl's equations

$$\begin{aligned} \sigma^{\mu}\, i\partial_\mu\, u_{\pm p}(x) &= 0 \qquad &(\,\forall\,\mathbf{p} \in \mathbb{R}^3\,,\ p_0 = |\,\mathbf{p}\,|\,) \\ \bar{\sigma}^{\mu}\, i\partial_\mu\, v_{\pm p}(x) &= 0 \qquad &(\,\forall\,\mathbf{p} \in \mathbb{R}^3\,,\ p_0 = |\,\mathbf{p}\,|\,) \end{aligned}$$

and do satisfy from (4.166) the orthogonality relations

$$\int \mathrm{d}\mathbf{x}\ \left(u_{\pm p}(t,\mathbf{x})\right)^\dagger\, u_{\pm q}(t,\mathbf{x}) = \delta(\mathbf{p} - \mathbf{q})$$

$$\int \mathrm{d}\mathbf{x}\; u_p^\dagger(t,\mathbf{x})\, u_{-q}(t,\mathbf{x}) = 0$$

$$\int \mathrm{d}\mathbf{x}\ \left(v_{\pm p}(t,\mathbf{x})\right)^\dagger\, v_{\pm q}(t,\mathbf{x}) = \delta(\mathbf{p} - \mathbf{q})$$

$$\int \mathrm{d}\mathbf{x}\; v_p^\dagger(t,\mathbf{x})\, v_{-q}(t,\mathbf{x})$$

Hence we can write the normal modes decomposition of the left-handed Weyl spinor quantum field in the form

$$\psi_L(x) = \sum_{\mathbf{p}} [\,\mathrm{c}_{\mathbf{p}}\, u_p(x) + \mathrm{d}^\dagger_{\mathbf{p}}\, u_{-p}(x)\,] \tag{4.167}$$

$$\psi_R(x) = \sum_{\mathbf{p}} [\,\mathrm{d}_{\mathbf{p}}\, v_p(x) + \mathrm{c}^\dagger_{\mathbf{p}}\, v_{-p}(x)\,] \tag{4.168}$$

the creation and destruction operators satisfying the usual canonical anti-commutation relations

$$\{\mathrm{c}_{\mathbf{p}}\,,\, \mathrm{c}^\dagger_{\mathbf{q}}\} = \{\mathrm{d}_{\mathbf{p}}\,,\, \mathrm{d}^\dagger_{\mathbf{q}}\} = \delta(\mathbf{p} - \mathbf{q}) \tag{4.169}$$

all the remaining anti-commutators being equal to zero.

Since the generators of the rotations group for both the irreducible Weyl spinor representations $\boldsymbol{\tau}_{\frac{1}{2}\,0}$ and $\boldsymbol{\tau}_{0\,\frac{1}{2}}$ are $S_{jk} = \frac{1}{2}\,\varepsilon_{jk\ell}\,\sigma_\ell$ it follows that the corresponding Pauli-Lubanski (1.55) operator reads

$$W_0 = \mathbf{P}\cdot\mathbf{S} = \mathbf{p}\cdot\tfrac{1}{2}\,\hbar\boldsymbol{\sigma}$$
$$W_j = \tfrac{1}{2}\,p\sigma_j - \tfrac{1}{2}\,i\varepsilon_{jk\ell}\,p_k\,\sigma_\ell$$

which entails

$$\begin{aligned} g^{\mu\nu}\,W_\mu W_\nu &= \tfrac{1}{4}\,p_j\,p_k\,\sigma_j\sigma_k - \tfrac{1}{4}\,(\,p\sigma_j - i\varepsilon_{jk\ell}\,p_k\,\sigma_\ell)\,(\,p\sigma_j - i\varepsilon_{jrs}\,p_r\,\sigma_s) \\ &= -\,\tfrac{1}{2}\,p^2 + \tfrac{1}{4}\,(\,\delta_{kr}\delta_{\ell s} - \delta_{ks}\delta_{\ell r})\,p_k\,p_r\,\sigma_\ell\,\sigma_s = 0 \end{aligned}$$

where the 2×2 identity matrix has been understood. Then, thanks to the light-like nature of the Pauli-Lubanski operator, we can identify the helicity operator to be

$$\mathrm{h} \equiv \frac{W_0}{|\,\mathbf{p}\,|} = \frac{|\mathbf{W}|}{p} = \tfrac{1}{2}\,\hbar\,\mathbf{n}\cdot\boldsymbol{\sigma} \qquad\qquad \mathbf{n}\equiv\widehat{\mathbf{p}} = \mathbf{p}/p$$

together with the Weyl's Hamiltonian operators

$$H_R = c\,\mathbf{p}\cdot\boldsymbol{\sigma} = -\,H_L$$

in such a manner that we find $H_L = -\,pc\mathrm{h}$, $H_R = pc\mathrm{h}$ and consequently

$$\begin{array}{ll} H_L\,u_L(\mathbf{p}) = pc\,u_L(\mathbf{p}) & \mathrm{h}u_L(\mathbf{p}) = -\,\tfrac{1}{2}\hbar\,u_L(\mathbf{p}) \\ H_R\,u_R(\mathbf{p}) = pc\,u_R(\mathbf{p}) & \mathrm{h}u_R(\mathbf{p}) = \tfrac{1}{2}\hbar\,u_R(\mathbf{p}) \\ H_L\,v_L(\mathbf{p}) = -\,pc\,v_L(\mathbf{p}) & \mathrm{h}v_L(\mathbf{p}) = \tfrac{1}{2}\hbar\,v_L(\mathbf{p}) \\ H_R\,v_R(\mathbf{p}) = -\,pc\,v_R(\mathbf{p}) & \mathrm{h}v_R(\mathbf{p}) = -\,\tfrac{1}{2}\hbar\,v_R(\mathbf{p}) \end{array}$$

It follows thereby that, mandatory, the positive energy solutions of the mass-less left Weyl spinor have negative helicity $\mathrm{h} = -\frac{1}{2}$, while the negative energy solutions exhibit positive helicity $\mathrm{h} = \frac{1}{2}$, the situation being reversed for the mass-less right Weyl spinors. Hence, it turns out that the particles with negative helicity correspond to the left-handed mass-less spinors, while the antiparticles to the right-handed mass-less spinors. Thus, the spin of left-handed mass-less Weyl spinor is always opposite to the direction of motion, while the spin of the right-handed mass-less Weyl spinor is always towards the direction of motion.

In Nature, whenever the masses of neutrinos and anti-neutrinos can be neglected [7], neutrinos are left-handed, *i.e.* with negative helicity, while anti-neutrinos are right-handed with positive helicity. Moreover, the mass-less

[7] According to J. Beringer *et al.* (Particle Data Group), Phys. Rev. D**86**, 010001 (2012) data from tritium beta decay experiments lead to the upper bound $m(\nu_e) = m(\bar{\nu}_e) < 2$ eV, while the present limits for heavier flavors are $m(\nu_\mu) = m(\bar{\nu}_\mu) < 170$ KeV and $m(\nu_\tau) = m(\bar{\nu}_\tau) < 18.2$ MeV.

Weyl fields are charged fields. In fact, the Weyl Lagrangian (4.156) turns out to be invariant with respect to the phase transformations

$$\begin{array}{rcl} \psi_L(x) & \mapsto & \psi'_L(x) = \mathrm{e}^{i\ell\theta}\psi_L(x) \\ \psi_R(x) & \mapsto & \psi'_R(x) = \mathrm{e}^{-i\ell\theta}\psi_R(x) \\ \psi^\dagger_L(x) & \mapsto & (\psi'_L(x))^\dagger = \mathrm{e}^{-i\ell\theta}\psi^\dagger_L(x) \\ \psi^\dagger_R(x) & \mapsto & (\psi'_R(x))^\dagger = \mathrm{e}^{i\ell\theta}\psi^\dagger_R(x) \end{array}$$

where ℓ is the so called lepton number of the Weyl spinor field while $0 \leq \theta \leq 2\pi$. Hence, from Noether's theorem, it follows that the left-handed and right-handed vector currents

$$J^\mu_L(x) = \ell\,\psi^\dagger_L(x)\,\sigma^\mu\psi_L(x) \qquad\qquad J^\mu_R(x) = (-\,\ell\,)\,\psi^\dagger_R(x)\,\bar\sigma^\mu\psi_R(x)$$

do satisfy the continuity equation

$$\partial_\mu J^\mu_L(x) = \partial_\mu J^\mu_R(x) = 0$$

As a consequence, the lepton number operator can be expressed in the form

$$\begin{aligned} Q_L &= \ell\int \mathrm{d}\mathbf{x}\,:\psi^\dagger_L(x)\psi_L(x): \\ &= \ell\sum_{\mathbf{p}}\left[\,\mathrm{c}^\dagger_{\mathbf{p}}\,\mathrm{c}_{\mathbf{p}} - \mathrm{d}^\dagger_{\mathbf{p}}\,\mathrm{d}_{\mathbf{p}}\,\right] \;=\; -\,Q_R \end{aligned}$$

In Nature there are three leptonic numbers $\ell_e, \ell_\mu, \ell_\tau$, one for each flavor. The assignments are $\ell_\imath = +1$ ($\imath = e, \mu, \tau$) for particles, *e.g.* electrons e^- and left-handed neutrinos ν_e , while $-\,\ell_\imath = -1$ for antiparticles like, for instance, the anti-muons μ^+ and the right-handed anti-neutrinos $\bar\nu_\mu$.

Notice that the charge conjugation transformation for the quantum fields is nothing but the exchange between the particle and antiparticle creation and destruction operators. For example, in the case of the mass-less right-handed quantum Weyl spinor we have

$$\psi^{\mathrm{c}}_R(x) = \sum_{\mathbf{p}}\left[\,\mathrm{c}_{\mathbf{p}}\,v_p(x) + \mathrm{d}^\dagger_{\mathbf{p}}\,v_{-\,p}(x)\,\right]$$

and from the relations (4.165) we immediately obtain

$$\begin{array}{rr} v_p(x) = -i\sigma_2 u^*_p(x) \qquad & v_{-\,p}(x) = -i\sigma_2 u^*_{-\,p}(x) \\ u_p(x) = i\sigma_2 v^*_p(x) \qquad & u_{-\,p}(x) = i\sigma_2 v^*_{-\,p}(x) \end{array}$$

that yields

$$\psi_R^c(x) = -i\sigma_2 \sum_{\mathbf{p}} [\, \mathrm{c}_{\mathbf{p}}\, u_p^*(x) + \mathrm{d}_{\mathbf{p}}^\dagger\, u_{-p}^*(x)\,] = -i\sigma_2 \left(\psi_L^\dagger(x)\right)^\top \tag{4.170}$$

$$\psi_L^c(x) = i\sigma_2 \sum_{\mathbf{p}} [\, \mathrm{d}_{\mathbf{p}}\, v_p^*(x) + \mathrm{c}_{\mathbf{p}}^\dagger\, v_{-p}^*(x)\,] = i\sigma_2 \left(\psi_R^\dagger(x)\right)^\top \tag{4.171}$$

The mass-less Weyl fields can also be represented by complex bispinors. To this concern, it appears that if $\Psi(x)$ is a four components complex bispinor solution of the mass-less Dirac equation

$$i\partial\!\!\!/\, \psi(x) = 0$$

then $\psi'(x) = \gamma_5\psi(x)$ is also a solution of the same equation owing to

$$\{\gamma_5\,,\,\gamma^\mu\} = 0$$

Thus we see that general bispinor solutions of the mass-less Dirac equation are actually characterized by their chirality as they are eigenfunctions of the chirality matrix γ_5 . It follows thereby that in the Weyl, spinorial, chiral representation of the Clifford algebra we find

$$\psi_\pm(x) \equiv \tfrac{1}{2}(\,\mathbb{I} \pm \gamma_5\,)\,\psi(x) \qquad\qquad \gamma_5\,\psi_\pm(x) = \pm\,\psi_\pm(x)$$

$$\psi_-(x) = \int \mathrm{d}\mathbf{p}\ \mathrm{c}_{\mathbf{p}} \begin{pmatrix} u_p(x) \\ 0 \end{pmatrix} + \int \mathrm{d}\mathbf{p}\ \mathrm{d}_{\mathbf{p}}^\dagger \begin{pmatrix} u_{-p}(x) \\ 0 \end{pmatrix}$$

$$\psi_+(x) = \int \mathrm{d}\mathbf{p}\ \mathrm{d}_{\mathbf{p}} \begin{pmatrix} 0 \\ v_p(x) \end{pmatrix} + \int \mathrm{d}\mathbf{p}\ \mathrm{c}_{\mathbf{p}}^\dagger \begin{pmatrix} 0 \\ v_{-p}(x) \end{pmatrix}$$

in such a manner that charge conjugation acts as follows. From the general relation (4.133) valid for any operator valued Dirac bispinor quantum field, *viz.*,

$$\psi^c(x) = \gamma^2 \left(\psi^\dagger(x)\right)^\top$$

if we make use of the real and symmetric chirality or helicity projectors

$$P_\pm \equiv \tfrac{1}{2}(\,\mathbb{I} \pm \gamma_5\,) = (P_\pm)^\top = (P_\pm)^\dagger$$

we can write

$$\begin{aligned} (\psi_\pm(x))^c &= \gamma^2 \left(\psi^\dagger(x) P_\pm\right)^\top = \gamma^2 P_\pm \left(\psi^\dagger(x)\right)^\top \\ &= P_\mp\, \gamma^2 \left(\psi^\dagger(x)\right)^\top = P_\mp\, \psi^c(x) \\ &\equiv \psi_\mp^c(x) \end{aligned} \tag{4.172}$$

in accordance with our previous eqs. (4.170,4.171). The charge conjugated of the left-handed component of a Dirac bispinor is the right component of the charge conjugated Dirac bispinor

$$(\psi_{-}(x))^{c} = (P_{-}\psi(x))^{c} = \psi_{+}^{c}(x) = P_{+}\,\psi^{c}(x)$$

the converse holding true for the right-handed component, the opposite charges of the charge conjugated bispinors being the lepton numbers.

To sum up, the quantum chiral fields $\psi_{\pm}(x)$ describe massless neutrinos ν_e, ν_μ, ν_τ of negative chirality/helicity but positive lepton number, together with the mass-less anti-neutrinos $\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau$ of positive chirality/helicity but negative lepton charge.

4.8.3 Majorana Spinor Field

One can write a relativistic invariant field equation for a massive 2-component left Weyl spinor wave field ψ_L that transforms according to (2.40). Call such a 2-component field $\chi_a(x)$ $(a = 1, 2)$. Let us consider the Weyl spinor wave field as a classical anti-commuting field, *i.e.* a Graßmann valued left Weyl spinor field function over the Minkowski space which satisfies

$$\{\,\chi_a(x)\,,\,\chi_b(y)\,\} = 0 \qquad\qquad (\,x, y \in \mathcal{M} \qquad a, b = 1, 2\,)$$

together with the complex conjugation rule

$$(\chi_1\chi_2)^* = \chi_2^*\chi_1^* = -\,\chi_1^*\chi_2^*$$

so to imitate the Hermitean conjugation of quantum fields.

(a) Show that the classical Lagrangian

$$\mathcal{L}_L = \frac{1}{2}\,\chi^{\dagger}(x)\,\sigma^{\mu}\,i\,\overleftrightarrow{\partial}_{\mu}\,\chi(x) + \;\frac{m}{2}\;\left[\,\chi^{\top}(x)\,\sigma_2\,\chi(x) + \chi^{\dagger}(x)\,\sigma_2\,\chi^{*}(x)\,\right]$$

is real with $\chi^{\dagger} = (\chi^{*})^{\top}$ and yields the Majorana wave field equation

$$i\,\sigma^{\mu}\,\partial_{\mu}\,\chi\,(x) + m\,\sigma_2\,\chi^{*}(x) = 0$$

where $\sigma^{\mu} = (\mathbf{1}\,,\,-\,\sigma_k)$. That is, show that this equation, named the Majorana field equation, is relativistically invariant and that it implies the Klein-Gordon equation $(\Box + m^2)\;\chi(x) \;=\; 0$. This form of the fermion mass is called a Majorana mass term.

(b) Show that all we have obtained before can be formulated in terms of a self-conjugated left handed real bispinor called the free Majorana spinor $\psi_L = \psi_L^* = \psi_L^c$ and derive the symmetries of the corresponding Action.

(c) Perform the quantum theory of the Majorana massive field.

Solution

(a) Consider the transformation rule (2.51)

$$\Lambda_L^{\dagger}\,\sigma^{\,\mu}\,\Lambda_L = \Lambda^{\mu}{}_{\nu}\,\sigma^{\,\nu}$$

Then we immediately get

$$\begin{aligned}
[\,\chi'(x'\,)\,]^{\dagger}\,\sigma^{\,\mu}\,i\,\overleftrightarrow{\partial}'_{\!\mu}\,\chi'(x'\,) &= \chi^{\dagger}\,(x)\,\Lambda_L^{\dagger}\,\sigma^{\,\mu}\,\Lambda_L\,i\,\overleftrightarrow{\partial}_{\!\rho}\,\chi\,(x)\,\Lambda_{\mu}{}^{\rho} \\
&= \chi^{\dagger}\,(x)\,\Lambda^{\mu}{}_{\nu}\,\sigma^{\,\nu}\,i\,\overleftrightarrow{\partial}_{\!\rho}\,\chi\,(x)\,\Lambda_{\mu}{}^{\rho} \\
&= \chi^{\dagger}\,(x)\,\sigma^{\,\mu}\,i\,\overleftrightarrow{\partial}_{\!\mu}\,\chi\,(x)
\end{aligned}$$

which vindicates once more the Poincaré invariance of the left Weyl kinetic Lagrangian

$$\mathfrak{T}_L\,[\,\chi\,] = \frac{1}{2}\,\chi^{\dagger}\,(x)\,\sigma^{\,\mu}\,i\,\overleftrightarrow{\partial}_{\!\mu}\,\chi\,(x)$$

Furthermore we have

$$\begin{aligned}
\chi'^{\top}(x'\,)\,\sigma_2\,\chi'(x'\,) &= \chi^{\top}\,(x)\,\Lambda_L^{\top}\,\sigma_2\,\Lambda_L\,\chi\,(x) \\
&= \chi^{\top}\,(x)\,\sigma_2\,\Lambda_L^{-1}\,\sigma_2^{2}\,\Lambda_L\,\chi\,(x) \\
&= \chi^{\top}(x)\,\sigma_2\,\chi(x) \\
&= -\,i\,\chi_1\,(x)\,\chi_2\,(x) +\; i\,\chi_2\,(x)\,\chi_1\,(x) \\
&= -\,2\,i\;\chi_1\,(x)\,\chi_2\,(x)
\end{aligned}$$

for anti-commuting Graßmann valued Weyl spinor fields. It follows therefrom that the mass term is the real Lorentz scalar

$$\begin{aligned}
\mathcal{L}_L^{m} &= -\,i\,m\,\Big[\,\chi_1\,(x)\,\chi_2\,(x) + \chi_1^{*}\,(x)\,\chi_2^{*}\,(x)\,\Big] \\
&= i\,m\,\Big[\,\chi_2^{*}(x)\,\chi_1^{*}\,(x) + \chi_2\,(x)\,\chi_1\,(x)\,\Big] \;=\; (\mathcal{L}_L^{m})^{*}
\end{aligned}$$

The Lagrange density can be rewritten as

$$\begin{aligned}
\mathcal{L}_L &= \frac{1}{2}\,\chi^{\dagger}(x)\,\sigma^{\,\mu}\,i\,\overleftrightarrow{\partial}_{\!\mu}\chi(x) + \frac{m}{2}\;[\,\chi^{\top}(x)\,\sigma_2\,\chi(x) + \chi^{\dagger}(x)\,\sigma_2\,\chi^{*}(x)\,] \\
&= \chi^{\dagger}(x)\,\sigma^{\,\mu}\,i\,\partial_{\mu}\chi(x) + \frac{m}{2}\;[\,\chi^{\top}(x)\,\sigma_2\,\chi(x) + \chi^{\dagger}(x)\,\sigma_2\,\chi^{*}(x)\,] \\
&- \;\frac{i}{2}\,\partial_{\mu}\Big(\chi^{\dagger}(x)\,\sigma^{\,\mu}\,\chi(x)\Big) \\
&\doteq \chi^{\dagger}(x)\,\sigma^{\,\mu}\,i\,\partial_{\mu}\chi(x) + \frac{m}{2}\;[\,\chi^{\top}(x)\,\sigma_2\,\chi(x) + \chi^{\dagger}(x)\,\sigma_2\,\chi^{*}(x)\,]
\end{aligned}$$

where $\doteq$ means that the 4-divergence term can be disregarded as it does not contribute to the equations of motion. If we treat $\chi_a(x)$ $(a = 1, 2)$ and $\chi^*_a(x)$ $(a = 1, 2)$ as independent field variables, then the Euler-Lagrange field equations yield

$$\partial_\mu \frac{\delta \mathcal{L}_L}{\delta \partial_\mu \chi} = i\, \partial_\mu \chi^\dagger \sigma^\mu = \frac{\delta \mathcal{L}_L}{\delta \chi} = m\, \chi^\top \sigma_2$$

hence, taking the transposed and complex conjugate equation

$$i\,\sigma^\mu \partial_\mu \chi(x) + m\,\sigma_2\, \chi^*(x) = 0 \qquad (I)$$

which is the Majorana field equation. Multiplication to the left by σ_2 and taking complex conjugation gives

$$i\,\sigma_2\, \partial_0\, \chi^*(x) - i\,\sigma_2\, \sigma^*_k\, \partial_k\, \chi^*(x) + m\,\chi(x) = 0$$

Remembering that $\sigma_2\, \sigma^*_k\, \sigma_2 = -\,\sigma_k$ and that $\bar{\sigma}^\mu = (\,\mathbf{1}\,,\,\sigma_k\,)$ we come to the equivalent form of the Majorana left equation, *i.e.*

$$i\,\bar{\sigma}^\mu\, \partial_\mu \sigma_2\, \chi^*(x) + m\,\chi(x) = 0 \qquad (II)$$

Now, if act from the left with the operator $i\,\bar{\sigma}^\nu\,\partial_\nu$ to equation $(\,I\,)$ and use equation $(\,II\,)$ we obtain

$$\bar{\sigma}^\nu \sigma^\mu\, \partial_\nu \partial_\mu\, \chi(x) + m^2\, \chi(x) = (\square + m^2\,)\,\chi(x) = 0$$

so that the left-handed Weyl massive spinor satisfies the Klein-Gordon wave equation.

(b) According to (2.80) we can introduce the Majorana left handed self-conjugated bispinor

$$\chi_L(x) = \begin{pmatrix} \chi(x) \\ -\,\sigma_2\,\chi^*(x) \end{pmatrix} = \chi^c_L(x) \qquad (III)$$

with the Lagrange density

$$\mathcal{L}_L = \tfrac{1}{4}\,\overline{\chi}_L(x)\, i\overleftrightarrow{\partial\!\!\!/}\, \chi_L(x) \;-\; \tfrac{1}{2}\, m\,\overline{\chi}_L(x)\,\chi_L(x)$$

in such a manner that the Majorana mass term can be written in the two equivalent forms

$$\mathcal{L}^m_L \;=\; \frac{m}{2}\,\big[\,\chi^\top(x)\,\sigma_2\,\chi(x) + \chi^\dagger(x)\,\sigma_2\,\chi^*(x)\,\big] \;=\; -\,\frac{m}{2}\,\bar{\chi}_L(x)\,\chi_L(x)$$

so that it is clear that the Majorana's Action $\int \mathrm{d}x\, \mathcal{L}_L$ is no longer invariant under the phase transformation

$$\chi(x) \longmapsto \chi'(x) = \chi(x)\, \mathrm{e}^{i\alpha}$$

Taking the Graßmann valued Majorana left handed spinor wave field $\chi_L(x)$ to be defined by the self-conjugation constraint $(\,III\,)$, it is easy to see that the pair of coupled Weyl equations $(\,I\,)$ and $(\,II\,)$ is equivalent to the single bispinor equation

$$(\alpha^\mu\, i\partial_\mu - \beta\, m)\, \chi_L(x) = 0 \qquad\qquad (\,IV\,)$$

in which

$$\alpha^\mu = \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \bar\sigma^\mu \end{pmatrix} \qquad \beta = \gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma^\mu = \beta\, \alpha^\mu$$

while the Majorana Lagrangian then becomes

$$\mathcal{L}_L = \frac{1}{4}\, \chi_L^\dagger(x)\, \alpha^\mu\, i\overleftrightarrow{\partial}_\mu \chi_L(x) - \frac{m}{2}\, \chi_L^\dagger(x)\, \beta\, \chi_L(x)$$

It is immediate to verify that the bispinor form $(\,IV\,)$ of the field equations does coincide with the two equivalent forms $(\,I\,)$ and $(\,II\,)$ of the Majorana wave field equation: namely,

$$\begin{cases} i\,\sigma^\mu\, \partial_\mu \chi(x) + m\, \sigma_2\, \chi^*(x) & = \; 0 \\ i\,\bar\sigma^\mu\, \sigma_2\, \partial_\mu \chi^*(x) + m\, \chi(x) & = \; 0 \end{cases}$$

Since the Majorana spinor wave field $\chi_L(x)$ has the constraint $(\,III\,)$ which relates the lower two components to the complex conjugate of the two upper components, a representation must exist which makes the Majorana spinor wave field real, with the previous two independent complex variables $\chi_a \in \mathbb{C}\ (a = 1, 2)$ replaced by the four real variables $\psi_{M,\alpha} \in \mathbb{R}\ (\alpha = 1, 2, 3, 4)$. To obtain this real representation, we note that

$$\chi_L = \begin{pmatrix} \chi \\ -\,\sigma_2\, \chi^* \end{pmatrix} \qquad \chi_L^* = \begin{pmatrix} 0 & -\,\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \chi_L$$

A transformation to left handed real bispinor fields $\psi_M = \psi_M^*$ can be made by writing

$$\chi_L = S\, \psi_M \qquad\qquad \chi_L^* = S^*\, \psi_M = S^*\, S^{-1}\, \chi_L$$

whence

$$S^* = \begin{pmatrix} 0 & -\,\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} S$$

Now, if we set

$$\begin{pmatrix} 0 & i\,\sigma_2 \\ -\,i\,\sigma_2 & 0 \end{pmatrix} \; = \; i\gamma^2 \;\equiv \rho_2$$

$$\rho_2 \;=\; \rho_2^\dagger \qquad\qquad \rho_2^2 = \mathbb{I}$$

so that

$$\exp\{i\,\rho_2\,\theta\} = \mathbb{I}\,\cos\theta + i\,\rho_2\,\sin\theta$$

then the solution for the above relation is the unitary matrix

$$S = \exp\{-\,\pi\,i\,\rho_2\,/\,4\} = \frac{\sqrt{2}}{2}\left(\mathbb{I} - i\,\rho_2\right)$$

which fulfills

$$S = \frac{\sqrt{2}}{2}\begin{pmatrix} \mathbf{1} & \sigma_2 \\ -\,\sigma_2 & \mathbf{1} \end{pmatrix} \; = \; \frac{\sqrt{2}}{2}\left(\mathbb{I} + \gamma^2\right)$$

or even more explicitly

$$S = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$$

Thus we can suitably make use of the so called Majorana representation for the Clifford algebra which is given by the similarity transformation acting on the $\gamma-$matrices in the Weyl representation, *viz.*,

$$\gamma_M^\mu \equiv \; S^\dagger\,\gamma^\mu\,S$$

$$\gamma_M^0 = \begin{pmatrix} -\,\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ -\,i & 0 & 0 & 0 \\ 0 & 0 & 0 & -\,i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\gamma_M^1 = \begin{pmatrix} -\,i\,\sigma_3 & 0 \\ 0 & -\,i\,\sigma_3 \end{pmatrix} = \begin{pmatrix} -\,i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -\,i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

$$\gamma_M^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\,\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\,i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -\,i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_M^3 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\gamma_M^5 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

which satisfy by direct inspection

$$\{\gamma_M^\mu\,,\,\gamma_M^\nu\} = 2\,g^{\mu\nu} \qquad \{\gamma_M^\nu\,,\,\gamma_M^5\} = 0$$

$$\gamma_M^0 = \gamma_M^{0\,\dagger} \qquad \gamma_M^k = -\,\gamma_M^{k\,\dagger} \qquad \gamma_M^5 = \gamma_M^{5\,\dagger}$$

$$\gamma_M^\nu \;=\; -\,\gamma_M^{\nu\,*} \qquad \gamma_M^5 \;=\; -\,\gamma_M^{5\,*}$$

The result is that, at the place of a complex self-conjugated bispinor, which corresponds to a left handed Weyl spinor, one can safely employ a real Majorana bispinor: namely,

$$\chi_L(x) = \chi_L^c(x) \qquad \leftrightarrow \qquad \psi_M\,(x) = S^\dagger\,\chi_L(x) = \psi_M^*\,(x)$$

A quite analogous construction can obviously be made, had we started from a right handed Weyl spinor. In so doing, the Majorana Lagrangian and the ensuing Majorana wave field equation take the form

$$\mathcal{L}_M = \tfrac{1}{4}\;\psi_M^\top\,(x)\,\alpha_M^\nu\,i\,\overset{\leftrightarrow}{\partial}_\nu\,\psi_M\,(x) - \tfrac{1}{2}\,m\;\psi_M^\top\,(x)\,\beta_M\,\psi_M\,(x)$$

$$(i\partial\!\!\!/_M - m)\,\psi_M\,(x) = 0 \qquad\qquad \psi_M\,(x) = \psi_M^*\,(x)$$

$$\alpha_M^\nu = \gamma_M^0\,\gamma_M^\nu \qquad \alpha_M^0 = \mathbb{I} \qquad \beta_M \equiv \gamma_M^0$$

The only relic internal symmetry of the Majorana's Action is the discrete $\mathbb{Z}_2$ symmetry, *i.e.* $\psi_M\,(x) \longmapsto -\,\psi_M\,(x)$. The Majorana Hamiltonian reads

$$H_M = \alpha_M^k\,\hat{p}^{\,k} + m\,\beta_M \qquad\qquad (\,\hat{p}^{\,k} = \,-\,i\,\nabla_k\,)$$

To solve the Majorana wave equation we set

$$\psi_M\,(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^{3/2}}\;\widetilde{\psi}_M\,(p)\,\exp\{-\,ip\cdot x\}$$

with the reality condition

$$\widetilde{\psi}_M^*\,(p) = \widetilde{\psi}_M\,(-\,p)$$

so that

$$(\not{p}_M - m)\,\widetilde{\psi}_M(p) = 0 \qquad \not{p}_M \equiv p_\nu \gamma_M^\nu$$

which implies

$$\begin{array}{ll} \widetilde{\psi}_{M,\alpha}(p) = (\not{p}_M + m)_{\alpha\beta}\,\widetilde{\phi}_\beta(p) & \\ \left(p^2 - m^2\right)\widetilde{\phi}_\alpha(p) = 0 & (\alpha = 1,2,3,4) \\ \widetilde{\phi}_\alpha(p) = \delta\left(p^2 - m^2\right)\, f_\alpha(p) & \end{array}$$

Furthermore, from the reality condition on the Majorana spinor field

$$\widetilde{\psi}^{\,*}_{M,\alpha}(p) = \widetilde{\psi}_{M,\alpha}(-\,p)$$

we find

$$\begin{aligned} \widetilde{\psi}^{\,*}_{M,\alpha}(p) &= (\not{p}^{\,*}_M + m)_{\alpha\beta}\,\widetilde{\phi}^{\,*}_\beta(p) \\ &= (-\not{p}_M + m)_{\alpha\beta}\,\widetilde{\phi}^{\,*}_\beta(p) \\ &= \widetilde{\psi}_{M,\alpha}(-\,p) \qquad \Longleftrightarrow \qquad f^{\,*}_\beta(p) = f_\beta(-\,p) \end{aligned}$$

thanks to the circumstance that the $\gamma-$matrices are purely imaginary in the Majorana representation. Then we can write

$$\begin{aligned} \psi_M(x) &= \int_{-\infty}^{\infty} \mathrm{d}p_0 \int \mathrm{d}\mathbf{p}\ \left[\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,\right]^{-1}\, \theta(p_0) \\ &\times\ (\not{p}_M + m)_{\alpha\beta}\, f_\beta(p)\, \delta\,(p_0 - \omega_{\mathbf{p}})\, \exp\{-\,ip\cdot x\} \\ &+\ \int_{-\infty}^{\infty} \mathrm{d}p_0 \int \mathrm{d}\mathbf{p}\ \left[\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,\right]^{-1}\, \theta(-\,p_0) \\ &\times\ (\not{p}_M + m)_{\alpha\beta}\, f_\beta(p)\, \delta\,(p_0 + \omega_{\mathbf{p}})\, \exp\{-\,ip\cdot x\} \\ &= 2m \int \mathrm{d}\mathbf{p}\ \left[\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,\right]^{-1} \mathcal{E}^{+}_M(p)\, f(p)\, e^{-\,ip\cdot x} \\ &+\ 2m \int \mathrm{d}\mathbf{p}\ \left[\,(2\pi)^3\, 2\omega_{\mathbf{p}}\,\right]^{-1} \mathcal{E}^{-}_M(p)\, f^{\,*}(p)\, e^{\,ip\cdot x} \\ &\stackrel{\mathrm{def}}{=} \sum_{\mathbf{p}} \left[\,\mathcal{E}^{+}_M(p)\ f_{\mathbf{p}}\, e^{-\,ip\cdot x} + \mathcal{E}^{-}_M(p)\ f^{\,*}_{\mathbf{p}}\, e^{\,ip\cdot x}\,\right] \\ &= \psi^{\,*}_M(x) \end{aligned}$$

where $p_0 = \omega_{\mathbf{p}}$, whereas $f_{\mathbf{p}} = 2mf(p)/\sqrt{(2\pi)^3\, 2\omega_{\mathbf{p}}}$. The projectors onto the spin states are

$$\mathcal{E}^{\pm}_M(p) = (\,m \pm \not{p}_M)\,/\,2m \qquad\qquad (p_0 = \omega_{\mathbf{p}})$$

with

$$[\,\mathcal{E}_M^{\pm}(p)\,]^* = \mathcal{E}_M^{\mp}(p) \qquad [\,\mathcal{E}_M^{\pm}(p)\,]^{\dagger} = \mathcal{E}_M^{\pm}(\tilde{p}) \qquad (\,\tilde{p}^{\mu} = p_{\mu}\,)$$

$$[\,\mathcal{E}_M^{\pm}(p)\,]^2 = \mathcal{E}_M^{\pm}(p) \qquad \mathcal{E}_M^{\pm}(p)\,\mathcal{E}_M^{\mp}(p) = 0$$

$$\operatorname{tr}\mathcal{E}_M^{\pm}(p) = 2 \qquad \mathcal{E}_M^{+}(p) + \mathcal{E}_M^{-}(p) = \mathbb{I}$$

Now, in order to set up the spin states of the Majorana real spinor field, let me start from the spin matrices in the Majorana representation

$$\Sigma_{M,1} = i\,\gamma_M^2\,\gamma_M^3 = \left(\begin{array}{cc} 0 & i\,\sigma_3 \\ -\,i\,\sigma_3 & 0 \end{array}\right)$$

$$\Sigma_{M,2} = i\,\gamma_M^3\,\gamma_M^1 = \left(\begin{array}{cc} \sigma_2 & 0 \\ 0 & \sigma_2 \end{array}\right)$$

$$\Sigma_{M,3} = i\,\gamma_M^1\,\gamma_M^2 = \left(\begin{array}{cc} 0 & -\,i\,\sigma_1 \\ i\,\sigma_1 & 0 \end{array}\right)$$

and from the common eigenvectors of the matrix β_M and of the diagonal spin matrix $\Sigma_{M,2}$ that are *e.g.*

$$\xi_+ \equiv \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ i \end{array}\right) \qquad \xi_- \equiv \left(\begin{array}{c} i \\ 1 \\ 0 \\ 0 \end{array}\right) \qquad \gamma_M^0\,\xi_{\pm} = \xi_{\pm}$$

or equivalently

$$\eta_+ \equiv \left(\begin{array}{c} 1 \\ i \\ 0 \\ 0 \end{array}\right) \qquad \eta_- \equiv \left(\begin{array}{c} 0 \\ 0 \\ i \\ 1 \end{array}\right) \qquad \gamma_M^0\,\eta_{\pm} = -\,\eta_{\pm}$$

which do indeed satisfy by direct inspection

$$\gamma_M^0\,\xi_{\pm} = \xi_{\pm} \qquad \xi_{\mp}^{\dagger}\,\xi_{\pm} = 0 \qquad (\,\Sigma_{M,2} \mp 1\,)\,\xi_{\pm} = 0$$

in such a manner that we have by construction

$$\xi_r^{\dagger}\,\xi_s = 2\,\delta_{rs} \qquad \xi_r^{\dagger}\,\gamma_M^k\,\xi_s = 0 \qquad \forall\, r,s = \pm\ \vee\ k = 1,2,3$$

Notice that, for example, the spin states ξ_r $(\,r = \pm\,)$ are in fact the two degenerate eigenstates of the Majorana Hamiltonian in the massive neutral spinor particle rest frame $\mathbf{p}=0$ with positive eigenvalue $p_0=m$ and with opposite spin projections on the OY axis, *viz.,*

$$\sigma_M^{31}\,\xi_\pm \equiv \tfrac{1}{4}\,i\,[\,\gamma_M^3\,,\,\gamma_M^1\,]\,\xi_\pm = \tfrac{1}{2}\,\Sigma_{M,2}\,\xi_\pm = \pm\,\tfrac{1}{2}\,\xi_\pm$$

Then we define the Majorana spin states to be

$$\begin{cases} u_r\,(p) \equiv 2m\,(2\omega_{\mathbf{p}}+2m)^{-\frac{1}{2}}\,\mathcal{E}_M^{+}\,(p)\,\xi_r \\ u_r^*\,(p) \equiv 2m\,(2\omega_{\mathbf{p}}+2m)^{-\frac{1}{2}}\,\mathcal{E}_M^{-}\,(p)\,\xi_r^* \end{cases} \qquad (\,r=\pm\,\vee\,p_0=\omega_{\mathbf{p}}\,)$$

which are the two eigenstates of the positive energy projector

$$\mathcal{E}_M^{+}\,(p)\,u_r\,(p) = u_r\,(p) \qquad\qquad (\,r=+,-\,\vee\;p_0=\omega_{\mathbf{p}}\,)$$

with

$$u_r^\dagger\,(p)\,u_s\,(p) = 2\,\omega_{\mathbf{p}}\,\delta_{rs}$$

In fact we have for instance

$$\begin{aligned} u_r^\dagger(p)\,u_s(p) &= (2\omega_{\mathbf{p}}+2m)^{-1}\;\xi_r^\dagger\,(m+\tilde{\not{p}}_M)\,(m+\not{p}_M)\,\xi_s \\ &= (2\omega_{\mathbf{p}}+2m)^{-1}\;\xi_r^\dagger\left(2\omega_{\mathbf{p}}^2+2m\,\omega_{\mathbf{p}}\right)\xi_s \\ &= 2\,\omega_{\mathbf{p}}\,\tfrac{1}{2}\,\xi_r^\dagger\,\xi_s = 2\,\omega_{\mathbf{p}}\,\delta_{rs} \end{aligned}$$

in which I have made use of the property

$$\xi_r^\dagger\,\gamma_M^k\,\gamma_M^0\,\xi_s = \xi_r^\dagger\,\gamma_M^k\,\xi_s = 0 \qquad\qquad \forall\,r,s=+,-\,\vee\;k=1,2,3$$

Moreover we can easily verify that

$$\bar{u}_r(p)\,u_s(p) = 2m\;\delta_{rs}$$

so that we can write

$$\sum_{r=+,-} u_r(p)\,\bar{u}_r(p) = m+\not{p}_M$$

Of course, a completely equivalent construction can be made, had we started from the further degenerate pair of the constant eigenspinors η_r $(r=+,-)$ of the matrix β_M .

In conclusion, the general normal mode decomposition of the Majorana real spinor wave field becomes

$$\psi_M\,(x) = \sum_{\mathbf{p}\,,\,r}\,\Big[\,a_{\mathbf{p}\,,\,r}\,u_{\mathbf{p}\,,\,r}(x) + a^*_{\mathbf{p}\,,\,r}\,u^*_{\mathbf{p}\,,\,r}(x)\,\Big] = \psi_M^*\,(x)$$

with

$$u_{\mathbf{p},r}(x) \equiv [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\;u_r(p)\;\exp\{-\,i\omega_{\mathbf{p}}\,t + i\,\mathbf{p}\cdot\mathbf{x}\}$$

$$\{a_{\mathbf{p},r}\,,\,a_{\mathbf{q},s}\} = \{a_{\mathbf{p},r}\,,\,a^*_{\mathbf{q},s}\} = \{a^*_{\mathbf{p},r}\,,\,a^*_{\mathbf{q},s}\} \;=\; 0$$
$$\forall\,\mathbf{p}\,,\,\mathbf{q}\in\mathbb{R}^3 \qquad \forall\,r\,,\,s=+,-$$

It is important to realize that the real Majorana massive spinor wave field does exhibit two opposite helicity states.

(c) The transition to the quantum theory is performed as usual by means of the creation annihilation operators $a_{\mathbf{p},r}$ and $a^\dagger_{\mathbf{p},s}$ which satisfy the canonical anti-commutation relations

$$\{a_{\mathbf{p},r}\,,\,a_{\mathbf{q},s}\} = 0 = \{a^\dagger_{\mathbf{p},r}\,,\,a^\dagger_{\mathbf{q},s}\}$$
$$\{a_{\mathbf{p},r}\,,\,a^\dagger_{\mathbf{q},s}\} \;=\; \delta_{rs}\,\delta\,(\mathbf{p}-\mathbf{q})$$
$$\forall\,\mathbf{p}\,,\,\mathbf{q}\in\mathbb{R}^3 \qquad \forall\,r\,,\,s=+,-$$

so that

$$\psi_M(x) = \sum_{\mathbf{p},r}\,\Big[\,a_{\mathbf{p},r}\,u_{\mathbf{p},r}(x) + a^\dagger_{\mathbf{p},r}\,u^*_{\mathbf{p},r}(x)\,\Big] = \psi^\dagger_M(x)$$

where the Hermitean conjugation refers to the creation destruction operators acting on the Fock space. Instead we have the expansion of the left Majorana adjoint spinor

$$\bar\psi_M(x) = \sum_{\mathbf{p},r}\,\Big[\,a^\dagger_{\mathbf{p},r}\,\bar u_{\mathbf{p},r}(x) + a_{\mathbf{p},r}\,\bar u^*_{\mathbf{p},r}(x)\,\Big] = \bar\psi^\dagger_M(x)$$

in which

$$\bar u_{\mathbf{p},r}(x) \equiv [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\;[\,u_r^\top(p)\,]^*\,\gamma^0_M\;\exp\{\,i\omega_{\mathbf{p}}\,t - i\,\mathbf{p}\cdot\mathbf{x}\}$$

Notice that from the normalization

$$(\,u_{\mathbf{p},r}\,,\,u_{\mathbf{q},s}\,) = \int \mathrm{d}\mathbf{x}\;\bar u_{\mathbf{p},r}(x)\,\gamma^0_M\,u_{\mathbf{q},s}(x) = \delta_{rs}\,\delta(\mathbf{p}-\mathbf{q})$$

one can easily obtain all the observable quantities involving the Majorana massive spinor field. For example the energy-momentum tetra-vector takes the form

$$\begin{aligned} P_\mu &= \frac{i}{2}\int \mathrm{d}\mathbf{x}\;\;:\bar\psi_M(x)\,\gamma^0_M\,\overleftrightarrow{\partial}_\mu\,\psi_M(x): \\ &= \sum_{\mathbf{p},r}\,p_\mu\,a^\dagger_{\mathbf{p},r}\,a_{\mathbf{p},r} \qquad\qquad (\,p_0=\omega_{\mathbf{p}}\,) \end{aligned}$$

Moreover, for example, if $\partial_1\psi_M = \partial_3\psi_M = 0$ that implies $\psi_M(t,y) = \psi_M(t,0,y,0)$, then we obtain for the energy-momentum tensor

$$
\begin{aligned}
2i\,T^{13}(t,y) &= \bar{\psi}_M(t,y)\,\gamma_M^1 \overset{\leftrightarrow}{\partial}_z \psi_M(t,y) = 0 \\
2i\,T^{31}(t,z) &= \bar{\psi}_M(t,y)\,\gamma_M^3 \overset{\leftrightarrow}{\partial}_x \psi_M(t,y) = 0
\end{aligned}
$$

and thereby

$$
\partial_\mu\, M_{13}^{\,\mu} = \partial_\mu\, S_{13}^{\,\mu} = 0 \tag{4.173}
$$

whence it follows that the helicity is conserved in time. After insertion of the normal modes expansion one gets

$$
\begin{aligned}
\mathrm{h} &= \int_{-\infty}^{\infty} \mathrm{d}y \;\; :\tfrac{1}{2}\,\psi_M(t,y)\,\Sigma_{M,2}\,\psi_M(t,y): \\
&= \int_{-\infty}^{\infty} \mathrm{d}y \;\; :\sum_{p\,,\,r} \Big[\, a_{\,p\,,\,r}\, u_{\,p\,,\,r}(t,y) + a^\dagger_{\,p\,,\,r}\, u^*_{\,p\,,\,r}(t,y) \Big] \\
&\times \;\; \tfrac{1}{2}\,\Sigma_{M,2} \sum_{q\,,\,s} \Big[\, a^\dagger_{\,q\,,\,s}\, u^*_{\,q\,,\,s}(t,y) + a_{\,q\,,\,s}\, u_{\,q\,,\,s}(t,y) \Big]:
\end{aligned}
$$

in which we have set

$$
\mathbf{p} = (0,p,0) \qquad\qquad \mathbf{q} = (0,q,0) \qquad\qquad \omega_p = \sqrt{p^2+m^2}
$$

$$
u_{\,p\,,\,r}(t,y) = [\,4\pi\omega_{\,p}\,]^{-1/2}\, u_r(p)\, \exp\{ipy - -it\omega_p\} \qquad (\,r = +,-\,)
$$

the normalization being now consistent with the occurrence that the spinor plane waves are independent of the transverse spatial coordinates $x_\perp = (x^1\,,\,x^3)$. From the commutation relation

$$
[\,\omega_p\gamma_M^0 - p\,\gamma_M^2\,,\,\Sigma_{M,2}\,] = 0
$$

together with the definition

$$
u_\pm(p) \equiv (2\omega_{\,p} + 2m)^{-1/2}\,\left(m + \omega_{\,p}\gamma_M^0 - p\,\gamma_M^2\right)\xi_\pm
$$

it can be readily derived that

$$
(\,\Sigma_{M,2} \mp 1\,)\,\xi_\pm = 0 \qquad\Rightarrow\qquad (\,\Sigma_{M,2} \mp 1\,)\,u_\pm(p) = 0
$$

which yields in turn

$$
\mathrm{h} = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p\, \Big[\, a^\dagger_{\,p\,,\,+}\, a_{\,p\,,\,+} - a^\dagger_{\,p\,,\,-}\, a_{\,p\,,\,-} \Big]
$$

The 1-particle states $a^\dagger_{\,\mathbf{p}\,,\,\pm}\,|\,0\,\rangle$ represent neutral Majorana massive particles with energy-momentum $p^{\,\mu} = (\omega_{\mathbf{p}},\,\mathbf{p})$ and positive/negative helicity.

4.8.4 Dirac and Majorana Mass Terms

Write the most general mass term involving left and right complex Weyl bispinors and real self-conjugated Majorana bispinors. Discuss its structure.

Solution. By making use of the real, symmetric and diagonal chirality or helicity projectors in the Weyl representation of the Clifford algebra

$$P_\pm \equiv \tfrac{1}{2}(\mathbb{I} \pm \gamma_5) = (P_\pm)^\top = (P_\pm)^\dagger$$

we can write

$$\begin{aligned}(\psi_\pm(x))^c &= \gamma^2 \left(\psi^\dagger(x) P_\pm\right)^\top = \gamma^2 P_\pm \left(\psi^\dagger(x)\right)^\top \\ &= P_\mp \gamma^2 \left(\psi^\dagger(x)\right)^\top = P_\mp \, \psi^c(x) \\ &\equiv \psi^c_\mp(x)\end{aligned}$$

Thus the charge conjugated of the left–handed component of a Dirac bispinor is the right component of the charge conjugated Dirac bispinor

$$(\psi_-(x))^c = (P_- \psi(x))^c = \psi^c_+(x) = P_+ \, \psi^c(x)$$

the converse holding true for the right-handed component, the opposite charges of the charge conjugated bispinors being the lepton numbers.

A real Dirac-type mass term in the Lagrangian does indeed connect the independent left and right components of the same Dirac bispinor

$$\begin{aligned}\mathcal{L}^m_D &= -\, m \, \bar{\psi}(x) \, \psi(x) \\ &= -\, m \left(\psi^\dagger_L(x) \, \psi_R(x) + \psi^\dagger_R(x) \, \psi_L(x)\right) = (\mathcal{L}^m_D)^*\end{aligned}$$

From a Dirac bispinor $\psi(x)$ one can always build up a pair of self-conjugated Majorana bispinors $\varphi(x)$ and $\chi(x)$ according to

$$\chi(x) = \psi_-(x) + (\psi_-(x))^c = \psi_-(x) + \psi^c_+(x) \tag{4.174}$$

$$\varphi(x) = \psi_+(x) + (\psi_+(x))^c = \psi_+(x) + \psi^c_-(x) \tag{4.175}$$

where use has been made of the charge conjugation properties connecting left-handed and right-handed Weyl spinor fields. Of course, by the very construction we have

$$\varphi(x) = \varphi^c(x) \qquad\qquad \chi(x) = \chi^c(x)$$

The inversion formulæ can be easily obtained through the application of the chirality or helicity projectors, that yields

$$\psi_-(x) = P_- \, \chi(x) \equiv \begin{pmatrix} \psi_L(x) \\ 0 \end{pmatrix}$$

$$\psi_+(x) = P_+\,\varphi(x) \equiv \begin{pmatrix} 0 \\ \psi_R(x) \end{pmatrix}$$

$$\psi_+^c(x) = P_+\,\chi(x) \equiv \begin{pmatrix} 0 \\ -\,\sigma_2\psi_L^*(x) \end{pmatrix}$$

$$\psi_-^c(x) = P_-\,\varphi(x) \equiv \begin{pmatrix} \sigma_2\psi_R^*(x) \\ 0 \end{pmatrix}$$

so that

$$\chi(x) = \begin{pmatrix} \psi_L(x) \\ -\,\sigma_2\psi_L^*(x) \end{pmatrix}$$

$$\varphi(x) = \begin{pmatrix} \sigma_2\psi_R^*(x) \\ \psi_R(x) \end{pmatrix}$$

It follows therefrom that the self-conjugated bispinor $\varphi(x)$ can be constructed from the left Weyl spinor $\psi_L(x)$, while the self-conjugated bispinor $\chi(x)$ out of the right Weyl spinor $\psi_R(x)$. Hence we can build up two Majorana-type mass terms which contain only left and right Weyl spinors respectively: namely,

$$\begin{aligned}
\mathcal{L}_L^m &= -\,\tfrac{1}{2}\,m_L\,\bar{\chi}(x)\,\chi(x) = (\mathcal{L}_L^m)^* \\
&= -\,\tfrac{1}{2}\,m_L\left(\psi_-^\dagger(x)\,\gamma^0\,\psi_+^c(x) + \overline{\psi_+^c}(x)\,\psi_-(x)\right) \\
&= \tfrac{1}{2}\,m_L\left[\,\psi_L^\dagger(x)\,\sigma_2\,\psi_L^*(x) + \psi_L^\top(x)\,\sigma_2\,\psi_L(x)\,\right] \\
\mathcal{L}_R^m &= -\,\tfrac{1}{2}\,m_R\,\bar{\varphi}(x)\,\varphi(x) = (\mathcal{L}_R^m)^* \\
&= -\,\tfrac{1}{2}\,m_R\left(\psi_+^\dagger(x)\,\gamma^0\,\psi_-^c(x) + \overline{\psi_-^c}(x)\,\psi_+(x)\right) \\
&= -\,\tfrac{1}{2}\,m_R\left[\,\psi_R^\top(x)\,\sigma_2\,\psi_R(x) + \psi_R^\dagger(x)\,\sigma_2\,\psi_R^*(x)\,\right]
\end{aligned}$$

This means that the Majorana mass terms do mix the mass-less chirality and helicity eigenstates $\psi_\pm(x)$, in such a manner that the lepton number conservation breaks down. The application of the chirality matrix γ_5 yields

$$\gamma_5\,\psi(x) = \psi_+(x) - \psi_-(x) = \psi'(x)$$

$$\gamma_5\,\chi(x) = \psi_+^c(x) - \psi_-(x) = \chi'(x)$$

$$\gamma_5\,\varphi(x) = \psi_+(x) - \psi_-^c(x) = \varphi'(x)$$

so that the the action of the chirality matrix γ_5 clearly changes the signs of all the mass terms. As a consequence, the bispinor fields $\psi'(x), \varphi'(x), \chi'(x)$ can be interpreted as the correct mass eigenstates for the minus values of the masses $m, m_\pm$ because *e.g.*

$$\mathcal{L}_R^m = -\,\tfrac{1}{2}\,m_+\,\bar{\varphi}(x)\,\varphi(x) = \tfrac{1}{2}\,m_+\,\bar{\varphi}'(x)\,\varphi'(x)$$

Notice as well that the right and left bispinors $\varphi'(x)$ and $\chi'(x)$ are in turn anti-self-conjugated because $\varphi'^c = -\varphi'$ and $\chi'^c = -\chi'$. Whenever both the Dirac and Majorana mass terms are simultaneously present we can write

$$\begin{aligned}\mathcal{L}_{DM} &= -\tfrac{1}{2}\,m\,(\bar{\chi}\varphi + \bar{\varphi}\chi) - \tfrac{1}{2}\,m_L\,\bar{\chi}\,\chi - \tfrac{1}{2}\,m_R\,\bar{\varphi}\,\varphi \\ &= -\tfrac{1}{2}\begin{pmatrix} \bar{\chi} & \bar{\varphi} \end{pmatrix}\begin{pmatrix} m_L & m \\ m & m_R \end{pmatrix}\begin{pmatrix} \chi \\ \varphi \end{pmatrix}\end{aligned}$$

the 2×2 mass matrix being readily made diagonal to yields the pair of distinct eigenvalues

$$M_\pm = \tfrac{1}{2}\Big[\,m_L + m_R \pm \sqrt{(m_R - m_L)^2 + 4\,m^2}\,\Big] \qquad (\,I\,)$$

In order to find the eigenstates, consider a general $O(2,\mathbb{R})$ transformation involving the left and right Majorana bispinors: namely,

$$\left.\begin{aligned}\chi(x) &= \cos\theta\,\eta(x) + \sin\theta\,\xi(x) \\ \varphi(x) &= -\,\sin\theta\,\eta(x) + \cos\theta\,\xi(x)\end{aligned}\right\}$$

After substitution into the mass Lagrangian $\mathcal{L}_{DM}$ we readily find

$$\begin{aligned}\mathcal{L}_{DM} &= -\tfrac{1}{2}\left(m_L\,\cos^2\theta + m_R\,\sin^2\theta - m\,\sin 2\theta\right)\bar{\eta}(x)\,\eta(x) \\ &- \tfrac{1}{2}\left(m_L\,\sin^2\theta + m_R\,\cos^2\theta + m\,\sin 2\theta\right)\bar{\xi}(x)\,\xi(x) \\ &- \tfrac{1}{2}[\,\bar{\eta}(x)\,\xi(x) + \bar{\xi}(x)\,\eta(x)\,]\left[\,m\,\cos 2\theta - \tfrac{1}{2}(\,m_R - m_L\,)\,\sin 2\theta\,\right]\end{aligned}$$

corresponding to the pair of Majorana bispinors mass eigenstates

$$\left.\begin{aligned}\eta(x) &= \cos\theta\,\chi(x) - \sin\theta\,\varphi(x) = \eta^c(x) \\ \xi(x) &= \sin\theta\,\chi(x) + \cos\theta\,\varphi(x) = \xi^c(x)\end{aligned}\right\} \qquad (\,II\,)$$

if and only if the following relation holds true, *viz.*,

$$\tan 2\theta = 2m/(m_R - m_L) \qquad (\,III\,)$$

Since any massive right-handed neutrino has never been detected so far, one is naturally led to assume that $m_R \gg m_L, m$ in such a manner that a pretty tiny angle $\theta \approx \frac{1}{2}\arctan(2m/m_R)$ is expected to yield the mixing which corresponds to the so called seesaw mechanism[8]. Notice that the eigenvalues (I) can be suitably rewritten in the form

$$M_\pm = \tfrac{1}{2}(\,m_L + m_R\,) \pm \frac{m}{\sin 2\theta}$$

[8]Rabindra Nath Mohapatra and Goran Senjanovic, *Exact Left-Right Symmetry and Spontaneous Violation of Parity*, The Physical Review D**12** (1975) 1512-1523.

for $\sin 2\theta > 0$, that yields in turn

$$2m = (\, M_+ - M_- \,)\, \sin 2\theta \qquad\qquad M_+ + M_- = m_R + m_L$$

In the limit of a tiny mass $m_L \longrightarrow 0$ of the left-handed Majorana spinor, one eventually gets

$$M_+ = m_L \, \sin^2\theta + m_R \, \cos^2\theta + m \, \sin 2\theta \,\simeq\, m_R$$

$$M_- = m_L \, \cos^2\theta + m_R \, \sin^2\theta - m \, \sin 2\theta \,\simeq\, m_L - \frac{m^2}{m_R}$$

Some plausibly realistic estimates might be $m \simeq M_Z = O(100 \text{ GeV})$ while $m_R = O(10^{16} \text{ GeV})$ so that $m_L \approx M_- = O(10^{-3} \text{ eV})$.

To sum up, the most general mass term for a four-component bispinor field $\psi(x)$ does actually describe two real Majorana bispinors with distinctive masses but mixed chirality or helicity eigenstates $\psi_\pm(x)$. Finally, taking an alternative though equivalent point of view, we can define the manifestly self-conjugated bispinor fields

$$\Psi_+(x) \equiv \psi(x) + \psi^c(x) \qquad\qquad \Psi_-(x) \equiv i\psi(x) - i\psi^c(x)$$

$$\Psi_\pm^c(x) = \Psi_\pm(x)$$

Then we obtain

$$\begin{aligned}\mathcal{L}_{DM} &= -\tfrac{1}{2}\, M \left(\overline{\psi}\,\psi + \overline{\psi}^{\,c}\psi^{\,c} \right) - \tfrac{1}{2}\, m \left(\overline{\psi}\,\psi^{\,c} + \overline{\psi}^{\,c}\psi \right) \\ &= -\tfrac{1}{2}(m+M)\overline{\Psi}_+\,\Psi_+ + \tfrac{1}{2}(M-m)\overline{\Psi}_-\,\Psi_-\end{aligned}$$

Since the Majorana-type mass terms do violate the conservation of any additive quantum number carried by the fermion field, such as electric charge, lepton number and so on, it follows that all elementary charged fermions must have $m_L = m_R = 0$. For neutrinos and anti-neutrinos, instead, a Majorana-type mass term violates the lepton number by two units because $L(\bar{\varphi}\,\varphi) = 2$. The presence of such a kind of Majorana masses leads, for example, to neutrino-less double beta decays $(Z-1) \to (Z+1)\, e^-\, e^-$, or Kaon decays such as $K^- \to \pi^+\, e^-\, e^-$ in which $\Delta\, L = -\,2$.

Chapter 5

The Vector Field

The quantum theory of a relativistic massive vector wave field has been first developed at the very early stage of the quantum field theory by the Romanian theoretical and mathematical great physicist

Alexandru Proca (Bucarest, 16.10.1897 – Paris, 13.12.1955)
Sur les equations fondamentales des particules elémentaires
Comptes Rendu Acad. Sci. Paris **202** (1936) 1490

Nearly twenty years after, a remarkable and far reaching generalization of the quantum theory for a massive vector relativistic wave field was discovered by the Swiss theoretician

E.C.G. Stueckelberg,
Théorie de la radiation de photons de masse arbitrairement petite
Helv. Phys. Acta **30** (1957) 209-215

whose clever construction is nowadays known as the Stueckelberg trick. The manifestly Lorentz invariant quantum theory of the radiation field has been first approached long time ago by Gupta and Bleuler

1. Sen N. Gupta, Proc. Phys. Soc. A**63** (1950) 681
2. K. Bleuler, Helv. Phys. Acta **23** (1950) 567

and further fully developed by Nakanishi and Lautrup

1. Noboru Nakanishi, Prog. Theor. Phys. **35** (1966) 1111; *ibid.* **49** (1973) 640; *ibid.* **52** (1974) 1929; Prog. Theor. Phys. Suppl. No.**51** (1972) 1
2. B. Lautrup, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **35** (1967) No. 11, 1

who completely clarified the subject. As we shall see below, there are many common features shared by the quantum dynamics of the massive and the mass-less relativistic vector fields, besides some crucial differences. Needless to say, the most important property of the mass-less vector field theory is its invariance under the so called gauge transformation of the first kind

$$A_\mu(x) \quad \mapsto \quad A'_\mu(x) = A_\mu(x) + \partial_\mu f(x)$$

where $f(x)$ is an arbitrary real function, its consequence being the exact null mass condition[1] of the photon and the transverse nature of its polarization. Conversely, this local symmetry is not an invariance of the massive vector field theory, so that a third longitudinal polarization indeed appears for the massive vector particles.

5.1 General Covariant Gauges

In what follow, on the one hand I will attempt to treat contextually the massive and mass-less cases though, on the other hand, I will likely to focus the key departures between the two items. The main novelty, with respect to the previously studied scalar and spinor relativistic wave fields, is the appearance of auxiliary, nonphysical ghost field to set up a general covariant and consistent quantization procedure, as well as the unavoidable presence of a space of the quantum states – the Fock space – with an indefinite metric.

We start from the classical Lagrange density

$$\begin{aligned} \mathcal{L}_{A,B} &= -\frac{1}{4}F^{\mu\nu}(x)\,F_{\mu\nu}(x) + \frac{m^2c^2}{2\hbar^2}A^\mu(x)\,A_\mu(x) \\ &+ A^\mu(x)\,\partial_\mu B(x) + \frac{\xi}{2}B^2(x) \end{aligned} \tag{5.1}$$

where the vector field $A^\mu(x)$ has canonical dimensions $[\,A^\mu\,] = \sqrt{\mathrm{eV/cm}}$ in physical units, whereas $B(x)$ is an auxiliary nonphysical scalar field of canonical engineering dimension $[\,B\,] = \mathrm{erg}^{\frac{1}{2}}\ \mathrm{cm}^{-\frac{3}{2}}$, while the dimensionless parameter $\xi \in \mathbb{R}$ is named the gauge fixing parameter, the Abelian field strength being as usual $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, in such a manner that the Action $S = \int \mathrm{d}t \int \mathrm{d}\mathbf{x}\,\mathcal{L}$ results to be Poincaré invariant. The variations with respect to the scalar field B and the vector potential A_μ drive to the Euler-Lagrange equations of motion

$$\left\{ \begin{array}{c} \partial_\mu F^{\mu\nu}(x) + (mc/\hbar)^2 A^\nu(x) + \partial^\nu B(x) = 0 \\ \partial_\mu A^\mu(x) = \xi\,B(x) \end{array} \right. \tag{5.2}$$

[1]The present day experimental limit on the photon mass is $m_\gamma < 6 \times 10^{-17}$ eV – see *The Review of Particle Physics* J. Beringer *et al.*, Phys. Rev. D**86**, 010001 (2012).

Taking the four divergence of the first equation and using the second equation we obtain

$$(mc/\hbar)^2 \, \partial \cdot A(x) = - \, \Box B(x) = (mc/\hbar)^2 \, \xi \, B(x)$$

which shows that the auxiliary field is a free real scalar field that satisfies the Klein-Gordon wave equation with a square mass $\xi \, m^2$, which is positive only for $\xi > 0$. This latter feature makes it apparent the nonphysical nature of the auxiliary $B-$field, for it becomes tachyon-like for negative values of the gauge fixing parameter. Nonetheless, it will become soon clear later on why the introduction of the auxiliary and nonphysical $B-$field turns out to bet very convenient and eventually unavoidable, to the aim of building up a covariant quantization of the real vector field, especially in the null mass gauge invariant limit $m \to 0$.

If we turn to natural units and rewrite the above equations of motion for the vector potential and the auxiliary scalar field we have

$$\begin{aligned} \left(\Box + m^2\right) A_\mu(x) + (1 - \xi) \, \partial_\mu B(x) &= 0 \qquad (5.3) \\ \partial \cdot A(x) &= \xi \, B(x) \qquad (5.4) \end{aligned}$$

or even more explicitly

$$\begin{aligned} &\left\{ g_{\mu\nu} \left(\Box + m^2\right) \; - \; \left(1 - \frac{1}{\xi}\right) \partial_\mu \, \partial_\nu \right\} A^\nu(x) = 0 \qquad (5.5) \\ &\left(\Box + m^2 \, \xi\right) B(x) = 0 \qquad (5.6) \\ &\partial \cdot A(x) = \xi \, B(x) \qquad (5.7) \end{aligned}$$

the very last relation being usually named the subsidiary condition.

The above system (5.2) of field equations, which includes the subsidiary condition, does nicely simplify for the particular value of the gauge fixing parameter $\xi = 1$: namely,

$$\left. \begin{array}{c} (\Box + m^2) \; A^\mu(x) = 0 \\ \partial \cdot A(x) = B(x) \\ (\Box + m^2) \, B(x) = 0 \end{array} \right\} \qquad (\xi = 1) \qquad (5.8)$$

and in the mass-less limit we come to the d'Alembert wave equation

$$\Box A^\mu(x) = 0 = \Box B(x) \qquad \qquad \partial \cdot A(x) = B(x) \qquad (5.9)$$

This especially simple and convenient choice of the gauge fixing parameter is named the Feynman gauge. If, instead, we put $\xi = 0$ in the Euler-Lagrange equations (5.2) we get

$$\left(\Box + m^2\right) A^\mu(x) + \partial^\mu B(x) = 0 \qquad \partial \cdot A(x) = 0 = \Box B(x) \tag{5.10}$$

and in the massless limit

$$\Box A^\mu(x) + \partial^\mu B(x) = 0 \qquad \partial \cdot A(x) = 0 = \Box B(x) \tag{5.11}$$

This latter choice is known as the Lorenz condition[2] in the context of classical electrodynamics or the Landau gauge in quantum electrodynamics, or even the renormalizable gauge in the massive case. The general case of a finite $\xi \neq 0, 1$ is called the general covariant gauge.

5.1.1 Conserved Quantities

The canonical energy-momentum tensor is obtained according to Noether theorem

$$\begin{aligned}
T_{\mu\nu} &= A_\mu \partial_\nu B - F_{\mu\lambda} \partial_\nu A^\lambda - g_{\mu\nu} \mathcal{L}_{A,B} \\
&= A_\mu \partial_\nu B - F_{\mu\lambda} F_\nu{}^\lambda - F_{\mu\lambda} \partial^\lambda A_\nu - g_{\mu\nu} \mathcal{L}_{A,B} \\
&= - F_{\mu\lambda} F_{\nu\kappa} g^{\lambda\kappa} - g_{\mu\nu} \mathcal{L}_{A,B} \\
&- \partial^\lambda (F_{\mu\lambda} A_\nu) + A_\nu \partial^\lambda F_{\mu\lambda} + A_\mu \partial_\nu B
\end{aligned}$$

and is not symmetric, just like in the gauge invariant Maxwell case (2.108). However, using the equation of motion

$$\dot{B} + m^2 A_0 + \nabla \cdot \mathbf{E} = 0$$

$$\partial^\lambda F_{\jmath\lambda} = \partial_\jmath B + m^2 A_\jmath \qquad (\jmath = 1, 2, 3)$$

we eventually obtain

$$T_{\mu\nu} = \Theta_{\mu\nu} - \partial^\lambda (F_{\mu\lambda} A_\nu)$$

in which the improved symmetric energy-momentum tensor appears to be

$$\begin{aligned}
\Theta_{\mu\nu} &\stackrel{\mathrm{def}}{=} A_\mu \partial_\nu B + A_\nu \partial_\mu B \\
&- g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} + m^2 A_\nu A_\mu - g_{\mu\nu} \mathcal{L}_{A,B} \\
&= A_\mu \partial_\nu B + A_\nu \partial_\mu B - g_{\mu\nu}\left(A \cdot \partial B + \tfrac{1}{2} \xi B^2\right) \\
&+ \tfrac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} - F_{\mu\lambda} F_\nu{}^\lambda + m^2\left(A_\nu A_\mu - \tfrac{1}{2} g_{\mu\nu} A^2\right)
\end{aligned} \tag{5.12}$$

[2]The proposal of solving the Maxwell equation with the condition $\partial \cdot A = 0$ was put forward in 1867 by the Danish mathematical physicist Ludvig Valentin Lorenz.

which satisfies

$$\partial_\mu T^{\,\mu\nu} = \partial_\mu \Theta^{\,\mu\nu} = \partial_\nu \Theta^{\,\mu\nu} = 0$$

The components of the improved symmetric energy-momentum tensor are

$$\begin{aligned}
\Theta_{\jmath k} \;&=\; \tfrac{1}{2}\,\delta_{\jmath k}\,(\,\mathbf{E}^2 + \mathbf{B}^2\,) - E_\jmath E_k - B_\jmath B_k \\
&+\; m^2 A_\jmath A_k + \tfrac{1}{2}\,\delta_{\jmath k}\, m^2\,(\,A_0^2 - \mathbf{A}^2\,) \\
&+\; A_\jmath\,\partial_k B + A_k\,\partial_\jmath B + \Big(A_0 \dot{B} + \mathbf{A}\cdot\boldsymbol{\nabla} B + \tfrac{1}{2}\,\xi\,B^2\Big)\,\delta_{\jmath k} \qquad (5.13)
\end{aligned}$$

which is the generalized spatial stress tensor,

$$\Theta_{\,0k} \;=\; A_0\,\partial_k B + A_k\,\dot{B} + F_{0\ell}\,F_{k\ell} + m^2\,A_0\,A_k \qquad (5.14)$$

that corresponds to the generalized Poynting vector and finally

$$\begin{aligned}
\Theta_{\,00} \;&=\; \tfrac{1}{2}\,(\,\mathbf{E}^2 + \mathbf{B}^2\,) + \tfrac{1}{2}\,m^2\,(\,A_0^2 + \mathbf{A}^2\,) \\
&+\; A_0 \dot{B} - \mathbf{A}\cdot\boldsymbol{\nabla} B - \tfrac{1}{2}\,\xi\,B^2 \qquad (5.15)
\end{aligned}$$

which is the energy density of the general Abelian vector field. Notice that we have the trace property

$$-\,\tfrac{1}{2}\,\Theta^{\,\mu}{}_{\mu} = A\cdot\partial B + \xi\,B^2 + \tfrac{1}{2}\,m^2\,A^2 = \partial_\mu\,(BA^\mu) + \tfrac{1}{2}\,m^2\,A^2$$

where use has been made of the equation of motion for the auxiliary field. It follows therefrom that the conserved Hamiltonian functional becomes

$$\begin{aligned}
cP_0 \;&=\; \int \mathrm{d}\mathbf{x}\;T_{00}\,(t,\mathbf{x}) = \int \mathrm{d}\mathbf{x}\;\Big\{A_{\,0}\,(t,\mathbf{x})\,\dot{B}\,(t,\mathbf{x}) + F_{\,0k}\,(t,\mathbf{x})\,\dot{A}_{\,k}\,(t,\mathbf{x}) \\
&+\; \tfrac{1}{4}\,F^{\,\rho\sigma}\,(t,\mathbf{x})\,F_{\rho\sigma}\,(t,\mathbf{x}) - \;\tfrac{1}{2}\,m^2\,A^{\,\lambda}\,(t,\mathbf{x})\,A_{\,\lambda}\,(t,\mathbf{x}) \\
&-\; A^{\,\mu}\,(t,\mathbf{x})\,\partial_{\,\mu}\,B\,(t,\mathbf{x}) - \;\tfrac{1}{2}\,\xi\,B^{\,2}\,(t,\mathbf{x})\Big\} \equiv H
\end{aligned}$$

The form of the canonical conjugated momenta can be derived from the Lagrange density (5.1) and reads

$$\Pi_{\,\mu} \;=\; \delta\,\mathcal{L}\,/\,\delta\,\dot{A}^{\,\mu} = \begin{cases} 0 & \text{for } \mu = 0 \\ F_{k0} \equiv E_{\,k} & \text{for } \mu = k = 1,2,3 \end{cases} \qquad (5.16)$$

$$\Pi \;=\; \delta\,\mathcal{L}\,/\,\delta\,\dot{B} = A_0 \qquad (5.17)$$

Notice that the canonical momentum $\Pi_\jmath(t,\mathbf{x})$ $(\,\jmath = 1,2,3\,)$, conjugated to the spatial vector potential $A^k(t,\mathbf{x})$ $(\,k = 1,2,3\,)$, is opposite to the electric field, *i.e.* $\Pi_\jmath(t,\mathbf{x}) = -\,E^\jmath(t,\mathbf{x})$, in such a manner that we have the canonical Poisson brackets

$$\big\{A^k(t,\mathbf{x})\,,\,\Pi_\jmath(t,\mathbf{y})\big\} = \delta^{\,k}_{\jmath}\,\delta(\mathbf{x}-\mathbf{y}) = \big\{A^k(t,\mathbf{x})\,,\,-\,E^\jmath(t,\mathbf{y})\big\}$$

Now, if we take into account that

$$
\begin{aligned}
F_{0k}(t,\mathbf{x})\,\dot{A}_k(t,\mathbf{x}) &= F_{0k}(t,\mathbf{x})\,F_{0k}(t,\mathbf{x}) + F_{0k}(t,\mathbf{x})\,\partial_k A_0(t,\mathbf{x}) \\
&= E^k(t,\mathbf{x})\,E^k(t,\mathbf{x}) - \ A_0(t,\mathbf{x})\,\partial_k E^k(t,\mathbf{x}) \\
&+ \ \partial_k\left[F_{0k}(t,\mathbf{x})\,A_0(t,\mathbf{x})\right]
\end{aligned}
$$

$$
\dot{B}(t,\mathbf{x}) + m^2\,\Pi(t,\mathbf{x}) + \nabla\cdot\mathbf{E}(t,\mathbf{x}) = 0
$$

we can recast the energy, up to an irrelevant spatial divergence, in the form

$$
\begin{aligned}
P_0 \ &\doteq \ \int \mathrm{d}\mathbf{x}\,\Big\{-\ \tfrac12\,m^2\,\Pi^2(t,\mathbf{x}) - \ \Pi(t,\mathbf{x})\,\partial_k E^k(t,\mathbf{x}) \\
&+ \ \tfrac12\,E^k(t,\mathbf{x})\,E^k(t,\mathbf{x}) + \tfrac14\,F_{jk}(t,\mathbf{x})\,F_{jk}(t,\mathbf{x}) - \tfrac12\,\xi\,B^2(t,\mathbf{x}) \\
&+ \ \tfrac12\,m^2\,A_k(t,\mathbf{x})\,A_k(t,\mathbf{x}) + A_k(t,\mathbf{x})\,\partial_k B(t,\mathbf{x})\Big\}
\end{aligned}
$$

Now we can rewrite the field equations (5.2) in the canonical form, that is

$$
\begin{aligned}
\dot{B}(x) &= \{\,B(x)\,,\,H\,\} = -\,m^2\,\Pi(x) - \nabla\cdot\mathbf{E}(x) \\
&= -\,m^2\,A_0(x) - \partial_k F_{0k}(x) && (5.18) \\
\dot{A}_k(x) &= \{\,A_k(x)\,,\,H\,\} = F_{0k}(x) + \partial_k\Pi(x) \\
&= F_{0k}(x) + \partial_k A_0(x) && (5.19) \\
\dot{\Pi}(x) &= \{\,\Pi(x)\,,\,H\,\} = \partial_k A_k(x) + \xi\,B(x) && (5.20) \\
\dot{F}_{0k}(x) &= \{\,E^k(x)\,,\,H\,\} \\
&= \partial_j F_{jk}(x) - \ m^2\,A_k(x) - \partial_k B(x) && (5.21)
\end{aligned}
$$

where $H \equiv P_0$ and I used the canonical Poisson brackets among all the independent pairs of canonical variables $(\,\mathbf{A}\,,\,B\,;\,-\,\mathbf{E}\,,\,\Pi\,)$: namely,

$$
\begin{gathered}
\{A^{\jmath}(t,\mathbf{x})\,,\,E^k(t,\mathbf{y})\} = g^{\jmath k}\,\delta(\mathbf{x}-\mathbf{y}) \\
\{\,B(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\} = \{\,B(t,\mathbf{x})\,,\,A_0(t,\mathbf{y})\} \ = \ \delta(\mathbf{x}-\mathbf{y})
\end{gathered}
\qquad (5.22)
$$

all the other Poisson brackets being equal to zero. Notice that the above set of Poisson's brackets holds true unchanged both in the massive and massless cases. Moreover, it's also very important to realize that, in the massive case, the Hamiltonian functional contains an unusual negative kinetic term $-\,\frac12\,m^2\,\Pi^2(t,\mathbf{x})$ for the auxiliary field.

The canonical total angular momentum density follows from the Noether theorem and reads

$$
M^{\,\mu\rho\sigma} \ \equiv \ x^{\rho}\,T^{\mu\sigma} \ - \ x^{\sigma}\,T^{\mu\rho} \ + \ S^{\mu\rho\sigma} \qquad (5.23)
$$

Hence we find

$$\begin{aligned}
M^{\mu\rho\sigma} &= x^{\rho}\left[\Theta^{\mu\sigma} + \partial_{\lambda}\left(F^{\lambda\mu}A^{\sigma}\right)\right] + F^{\mu\sigma}A^{\rho} - \{\rho \leftrightarrow \sigma\} \\
&= x^{\rho}\,\Theta^{\mu\sigma} - x^{\sigma}\,\Theta^{\mu\rho} + \partial_{\lambda}\{F^{\lambda\mu}\,(x^{\rho}A^{\sigma} \;-\; x^{\sigma}A^{\rho})\} \\
&= \mathfrak{M}^{\mu\rho\sigma} + \partial_{\lambda}\{F^{\lambda\mu}\,(x^{\rho}A^{\sigma} \;-\; x^{\sigma}A^{\rho})\}
\end{aligned}$$

Since the very last term does not contribute to the continuity equation $\partial_{\mu}\,M^{\mu\rho\sigma} = 0$, we see that the total angular momentum tensor can be always written in the purely orbital form, just like in the classical gauge invariant Maxwell case

$$M^{\rho\sigma} = \int \mathrm{d}\mathbf{x}\;\left[x^{\rho}\,\Theta^{0\sigma}(t,\mathbf{x}) - x^{\sigma}\,\Theta^{0\rho}(t,\mathbf{x})\right]$$

which satisfies

$$\dot{M}^{\rho\sigma} = 0$$

As a consequence we get the three spatial components

$$M^{\imath\jmath} = \int \mathrm{d}\mathbf{x}\;\left\{x^{\imath}\,\Theta^{0\jmath}(t,\mathbf{x}) - x^{\jmath}\,\Theta^{0\imath}(t,\mathbf{x})\right\}$$

which corresponds to an orbital angular momentum from the generalized Poynting vector, while the spatial temporal components

$$M^{0k} = x^{0}\,P^{k} - \frac{1}{c}\int \mathrm{d}\mathbf{x}\;x^{k}\,\Theta_{00}(t,\mathbf{x})$$

lead to the definition of the center of the energy for the radiation, *viz.,*

$$\mathbf{X}_{t} \equiv \int \mathrm{d}^{3}x\;\mathbf{x}\;\Theta_{00}(t,\mathbf{x})/cP_{0}$$

that satisfies the suggestive particle velocity relationship

$$\dot{M}^{0k} = 0 \quad\Leftrightarrow\quad \dot{\mathbf{X}}_{t} = c\,\frac{\mathbf{P}}{P_{0}}$$

5.2 Normal Modes Decomposition

To solve the Euler-Lagrange system of equations (5.2) in the general case, it is very convenient to decompose the vector potential according to the definition

$$A^{\mu}(x) \stackrel{\text{def}}{=} V^{\mu}(x) - \begin{cases} m^{-2}\,\partial^{\mu} B(x) & m \neq 0 \\ -\xi\,\partial^{\mu}\,\mathcal{D} * B(x) & m = 0 \end{cases} \tag{5.24}$$

where the integral-differential operator $\mathcal{D}$ defined by

$$\mathcal{D} \stackrel{\text{def}}{=} \tfrac{1}{2}\left(\nabla^2\right)^{-1}(x_0\,\partial_0 - \mathrm{C}) \tag{5.25}$$

for arbitrary constant C works as an inverse of the d'Alembert wave operator $\Box$ in front of any solution of the wave equation: namely,

$$\Box\,\mathcal{D} * f(x) = f(x) \qquad \text{iff} \quad \Box\, f(x) = 0$$

as it can be readily checked by direct inspection.

`Proof.` The Green's function which corresponds to the inverse Laplace operator can be readily obtained in the Fourier integral representation: namely,

$$\nabla^2 G(\mathbf{x}-\mathbf{y}) = \triangle G(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$

$$\langle\,\mathbf{y}|\left(\nabla^2\right)^{-1}|\mathbf{x}\,\rangle \equiv G(\mathbf{x}-\mathbf{y}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\,\widetilde{G}(\mathbf{k})\;\mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \qquad \widetilde{G}(\mathbf{k}) = -\,\frac{1}{\mathbf{k}^2}$$

$$G(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi^2}\int_0^{\infty}\mathrm{d}k\int_0^{\pi}\mathrm{d}\cos\theta\;\exp\{-ik|\mathbf{x}-\mathbf{y}|\cos\theta\}$$

$$\equiv \frac{-1}{4\pi^2|\mathbf{x}-\mathbf{y}|}\,\Im\mathrm{m}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{k-i0}\;\exp\{ik|\mathbf{x}-\mathbf{y}|\} = \frac{1}{2\pi|\mathbf{x}-\mathbf{y}|}$$

Then we get for $\partial_0^2 f(x) = \triangle f(x)$

$$\begin{aligned} \partial_0\,\mathcal{D} * f(x) &= \tfrac{1}{2}\left(\nabla^2\right)^{-1}\left((1-\mathrm{C})\,\partial_0 + x_0\,\partial_0^2\right) f(x) \\ &= \tfrac{1}{2}\left(\nabla^2\right)^{-1}\partial_0 f(x)\,(1-\mathrm{C}) + \tfrac{1}{2}\,x_0 f(x) \\ \partial_0^2\,\mathcal{D} * f(x) &= (1-\mathrm{C}/2)\,f(x) + \tfrac{1}{2}\,x_0\,\partial_0 f(x) \end{aligned}$$

$$\triangle\,\mathcal{D} * f(x) = \tfrac{1}{2}\,(x_0\,\partial_0 - \mathrm{C})\,f(x) \qquad \Longrightarrow \qquad \Box\,\mathcal{D} * f(x) = f(x)$$

q.e.d.

Hence, the subsidiary condition (5.7) entails the transverse condition, *i.e.*

$$\partial_{\mu} A^{\mu} = \xi\, B \quad \Leftrightarrow \quad \partial_{\mu} V^{\mu} = 0$$

so that we eventually obtain from the equations of motion (5.4)

$$\begin{cases} (\Box + m^2)\,V_{\mu}(x) = 0 \\ \partial^{\mu} V_{\mu}(x) = 0 \\ (\Box + \xi\, m^2)\,B(x) = 0 \end{cases} \qquad (m \neq 0) \tag{5.26}$$

$$\begin{cases} \Box V^{\mu}(x) + \partial^{\mu} B(x) = 0 \\ \partial^{\mu} V_{\mu}(x) = 0 \\ \Box B(x) = 0 \end{cases} \qquad (m = 0) \tag{5.27}$$

It is worthwhile to realize that the field strength does not depend upon the nonphysical auxiliary field $B(x)$, albeit merely on the transverse vector field $V_{\mu}(x)$ since we have $F_{\mu\nu}(x) = \partial_{\mu} V_{\nu}(x) - \partial_{\nu} V_{\mu}(x)$. This means that we can also write

$$\left(\Box + m^2\right) F_{\mu\nu}(x) = 0 \qquad \text{or} \qquad \Box F_{\mu\nu}(x) = 0$$

The transverse real vector field $V_{\mu}(x)$ is also named the Proca vector field in the massive case, while it is called the transverse vector potential in the null mass case. We shall now find the general solution of above system of the equations of motion in both cases, *i.e.* , the massive and the mass-less cases.

5.2.1 Normal Modes of the Massive Vector Field

Let me first discuss the normal mode decomposition in the massive case. To this purpose, if we set

$$V_{\mu}(x) = (2\pi)^{-3/2} \int \mathrm{d}k \ \widetilde{V}_{\mu}(k) \ \exp\{-\, i\, k \cdot x\} \tag{5.28}$$

$$\widetilde{V}^{\,*}_{\mu}(k) = \widetilde{V}_{\mu}(-k)$$

then we find

$$\widetilde{V}_{\mu}(k) = f_{\mu}(k)\, \delta\left(k^2 - m^2\right) \qquad k \cdot f(k) = 0 \tag{5.29}$$

where $f_{\mu}(k)$ are regular functions on the hyperbolic manifold $k^2 = m^2$, which are any arbitrary functions, but for the transverse condition $k \cdot f(k) = 0$ and the reality condition $f^{\,*}_{\mu}(k) = f_{\mu}(-k)$.

Next, it is convenient to introduce the three linear polarization real unit vectors $e^{\,\mu}_{r}(\mathbf{k})$ $(\, r = 1, 2, 3\,)$ which are dimensionless and determined by the properties

$$k_{\mu}\, e^{\,\mu}_{r}(\mathbf{k}) = 0 \qquad (\, r = 1, 2, 3\,) \qquad k_0 \equiv \omega_{\mathbf{k}} = \left(\mathbf{k}^{\,2} + m^2\right)^{1/2}$$

$$-\, g_{\mu\nu}\, e^{\,\mu}_{r}(\mathbf{k})\, e^{\,\nu}_{s}(\mathbf{k}) = \delta_{rs} \qquad (\text{orthogonality relation}) \tag{5.30}$$

$$\begin{aligned} &\sum_{r=1}^{3} e^{\,\mu}_{r}(\mathbf{k})\, e^{\,\nu}_{r}(\mathbf{k}) \\ &= \; -\, g^{\,\mu\nu} \; + \; k^{\mu}\, k^{\nu} / k^2 \\ &= \; -\, g^{\,\mu\nu} \; + \; k^{\mu}\, k^{\nu} / m^2 \qquad (\text{closure relation}) \end{aligned} \tag{5.31}$$

A suitable explicit choice is provided by

$$\left.\begin{array}{r} \mathrm{e}_r^0(\mathbf{k}) = 0 \\ \mathbf{k}\cdot\mathbf{e}_r(\mathbf{k}) = 0 \\ \mathbf{e}_r(\mathbf{k})\cdot\mathbf{e}_s(\mathbf{k}) = \delta_{rs} \end{array}\right\} \quad \text{for } r,s = 1,2$$

$$\mathrm{e}_3^0(\mathbf{k}) = \frac{|\mathbf{k}|}{m} \qquad \mathbf{e}_3(\mathbf{k}) = \frac{\widehat{\mathbf{k}}}{m}\,\omega_{\mathbf{k}}$$

in such a manner that we can write the normal mode decomposition of the classical Proca real vector field in the form

$$V^{\nu}(x) = \sum_{\mathbf{k},r}\left[\,f_{\mathbf{k},r}\,u^{\nu}_{\mathbf{k},r}(x) + f^{*}_{\mathbf{k},r}\,u^{\nu\,*}_{\mathbf{k},r}(x)\,\right] \tag{5.32}$$

$$u^{\nu}_{\mathbf{k},r}(x) = [\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{-1/2}\,e^{\nu}_{r}(\mathbf{k})\,\exp\{-\,i\,\omega_{\mathbf{k}}t + i\,\mathbf{k}\cdot\mathbf{x}\} \tag{5.33}$$

where we have denoted as usual

$$\sum_{\mathbf{k},r} \stackrel{\mathrm{def}}{=} \int \mathrm{d}\mathbf{k}\sum_{r=1}^{3}$$

Notice that the set of the vector wave functions $u^{\nu}_{\mathbf{k},r}(x)$ does satisfy the orthogonality and closure relations

$$\begin{aligned}
-\,g_{\lambda\sigma}\left(u^{\sigma}_{\mathbf{k},s}\,,\,u^{\lambda}_{\mathbf{h},r}\right) &\equiv -\,g_{\lambda\sigma}\int \mathrm{d}\mathbf{x}\;u^{\sigma\,*}_{\mathbf{k},s}(y)\,i\overleftrightarrow{\partial}_0\,u^{\lambda}_{\mathbf{h},r}(x) \\
&= \delta(\mathbf{h}-\mathbf{k})\;\delta_{rs} &(5.34)\\
g_{\lambda\sigma}\left(u^{\sigma}_{\mathbf{k},s}\,,\,u^{\lambda\,*}_{\mathbf{h},r}\right) &= g_{\lambda\sigma}\left(u^{\sigma\,*}_{\mathbf{k},s}\,,\,u^{\lambda}_{\mathbf{h},r}\right) = 0 &(5.35)\\
g_{\lambda\sigma}\left(u^{\sigma\,*}_{\mathbf{k},s}\,,\,u^{\lambda\,*}_{\mathbf{h},r}\right) &= \delta(\mathbf{h}-\mathbf{k})\;\delta_{rs} &(5.36)\\
\sum_{\mathbf{k},r} u^{\lambda}_{\mathbf{k},r}(x)\,u^{\nu\,*}_{\mathbf{k},r}(y) &= \left(g^{\lambda\nu} - m^{-2}\,\partial^{\lambda}_{x}\,\partial^{\nu}_{y}\right)\,iD^{(-)}(x-y) &(5.37)
\end{aligned}$$

Next we have

$$\begin{aligned}
B(x) &= m\sum_{\mathbf{k}}\left[\,b_{\mathbf{k}}\,u_{\mathbf{k}}(x) + b^{*}_{\mathbf{k}}\,u^{*}_{\mathbf{k}}(x)\right] \\
u_{\mathbf{k}}(x) &= [\,(2\pi)^3\,2\,\omega'_{\mathbf{k}}\,]^{-1/2}\,\exp\{-\,i\,\omega'_{\mathbf{k}}\,x^0 + i\,\mathbf{k}\cdot\mathbf{x}\} \\
\omega'_{\mathbf{k}} &\equiv \left(\mathbf{k}^2 + \xi\,m^2\right)^{1/2} &(5.38)
\end{aligned}$$

so that from eq. (5.24) we eventually come to the normal mode decomposition of the classical real vector potential

$$A^{\mu}(x) \;=\; \sum_{\mathbf{k},r}\left[\,f_{\mathbf{k},r}\,u^{\mu}_{\mathbf{k},r}(x) + f^{*}_{\mathbf{k},r}\,u^{\mu\,*}_{\mathbf{k},r}(x)\,\right]$$

$$
\begin{aligned}
& + \; \frac{i}{m} \sum_{\mathbf{k}} k'^{\,\mu} \Big[b_{\mathbf{k}}\, u_{\mathbf{k}}(x) \; - \; b^*_{\mathbf{k}}\, u^*_{\mathbf{k}}(x) \Big] \\
k'^{\,\mu} \; &= \; (\omega'_{\mathbf{k}} \,,\, \mathbf{k}) \qquad (5.39)
\end{aligned}
$$

The Proca transverse vector field. As a final remark about the massive case, it is very instructive to rewrite the classical Lagrange density and the canonical energy-momentum tensor as functional of the transverse Proca vector field $V_\mu\,(x)$ and of the nonphysical auxiliary scalar field $B\,(x)$. Actually, making use of the decomposition (5.24) we obtain

$$
\begin{aligned}
\mathcal{L}_{A\,,\,B} \; &= \; \mathcal{L}_V + \mathcal{L}_B & (5.40) \\
\mathcal{L}_V \; &= \; -\,\tfrac{1}{4}\,F^{\,\mu\nu}\,(x)\,F_{\mu\nu}\,(x) + \,\tfrac{1}{2}\,m^2\,V^{\,\mu}\,(x)\,V_\mu\,(x) & (5.41) \\
\mathcal{L}_B \; &= \; -\,\left(1/2m^2\right)\,\partial^{\,\mu}B\,(x)\,\partial_\mu\,B\,(x) + \tfrac{1}{2}\,\xi\,B^2\,(x) & (5.42)
\end{aligned}
$$

The Lagrange density $\mathcal{L}_V$ is also called the Proca Lagrangian, as it involves solely the massive transverse vector field. Notice that the Lagrangian of the transverse vector field $V^{\,\mu}\,(x)$ entails the Euler-Lagrange equations

$$
\partial_\mu\,F^{\,\mu\nu}\,(x) + m^2\,V^{\,\nu}\,(x) = 0 \qquad\qquad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu
$$

so that the transverse-like condition

$$
\partial_\mu\,V^{\,\mu}\,(x) = 0
$$

does indeed follow from the equations of motion that can be consequently written in the simplest form

$$
\left(\Box + m^2\right)V^\mu(x) = 0 \qquad\qquad \partial_\mu V^\mu(x) = 0 \qquad\qquad \left(\Box + \xi m^2\right)B(x) = 0 \qquad (5.43)
$$

The canonical conjugate momenta are given by

$$
\Pi^{\,\mu}\,(x) = \delta\,\mathcal{L}_V\,/\,\delta\,\dot{V}_\mu\,(x) = \begin{cases} \quad 0 & \text{for } \mu = 0 \\ F_{0k} = E^{\,k} & \text{for } \mu = k = 1,2,3 \end{cases} \qquad (5.44)
$$

$$
\Pi\,(x) = \delta\,\mathcal{L}_B\,/\,\delta\,\dot{B}\,(x) = -\,m^{-2}\,\dot{B}\,(x) \qquad (5.45)
$$

whence we get the Poisson's brackets

$$
\begin{aligned}
\{V_k(t,\mathbf{x})\,,\,E^{\,\ell}(t,\mathbf{y})\} &= \delta^{\,\ell}_k\,\delta(\mathbf{x}-\mathbf{y}) & (5.46) \\
\{B(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\} &= \delta(\mathbf{x}-\mathbf{y}) & (5.47)
\end{aligned}
$$

all the remaining ones being equal to zero. Also the canonical energy-momentum tensor can be conveniently expressed in terms of the transverse vector field according to

$$
\begin{aligned}
T^{\,\mu}{}_{\nu} \; &= \; \tfrac{1}{4}\,\delta^{\,\mu}{}_{\nu}\left(F^{\,\rho\sigma}\,F_{\rho\sigma} - 2m^2\,V^{\,\lambda}\,V_\lambda\right) + m^2\,V_\nu\,V^{\,\mu} - F^{\,\mu\lambda}\,F_{\nu\lambda} - \partial_\lambda\left(V_\nu\,F^{\,\mu\lambda}\right) \\
&- \; \frac{1}{m^2}\,\partial^{\,\mu}B\,\partial_\nu\,B + \left(\frac{1}{2m^2}\,\partial^{\,\lambda}B\,\partial_{\,\lambda}B - \tfrac{1}{2}\,\xi\,B^2\right)\delta^{\,\mu}{}_{\nu} \\
&\equiv \; \Theta^{\,\mu}{}_{\nu} - \partial_\lambda\left(V_\nu\,F^{\,\mu\lambda}\right) \qquad (5.48)
\end{aligned}
$$

where use has been made of the field equations. It is apparent that the symmetric energy momentum tensor $\Theta^{\mu\nu} = \Theta^{\nu\mu}$ turns out to be the sum of the transverse vector and of the auxiliary scalar parts. As a consequence, when the total angular momentum density is expressed as a functional of the transverse vector field $V_\mu\,(x)$ and of the nonphysical auxiliary field $B\,(x)$ we obtain

$$\begin{aligned}
M^{\,\mu\rho\sigma} &= x^{\rho}\,T^{\,\mu\sigma} - x^{\sigma}\,T^{\,\mu\rho} - F^{\,\mu\rho}\,V^{\,\sigma} + F^{\,\mu\sigma}\,V^{\,\rho} \\
&= x^{\rho}\,\Theta^{\,\mu\sigma} - x^{\sigma}\,\Theta^{\,\mu\rho} \\
&- F^{\,\mu\rho}\,V^{\,\sigma} + F^{\,\mu\sigma}\,V^{\,\rho} - x^{\rho}\,\partial_\lambda\left(V^{\,\sigma}\,F^{\,\mu\lambda}\right) + x^{\sigma}\,\partial_\lambda\left(V^{\,\rho}\,F^{\,\mu\lambda}\right) \\
&= x^{\rho}\,\Theta^{\,\mu\sigma} - x^{\sigma}\,\Theta^{\,\mu\rho} - \partial_\lambda\left(x^{\rho}\,V^{\,\sigma}\,F^{\,\mu\lambda} - x^{\sigma}\,V^{\,\rho}\,F^{\,\mu\lambda}\right)
\end{aligned}$$

Hence, once more, since the very last term does not contribute to the continuity equation $\partial_\mu\,M^{\,\mu\rho\sigma} = 0$, we see that the total angular momentum tensor can be always written in the purely orbital form, just like in the classical gauge invariant Maxwell case

$$M^{\,\rho\sigma} = \int \mathrm{d}\mathbf{x}\;\left[\,x^{\rho}\,\Theta^{\,0\sigma}\,(t,\mathbf{x}) - x^{\sigma}\,\Theta^{\,0\rho}\,(t,\mathbf{x})\,\right]$$

which satisfies

$$\dot{M}^{\,\rho\sigma} = 0$$

As a consequence we get the three spatial components

$$M^{\,\imath\jmath} = \int \mathrm{d}\mathbf{x}\;\left\{x^{\imath}\,\Theta^{0\jmath}(t,\mathbf{x}) - x^{\jmath}\,\Theta^{0\imath}(t,\mathbf{x})\right\}$$

which corresponds to an orbital angular momentum from the sum of the Poynting vector and of the auxiliary scalar parts, while the spatial temporal components

$$M^{\,0k} = x^{0}\,P^{\,k} - \int \mathrm{d}\mathbf{x}\;x^{\,k}\,\Theta_{00}(t,\mathbf{x})$$

lead to the definition of the center of the energy for the full system of transverse vector and auxiliary scalar fields

$$X_t^{\,k} \equiv \int \frac{\mathrm{d}\mathbf{x}}{P_0}\;x^{\,k}\,\Theta_{00}(t,\mathbf{x})$$

that still satisfies the suggestive particle velocity relationship

$$\dot{M}^{\,0k} = 0 \quad\Leftrightarrow\quad \dot{X}_t^{\,k} = \frac{P^{\,k}}{P_0}$$

5.2.2 Normal Modes of the Gauge Potential

Let me now turn to the discussion of the null mass case. Here, it is very important to gather that the transverse-like condition

$$0 = \partial^{\,\mu}\,V_\mu\,(x) = (2\pi)^{-3/2}\int \mathrm{d}\,k\;(-\,i\,)\,k^{\,\mu}\,\widetilde{V}_\mu\,(k)\;\exp\{-\,i\,k\cdot x\} \tag{5.49}$$

$$\widetilde{V}^{\,*}_\mu\,(k) = \widetilde{V}_\mu\,(-\,k)$$

on the light-cone $k^2 = 0$ is fulfilled by three independent **linear polarization** unit **real** vectors $\varepsilon_A^{\,\mu}(\mathbf{k}) \quad (A = 1, 2, L)$ which are again dimensionless and defined by the properties

$$k_\mu \,\varepsilon_A^{\,\mu}(\mathbf{k}) = 0 \qquad\qquad (A = 1, 2, L) \qquad\quad k_0 \equiv \omega_{\mathbf{k}} = |\,\mathbf{k}\,|$$

$$\left.\begin{array}{c} \varepsilon_A^{\,0}(\mathbf{k}) = 0 \\ \mathbf{k}\cdot\boldsymbol{\varepsilon}_A(\mathbf{k}) = 0 \\ \boldsymbol{\varepsilon}_A(\mathbf{k})\cdot\boldsymbol{\varepsilon}_B(\mathbf{k}) = \delta_{AB} \end{array}\right\} \quad \text{for } A, B = 1, 2$$

$$\varepsilon_L^{\,\mu}(\mathbf{k}) \equiv k^\mu / \,|\,\mathbf{k}\,| = \left(1\,,\,\widehat{\mathbf{k}}\right)$$

Now, if we introduce a further light-like real polarization vector

$$\begin{aligned} \varepsilon_S^{\,\lambda}(\mathbf{k}) \;&=\; \tfrac{1}{2}\,(\,|\,\mathbf{k}\,|\;,\,-\,\mathbf{k}\,)\,/\,|\,\mathbf{k}\,| \;\equiv\; k_*^{\,\lambda}/\sqrt{2k\cdot k_*} \\ &=\; \tfrac{1}{2}\left(1\,,\,-\,\widehat{\mathbf{k}}\right) \qquad\qquad \varepsilon_L\cdot\varepsilon_S = 1 \end{aligned} \tag{5.50}$$

where $k_*^{\,\lambda} = (\,|\,\mathbf{k}\,|\;,\,-\,\mathbf{k}\,)$ is the so called dual null vector, then we can write

$$-\,g_{\mu\nu}\,\varepsilon_A^{\,\mu}(\mathbf{k})\,\varepsilon_B^{\,\nu}(\mathbf{k}) = \eta_{AB} \qquad\qquad (\text{orthonormality relation}) \tag{5.51}$$

$$\eta_{AB}\,\varepsilon_A^{\,\mu}(\mathbf{k})\,\varepsilon_B^{\,\nu}(\mathbf{k}) = -\,g^{\,\mu\nu} \qquad\qquad (\text{closure relation}) \tag{5.52}$$

where

$$\eta_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad\qquad (A, B = 1, 2, L, S)$$

According to the conventional wisdom, the labels L and S do correspond respectively to the longitudinal polarization and the scalar polarization of the mass-less vector particles. Hence, it turns out that the two pairs of space-like and light-like polarization vectors $\varepsilon_A^{\,\mu}(\mathbf{k}) \quad (A = 1, 2; L, S)$ do represent a complete base in the vector space spanned by $\widetilde{V}^{\,\mu}(k)$ for any given light-like tetra-momentum $k^\mu = (\,|\mathbf{k}|, \mathbf{k}\,)$.

Moreover, let me introduce the transverse and the light-cone projectors

$$\begin{aligned} \Pi_\perp^{\lambda\nu}(k) \;&=\; g^{\lambda\nu} \;-\; \frac{k^\lambda k_*^\nu + k^\nu k_*^\lambda}{k\cdot k_*} \\ &=\; -\sum_{A=1,2} \varepsilon_A^{\,\lambda}(\mathbf{k})\,\varepsilon_A^{\,\nu}(\mathbf{k}) \end{aligned} \tag{5.53}$$

$$\begin{aligned} \Pi_\vee^{\lambda\nu}(k) \;&=\; \frac{k^\lambda k_*^\nu + k^\nu k_*^\lambda}{k\cdot k_*} \\ &=\; \varepsilon_L^{\,\lambda}(\mathbf{k})\,\varepsilon_S^{\,\nu}(\mathbf{k}) + \varepsilon_S^{\,\lambda}(\mathbf{k})\,\varepsilon_L^{\,\nu}(\mathbf{k}) \end{aligned} \tag{5.54}$$

which satisfy by construction

$$\Pi_{\perp}^{\lambda\nu}(k) = \left(\Pi_{\perp}^{\nu\lambda}(k)\right)^{*} \qquad g_{\mu\rho}\,\Pi_{\perp}^{\mu\nu}(k)\,\Pi_{\perp}^{\rho\sigma}(k) = \Pi_{\perp}^{\nu\sigma}(k)$$

$$\Pi_{\vee}^{\lambda\nu}(k) = \left(\Pi_{\vee}^{\nu\lambda}(k)\right)^{*} \qquad g_{\mu\rho}\,\Pi_{\vee}^{\mu\nu}(k)\,\Pi_{\vee}^{\rho\sigma}(k) = \Pi_{\vee}^{\nu\sigma}(k)$$

$$\mathrm{tr}\,\Pi_{\perp}(k) \;=\; g_{\lambda\nu}\,\Pi_{\perp}^{\lambda\nu}(k) = 2 = \mathrm{tr}\,\Pi_{\vee}(k) \;=\; g_{\lambda\nu}\,\Pi_{\vee}^{\lambda\nu}(k)$$

$$\Pi_{\perp}^{\mu\nu}(k) + \Pi_{\vee}^{\mu\nu}(k) = g^{\mu\nu}$$

as it can be readily verified by direct inspection. As a consequence, for any given light-like momentum k^{μ} $(k^2 = 0)$ the physical photon polarization density matrix

$$\rho_{\perp}^{\lambda\nu}(k) \stackrel{\mathrm{def}}{=} -\tfrac{1}{2}\sum_{A=1,2}\varepsilon_{A}^{\lambda}(\mathbf{k})\,\varepsilon_{A}^{\nu}(\mathbf{k}) \tag{5.55}$$

$$\rho_{\perp}^{\lambda\nu}(k) \;=\; \left(\rho_{\perp}^{\nu\lambda}(k)\right)^{*} \qquad \mathrm{tr}\,\rho_{\perp} \;=\; 1 \tag{5.56}$$

will represent a mixed state corresponding to non-polarized monochromatic photons. In conclusion, we can finally write the normal mode decomposition of the classical real transverse mass-less vector field which satisfies the Lorenz condition $\partial\cdot V(x) = 0$: namely,

$$\begin{aligned} V^{\lambda}(x) \;&=\; \sum_{\mathbf{k},A}\left[\,g_{\mathbf{k},A}\,u_{\mathbf{k},A}^{\lambda}(x) + g_{\mathbf{k},A}^{*}\,u_{\mathbf{k},A}^{\lambda\,*}(x)\,\right] \\ &-\; \partial^{\lambda}\,\mathcal{D} * B(x) \end{aligned} \tag{5.57}$$

$$\begin{aligned} u_{\mathbf{k},A}^{\lambda}(x) \;&=\; [\,(2\pi)^3\,2\,|\mathbf{k}|\,]^{-1/2}\,\varepsilon_{A}^{\lambda}(\mathbf{k})\,\exp\{\,i\,|\mathbf{k}|(\widehat{\mathbf{k}}\cdot\mathbf{x} - ct\,)\} \\ &=\; \varepsilon_{A}^{\lambda}(\mathbf{k})\,u_{\mathbf{k}}(x) \qquad A = 1,2,L,S \qquad k_0 = |\,\mathbf{k}\,| \\ k_{\lambda}\,\varepsilon_{A}^{\lambda}(\mathbf{k}) \;&=\; |\,\mathbf{k}\,|\,\delta_{AS} \qquad i\partial_{\lambda}\,u_{\mathbf{k},A}^{\lambda}(x) = |\,\mathbf{k}\,|\;u_{\mathbf{k}}(x)\;\delta_{AS} \qquad k_0 = |\,\mathbf{k}\,| \end{aligned}$$

together with

$$\begin{aligned} B(x) \;&=\; \partial_{\lambda}\sum_{\mathbf{k},A}\;\left[\,g_{\mathbf{k},A}\,u_{\mathbf{k},A}^{\lambda}(x) + g_{\mathbf{k},A}^{*}\,u_{\mathbf{k},A}^{\lambda\,*}(x)\,\right] \\ &=\; (-i\,)\sum_{\mathbf{k}}\;|\mathbf{k}|\left[\,u_{\mathbf{k}}(x)\,g_{\mathbf{k},S} - \,u_{\mathbf{k}}^{*}(x)\,g_{\mathbf{k},S}^{*}\,\right] \end{aligned} \tag{5.58}$$

The transverse and mass-less real vector field $V_{\lambda}(x)$ is also named the vector potential in the Lorenz or Landau gauge $\xi = 0$. Then, from eq. (5.24) we eventually come to the normal mode decomposition of the classical real mass-less vector potential

$$
\begin{aligned}
A^{\lambda}(x) &= V^{\lambda}(x) + \xi\,\partial^{\lambda}\,\mathcal{D} * B(x) \\
&= \sum_{\mathbf{k},A}\left[\,g_{\mathbf{k},A}\,u^{\lambda}_{\mathbf{k},A}(x) + g^{*}_{\mathbf{k},A}\,u^{\lambda *}_{\mathbf{k},A}(x)\,\right] \\
&- (1-\xi)\,\partial^{\lambda}\,\mathcal{D} * B(x)
\end{aligned} \tag{5.59}
$$

Notice that the complete and orthogonal system of the positive frequency plane wave solutions $u^{\lambda}_{\mathbf{k},A}(x)$ for the null mass gauge vector potential does satisfy

$$
\begin{aligned}
g_{\lambda\nu}\left(u^{\lambda}_{\mathbf{h},A}\,,\,u^{\nu}_{\mathbf{k},B}\right) &\equiv g_{\lambda\nu}\int \mathrm{d}\mathbf{x}\;u^{\lambda *}_{\mathbf{h},A}(x)\,i\overleftrightarrow{\partial}_{0}\,u^{\nu}_{\mathbf{k},B}(x) \\
&= g_{\lambda\nu}\,\varepsilon^{\lambda}_{A}(\mathbf{h})\,\varepsilon^{\nu}_{B}(\mathbf{k})\,(u_{\mathbf{h}}\,,\,u_{\mathbf{k}}) \\
&= -\,\delta(\mathbf{h}-\mathbf{k})\;\eta_{AB} \\
&= -\,g_{\lambda\nu}\left(u^{\lambda *}_{\mathbf{h},A}\,,\,u^{\nu *}_{\mathbf{k},B}\right)
\end{aligned} \tag{5.60}
$$

$$
g_{\lambda\nu}\left(u^{\lambda *}_{\mathbf{h},A}\,,\,u^{\nu}_{\mathbf{k},B}\right) = g_{\lambda\nu}\left(u^{\lambda}_{\mathbf{h},A}\,,\,u^{\nu *}_{\mathbf{k},B}\right) = 0 \tag{5.61}
$$

$$
\begin{aligned}
\eta_{AB}\sum_{\mathbf{k}} u^{\lambda}_{\mathbf{k},A}(x)\,u^{\nu *}_{\mathbf{k},B}(y) &= \eta_{AB}\sum_{\mathbf{k}}\varepsilon^{\lambda}_{A}(\mathbf{k})\,\varepsilon^{\nu}_{B}(\mathbf{k})\,u_{\mathbf{k}}(x)\,u^{*}_{\mathbf{k}}(y) \\
&= i\,g^{\lambda\nu}\,D^{(-)}_{0}(x-y)
\end{aligned} \tag{5.62}
$$

$$
\sum_{\mathbf{k}}\;u^{\lambda}_{\mathbf{k},L}(x)\,u^{\nu *}_{\mathbf{k},S}(y) = \frac{1}{i}\,\partial^{\lambda}_{x}\,\partial^{\nu}_{*y}\,D^{(-)}_{0}(x-y) \tag{5.63}
$$

where the mass-less scalar positive frequency distribution is given by

$$
D^{(-)}_{0}(x) \equiv \lim_{m\to 0}\frac{i}{(2\pi)^{3}}\int \mathrm{d}^{4}k\;\delta\left(k^{2}-m^{2}\right)\,\theta(k_{0})\,\exp\{-\,i\,k\cdot x\}
$$

whereas I have set

$$
\partial^{\lambda}_{x}\,\partial^{\nu}_{*y}\,D^{(-)}_{0}(x-y) \equiv \frac{i}{(2\pi)^{3}}\int \mathrm{d}\,k\;\frac{k^{\lambda}k^{\nu}_{*}}{k\cdot k_{*}}\;\delta\left(k^{2}\right)\,\theta(k_{0})\,\exp\{-\,i\,k\cdot x\}
$$

We are now ready to perform the general covariant canonical quantization of the free vector fields, both in the massive and mass-less cases.

5.3 Covariant Canonical Quantum Theory

The manifestly covariant canonical quantization of the massive and massless real vector free fields can be done in a close analogy with the canonical quantization of the real scalar free field, just like I have done in Section 3.3.

5.3.1 The Massive Vector Field

Let us first consider the transverse vector free fields. The simplest way to obtain the canonical quantization of the transverse massive Proca field is based upon the Lorentz covariance. As a matter of fact, it is an easy exercise to realize that the covariant canonical commutator – in physical units

$$[\, V_\mu(x)\,,\, V_\nu(y)\,] = i\hbar c \left\{ g_{\mu\nu} + \left(\frac{\hbar}{mc}\right)^2 \frac{\partial^2}{\partial x^\mu\, \partial x^\nu} \right\} D\,(x-y\,;\, m) \tag{5.64}$$

where $D\,(x-y\,;\,m)$ denotes the Pauli-Jordan distribution (3.112), does indeed fulfill all the fundamental requirements: namely,

1. it is a solution of the Klein-Gordon equation
$$\left(\Box_x + m^2\right)[\, V_\mu(x)\,,\, V_\nu(y)\,] = \left(\Box_y + m^2\right)[\, V_\mu(x)\,,\, V_\nu(y)\,] = 0$$
2. it fulfills the transverse-like condition
$$\partial_x^\mu\,[\, V_\mu(x)\,,\, V_\nu(y)\,] = \partial_y^\nu\,[\, V_\mu(x)\,,\, V_\nu(y)\,] = 0$$
3. it encodes the symmetry, Hermitean and micro-causality properties
$$[\, V_\mu(x)\,,\, V_\nu(y)\,] = -\,[\, V_\mu(x)\,,\, V_\nu(y)\,]^\dagger = -\,[\, V_\nu(y)\,,\, V_\mu(x)\,]$$
$$[\, V_\imath(x)\,,\, V_\jmath(y)\,] = 0 \qquad \forall\,(x-y)^2 < 0\,,\ \imath,\jmath = 1,2,3$$
$$\lim_{x_0 \to y_0}\,[\, V_0(x)\,,\, V_\jmath(y)\,] = -\,\frac{i}{m^2}\,\partial_\jmath\,\delta(\mathbf{x}-\mathbf{y})$$
4. from the expression of the conjugated momentum Proca field operator
$$E^\jmath(y) = \partial_0 V_\jmath(y) - \partial_\jmath V_0(y)$$

we obtain the commutator

$$
\begin{aligned}
\left[\,V_\jmath(x)\,,\,E^\ell(y)\,\right] &= [\,V_\jmath(x)\,,\,\partial_0 V_\ell(y) - \partial_\ell V_0(y)\,] \\
&= \frac{\partial}{\partial y_0}\,[\,V_\jmath(x)\,,\,V_\ell(y)\,] - \frac{\partial}{\partial y^\ell}\,[\,V_\jmath(x)\,,\,V_0(y)\,] \\
&= \frac{i\partial}{\partial y_0}\left\{\left(g_{\jmath\ell} + m^{-2}\,\frac{\partial^2}{\partial x^\jmath\,\partial x^\ell}\right)\right\} D(x-y\,;\,m) \\
&- m^{-2}\,\frac{\partial^2}{\partial x^\jmath\,\partial x_0}\,\frac{i\partial}{\partial y^\ell}\,D(x-y\,;\,m) \\
&= i\,\delta_\jmath^\ell\,\frac{\partial}{\partial x_0}\,D(x-y\,;\,m)
\end{aligned}
$$

that yields

$$\lim_{x_0\to y_0}\left[\,V_\jmath(x)\,,\,E^\ell(y)\,\right] = i\,\delta_\jmath^\ell\,\delta(\mathbf{x}-\mathbf{y})$$

In a quite analogous way it is straightforward to verify that

$$\lim_{x_0\to y_0}\left[\,E^\jmath(x)\,,\,E^\ell(y)\,\right] = 0$$

Hence the following initial conditions hold true, *viz.*,

$$\lim_{x_0\to y_0}\,[\,V_\imath(x)\,,\,V_\jmath(y)\,] = 0 \tag{5.65}$$

$$\lim_{x_0\to y_0}\left[\,V_\jmath(x)\,,\,E^\ell(y)\,\right] = i\,\delta_\jmath^\ell\,\delta(\mathbf{x}-\mathbf{y}) \tag{5.66}$$

$$\lim_{x_0\to y_0}\left[\,E^\jmath(x)\,,\,E^\ell(y)\,\right] = 0 \tag{5.67}$$

which are nothing but the equal time canonical commutation relations that arise from the Dirac correspondence principle $\{\cdot\,,\cdot\} \longrightarrow [\,\cdot\,,\cdot\,]/i\hbar$ when applied to the classical Poisson's brackets (5.46).

As a consequence we safely come to the conclusion that the above covariant canonical commutation relations (5.64) are the unique operator solution of the equations of motion that endorses all the requirements demanded by the first principles of Quantum Field Theory.

In the massive case, the classical Proca vector free field (5.33) is turned into an operator valued tempered distribution: namely,

$$V^\nu(y) = \sum_{\mathbf{k}\,,\,r}\left[\,f_{\mathbf{k}\,,\,r}\,u^\nu_{\mathbf{k}\,,\,r}(y) + f^\dagger_{\mathbf{k}\,,\,r}\,u^{\nu\,*}_{\mathbf{k}\,,\,r}(y)\right] \tag{5.68}$$

$$u^\nu_{\mathbf{k}\,,\,r}(y) = [\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{-1/2}\,e^\nu_r(\mathbf{k})\,\exp\{-\,i\,\omega_{\mathbf{k}}\,y_0 + i\,\mathbf{k}\cdot\mathbf{y}\} \tag{5.69}$$

Now it is straightforward to derive the algebra of the creation and destruction operators

$$[\, f_{\mathbf{h}\,,r}\,,\; f_{\mathbf{k}\,,s}\,] = 0 \qquad\qquad [\, f^{\dagger}_{\mathbf{h}\,,r}\,,\; f^{\dagger}_{\mathbf{k}\,,s}\,] = 0$$

$$[\, f_{\mathbf{h}\,,r}\,,\; f^{\dagger}_{\mathbf{k}\,,s}\,] = \delta_{rs}\;\delta\,(\mathbf{h}-\mathbf{k})$$

`Proof.` Consider the covariant canonical commutator (5.64) and insert the normal modes expansions of the Proca vector field: then we obtain

$$\begin{aligned}
&[\,V^{\,\mu}\,(x)\,,\,V^{\,\nu}\,(y)\,] = i\left(g^{\mu\nu} - m^{-2}\,\partial^{\mu}_{x}\partial^{\nu}_{y}\right)\Big\{D^{(-)}\,(x-y\,;\,m) + D^{(+)}\,(x-y\,;\,m)\Big\} \\
=\;& \sum_{\mathbf{h}\,,r}\sum_{\mathbf{k}\,,s}\;\Big[\,f_{\mathbf{h}\,,r}\,u^{\mu}_{\mathbf{h}\,,r}(x) + f^{\dagger}_{\mathbf{h}\,,r}\,u^{\mu\,*}_{\mathbf{h}\,,r}\,(x)\,,\; f_{\mathbf{k}\,,s}\,u^{\nu}_{\mathbf{k}\,,s}\,(y) + f^{\dagger}_{\mathbf{k}\,,s}\,u^{\nu\,*}_{\mathbf{k}\,,s}\,(y)\,\Big] \\
=\;& \sum_{\mathbf{h}\,,r}\sum_{\mathbf{k}\,,s}\Big\{\Big[\,f_{\mathbf{h}\,,r}\,,\,f_{\mathbf{k}\,,s}\Big]\,u^{\mu}_{\mathbf{h}\,,r}\,(x)\,u^{\nu}_{\mathbf{k}\,,s}(y) + \Big[\,f_{\mathbf{h}\,,r}\,,\,f^{\dagger}_{\mathbf{k}\,,s}\Big]\;u^{\mu}_{\mathbf{h}\,,r}\,(x)\,u^{\nu\,*}_{\mathbf{k}\,,s}(y) \\
+\;& \Big[\,f^{\dagger}_{\mathbf{h}\,,r}\,,\,f_{\mathbf{k}\,,s}\Big]\;u^{\mu\,*}_{\mathbf{h}\,,r}\,(x)\,u^{\nu}_{\mathbf{k}\,,s}\,(y) + \Big[\,f^{\dagger}_{\mathbf{h}\,,r}\,,\,f^{\dagger}_{\mathbf{k}\,,s}\Big]\;u^{\mu\,*}_{\mathbf{h}\,,r}\,(x)\,u^{\nu\,*}_{\mathbf{k}\,,s}\,(y)\Big\}
\end{aligned}$$

On the other side, from the closure relations (5.37) we can write

$$\sum_{\mathbf{k}\,,r}\Big(u^{\mu}_{\mathbf{k}\,,r}\,(x)\,u^{\nu\,*}_{\mathbf{k}\,,r}\,(y) - u^{\mu\,*}_{\mathbf{k}\,,r}\,(x)\,u^{\nu}_{\mathbf{k}\,,r}\,(y)\Big) = [\,V^{\,\mu}\,(x)\,,\,V^{\,\nu}\,(y)\,]$$

so that, by comparison, equality occurs iff the creation and destruction operator algebra

$$[\, f_{\mathbf{h}\,,r}\,,\; f_{\mathbf{k}\,,s}\,] = 0 \qquad\qquad [\, f^{\dagger}_{\mathbf{h}\,,r}\,,\; f^{\dagger}_{\mathbf{k}\,,s}\,] = 0$$

$$[\, f_{\mathbf{h}\,,r}\,,\; f^{\dagger}_{\mathbf{k}\,,s}\,] = \delta_{rs}\;\delta\,(\mathbf{h}-\mathbf{k})$$

holds true. □

As in the scalar and spinor cases, the wave functions of the neutral massive Proca particle of spin one are provided by

$$\langle\,0\,|\;V^{\,\nu}\,(x)\,|\,\mathbf{h}\,,\;s\,\rangle = \langle\,0\,|\;V^{\,\nu}\,(x)\,f^{\dagger}_{\mathbf{h}\,,s}\,|\,0\,\rangle = u^{\nu}_{\mathbf{h}\,,s}\,(x)$$

Moreover, the normal mode expansion of the field strengths becomes

$$\begin{aligned}
\mathbf{E}\,(x) \;&=\; i\sum_{\mathbf{k}\,,r}\;\Big\{f_{\mathbf{k}\,,r}\;\big[\,\omega_{\mathbf{k}}\,\mathbf{u}_{\mathbf{k}\,,r}\,(x) - \mathbf{k}\,u^{0}_{\mathbf{k}\,,r}\,(x)\,\big] \\
&-\; f^{\dagger}_{\mathbf{k}\,,r}\;\big[\,\omega_{\mathbf{k}}\,\mathbf{u}^{*}_{\mathbf{k}\,,r}\,(x) - \mathbf{k}\,u^{0\,*}_{\mathbf{k}\,,r}\,(x)\,\big]\Big\} \qquad (5.70) \\
\mathbf{B}\,(x) \;&=\; i\sum_{\mathbf{k}\,,r}\;\mathbf{k}\,\times\,\mathbf{u}_{\mathbf{k}\,,r}\,(x)\,f_{\mathbf{k}\,,r} \;+\; \text{c.c.} \qquad (5.71)
\end{aligned}$$

where the massive electric and massive magnetic fields are defined by

$$E^{\,k} = F^{\,k0} = -\,\left(\partial_0\,A^{\,k} + \partial_k\,A_{\,0}\right) \qquad\qquad B^{\,k} = \tfrac{1}{2}\,\varepsilon^{\,j\ell k}\,F_{j\ell}$$

It is in fact a straightforward exercise to show that, by inserting the normal mode expansion (5.69) and by making use of the orthogonality relations among the polarization vectors $e_r^{\,\mu}(\mathbf{k})$, the energy momentum operator takes the expected diagonal form, which corresponds to the sum over an infinite set of independent linear harmonic oscillators, one for each component of the wave vector $\mathbf{k}$ and for each one of the three independent physical polarization $e_r^{\,\mu}(\mathbf{k})$ $(r = 1, 2, 3)$. Actually, if we recast the equations of motion (5.26) for the Proca wave field in the Maxwell-like form

$$\begin{cases} \partial_j F_{jk} = \dot{F}_{0k} + m^2 V_k & \text{(displacement current)} \\ \partial_k F_{0k} = -\,m^2 V_0 & \text{(Gauss law)} \\ \dot{V}_0 = \partial_k V_k & \text{(subsidiary condition)} \end{cases} \tag{5.72}$$

then we get the quite simple expression for the energy operator

$$\begin{aligned} P_0 &= H = \int \mathrm{d}\mathbf{x}\ :\Theta_{00}(t,\mathbf{x}): \\ &= \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\ :E^k(x)\,E^k(x) + \tfrac{1}{2}\,F_{jk}(x)\,F_{jk}(x): \\ &+ \tfrac{1}{2}\,m^2 \int \mathrm{d}\mathbf{x}\ :V_0^2(x) + V_k(x)\,V_k(x): \\ &\doteq \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\ :F_{0k}(x)\,\overleftrightarrow{\partial}_0\,V_k(x): \\ &\doteq \int \mathrm{d}\mathbf{x} : \tfrac{1}{2}\,V_\mu(x)\,\overleftrightarrow{\partial}_0\,\dot{V}^{\mu}(x): \end{aligned}$$

where $\doteq$ means, as usual, that I have dropped some spatial divergence term and I have repeatedly made use of the equations of motion. Now, from the orthogonality relation (5.30) we can easily recognize that

$$\begin{aligned} g_{\lambda\nu}\int \mathrm{d}\mathbf{x}\ u^{\lambda\,*}_{\mathbf{k},r}(x)\,\overleftrightarrow{\partial}_0\,\dot{u}^{\nu}_{\mathbf{h},s}(x) &= \omega_{\mathbf{k}}\,\delta_{rs}\,\delta(\mathbf{h}-\mathbf{k}) \\ g_{\lambda\nu}\int \mathrm{d}\mathbf{x}\ u^{\lambda}_{\mathbf{k},r}(x)\,\overleftrightarrow{\partial}_0\,\dot{u}^{\nu}_{\mathbf{h},s}(x) &\equiv 0 \end{aligned}$$

and consequently

$$\begin{aligned} P_0 &= \sum_{\mathbf{k},r} \omega_{\mathbf{k}}\,f^{\dagger}_{\mathbf{k},r}\,f_{\mathbf{k},r} \\ \mathbf{P} &= \int \mathrm{d}\mathbf{x}\ :\mathbf{E}(x)\times\mathbf{B}(x) + m^2\,V_0(x)\,\mathbf{V}(x): \\ &\doteq \int \mathrm{d}\mathbf{x}\ :\tfrac{1}{2}\,V_\mu(x)\,\overleftrightarrow{\partial}_k\,\dot{V}^{\mu}(x): \\ &= \sum_{\mathbf{k},r} \mathbf{k}\,f^{\dagger}_{\mathbf{k},r}\,f_{\mathbf{k},r} \end{aligned} \tag{5.73}$$

On the other hand, the energy-momentum tensor of the auxiliary field is provided by the $B-$dependent part of the general expression (5.48): namely,

$$\Theta_{B}^{\mu\nu} \;=\; m^{-2} \;:\; \tfrac{1}{2}\, g^{\mu\nu}\, \partial^{\,\lambda} B\, \partial_{\lambda} B - \tfrac{1}{2}\, \xi m^2\, g^{\mu\nu}\, B^2 - \partial^{\,\mu} B\, \partial^{\,\nu} B \;:$$

which drives to the conserved energy-momentum operators

$$P_0 = \frac{1}{2m^2} \int \mathrm{d}\mathbf{x} \;:\; B(x)\, \ddot{B}(x) - \dot{B}^2(x) \;: \tag{5.74}$$

$$\mathbf{P} = \frac{1}{m^2} \int \mathrm{d}\mathbf{x} \;:\; \dot{B}(x)\, \boldsymbol{\nabla} B(x) \;: \tag{5.75}$$

5.3.2 The Stückelberg Ghost Scalar

It is important to gather that from the Lagrange density (5.42) it follows that the conjugate momentum of the auxiliary field gets the wrong sign, *i.e.*

$$\Pi(x) \;=\; -\, \dot{B}(x)/m^2$$

From the normal mode decomposition of the auxiliary nonphysical field and its conjugate momentum

$$\begin{aligned}
B(x) &= m \sum_{\mathbf{h}} \left[b_{\mathbf{h}}\, u_{\mathbf{h}}(x) + b^{\dagger}_{\mathbf{h}}\, u^{*}_{\mathbf{h}}(x) \right] \\
\Pi(y) &= \frac{i}{m} \sum_{\mathbf{k}} \omega'_{\mathbf{k}} \left[b_{\mathbf{k}}\, u_{\mathbf{k}}(y) - b^{\dagger}_{\mathbf{k}}\, u^{*}_{\mathbf{k}}(y) \right] \\
u_{\mathbf{k}}(x) &= [\,(2\pi)^3\, 2\omega'_{\mathbf{k}}\,]^{-1/2} \exp\{-\, i\, \omega'_{\mathbf{k}} t + i\, \mathbf{k}\cdot\mathbf{x}\} \\
&\qquad\qquad\qquad \omega'_{\mathbf{k}} \;\equiv\; \left(\mathbf{k}^2 + \xi\, m^2\right)^{1/2}
\end{aligned}$$

it is evident that in order to recover the canonical commutation relation

$$[\, B(t,\mathbf{x})\, ,\, \Pi(t,\mathbf{y})\,] = i\hbar\, \delta(\mathbf{x}-\mathbf{y})$$

that corresponds to the classical Poisson bracket (5.47) we have to require

$$[\, b^{\dagger}_{\mathbf{h}}\, ,\, b_{\mathbf{k}}\,] \;=\; \delta(\mathbf{h}-\mathbf{k}) \tag{5.76}$$

all the other commutators vanishing.

`Proof.` On the one hand we find

$$
\begin{aligned}
& [\,B\,(t,\mathbf{x})\,,\,\Pi\,(t,\mathbf{y})\,] = \\
= & \sum_{\mathbf{h}}\sum_{\mathbf{k}} i\,\omega'_{\mathbf{k}}\,\Big[\,b_{\mathbf{h}}\,u_{\mathbf{h}}\,(t,\mathbf{x}) + b^{\dagger}_{\mathbf{h}}\,u^{*}_{\mathbf{h}}\,(t,\mathbf{x})\,,\,b_{\mathbf{k}}\,u_{\mathbf{k}}\,(t,\mathbf{y}) - b^{\dagger}_{\mathbf{k}}\,u^{*}_{\mathbf{k}}\,(t,\mathbf{y})\,\Big] \\
= & \sum_{\mathbf{h}}\sum_{\mathbf{k}} i\,\omega'_{\mathbf{k}}\,\Big\{[\,b_{\mathbf{h}}\,,\,b_{\mathbf{k}}\,]\,u_{\mathbf{h}}\,(t,\mathbf{x})\,u_{\mathbf{k}}\,(t,\mathbf{y}) - [\,b_{\mathbf{h}}\,,\,b^{\dagger}_{\mathbf{k}}\,]\,u_{\mathbf{h}}\,(t,\mathbf{x})\,u^{*}_{\mathbf{k}}\,(t,\mathbf{y})\Big\} \\
+ & \sum_{\mathbf{h}}\sum_{\mathbf{k}} i\,\omega'_{\mathbf{k}}\,\Big\{[\,b^{\dagger}_{\mathbf{h}}\,,\,b_{\mathbf{k}}\,]\,u^{*}_{\mathbf{h}}\,(t,\mathbf{x})\,u_{\mathbf{k}}\,(t,\mathbf{y}) - [\,b^{\dagger}_{\mathbf{h}}\,,\,b^{\dagger}_{\mathbf{k}}\,]\,u^{*}_{\mathbf{h}}\,(t,\mathbf{x})\,u^{*}_{\mathbf{k}}\,(t,\mathbf{y})\Big\}
\end{aligned}
$$

On the other hand we can always write the identities

$$
\begin{aligned}
[\,B\,(t,\mathbf{x})\,,\,\Pi\,(t,\mathbf{y})\,] &= i\,\delta(\mathbf{x}-\mathbf{y}) \\
&= i\int\frac{\mathrm{d}\mathbf{k}}{2(2\pi)^3}\Big(\exp\{i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\} + \exp\{i\,\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})\}\Big) \\
&= \sum_{\mathbf{k}} i\,\omega'_{\mathbf{k}}\,\Big\{u^{*}_{\mathbf{k}}\,(t,\mathbf{x})\,u_{\mathbf{k}}\,(t,\mathbf{y}) + u_{\mathbf{k}}\,(t,\mathbf{x})\,u^{*}_{\mathbf{k}}\,(t,\mathbf{y})\Big\}
\end{aligned}
$$

By comparison, owing to the inversion of the normal mode expansions, we are led to the ghost-like canonical commutation relations

$$
[\,b_{\mathbf{h}}\,,\,b^{\dagger}_{\mathbf{k}}\,] = -\,\delta\,(\mathbf{h}-\mathbf{k}) \qquad [\,b_{\mathbf{h}}\,,\,b_{\mathbf{k}}\,] = [\,b^{\dagger}_{\mathbf{h}}\,,\,b^{\dagger}_{\mathbf{k}}\,] = 0 \qquad (\,\forall\,\mathbf{h},\mathbf{k}\in\mathbb{R}^3\,)
$$

which is what we had to prove. □

Thus we can eventually write

$$
\begin{aligned}
P_0 &= -\sum_{\mathbf{k}}\sqrt{\mathbf{k}^2+\xi\,m^2}\;b^{\dagger}_{\mathbf{k}}\,b_{\mathbf{k}} \;\equiv\; H_B \\
\mathbf{P} &= \sum_{\mathbf{k}}\mathbf{k}\,b^{\dagger}_{\mathbf{k}}\,b_{\mathbf{k}}
\end{aligned}
\tag{5.77}
$$

From the above expression (5.77) for the conserved Hamiltonian of the auxiliary $B-$field, as well as from the unconventional nature of the canonical commutation relations (5.76), it turns out that no physical meaning can be assigned to the Hamiltonian operator of the auxiliary scalar field. As a matter of fact, for $\xi \geq 0$ the Hamiltonian operator becomes positive semi-definite - thanks to the unconventional canonical commutation relations - while for $\xi < 0$ we see that at low momenta $\mathbf{k}^2 < |\,\xi\,|\,m^2$ the energy becomes imaginary. In all cases, any physical interpretation of the Hamiltonian operator breaks down.

Moreover, from the conventional definition of the Fock vacuum

$$
f_{\mathbf{k}\,,\,r}\;|\,0\,\rangle = 0 \qquad b_{\mathbf{k}}\;|\,0\,\rangle = 0 \qquad \forall\,\mathbf{k}\in\mathbb{R}^3 \quad r = 1,2,3
$$

and the unconventional canonical commutation relations (5.76), it follows that *e.g.* the proper 1-particle states of the auxiliary field

$$|\,b\,\rangle \stackrel{\text{def}}{=} \int \mathrm{d}\mathbf{k}\;\tilde{b}(\mathbf{k})\,b_{\mathbf{k}}^{\dagger}\,|\,0\,\rangle \qquad\qquad \int \mathrm{d}\mathbf{k}\;|\,\tilde{b}(\mathbf{k})\,|^2 = 1$$

do exhibit **negative norm**, that is

$$\langle\, b\,|\,b\,\rangle \;=\; -\,1$$

Hence the Fock space $\mathcal{F}$ of the quantum states for the free massive vector field in the general covariant gauge is equipped with an indefinite metric, which means that it contains normalizable states with positive, negative and null norm. The auxiliary $B-$field is named a ghost field: its presence entails a Hamiltonian operator which is unbounded from below, leading thereby to the instability, or even imaginary energy eigenvalues, that means meta-stable states when $\exp\{-\,i\,\omega_{\mathbf{k}}'\,t\}$ is less than one, or even runaway solutions when $\exp\{-\,i\,\omega_{\mathbf{k}}'\,t\}$ becomes very large.

Thus, in order to ensure some meaningful and sound quantum mechanical interpretation of the free field theory of a massive vector field in the general covariant gauge, we are necessarily led to select a physical subspace $\mathcal{H}_{\text{phys}}$ of the large Fock space $\mathcal{F}$, in which no quanta of the auxiliary field are allowed. This can be achieved by imposing the subsidiary condition

$$B^{(-)}(x)\,|\,\text{phys}\,\rangle \;=\; 0 \qquad\qquad \forall\,|\,\text{phys}\,\rangle \in \mathcal{H}_{\text{phys}} \tag{5.78}$$

where $B^{(-)}(x)$ denotes, as usual, the positive frequency destruction part of the auxiliary scalar Klein-Gordon ghost field *i.e.*

$$B^{(-)}(x) = m\sum_{\mathbf{k}}\; b_{\mathbf{k}}\,u_{\mathbf{k}}(x)$$

This entails that the vacuum state is physical and cyclic, in such a manner that all the physical states are generated by the repeated action of the massive Proca field creation operators $f_{\mathbf{k}\,,\,r}^{\dagger}$ on the vacuum.

5.3.3 The Stückelberg Vector Propagator

Now we are ready to obtain the covariant commutation relations and the causal Green's functions for the Stückelberg vector field, which is defined by the decomposition (5.24)

$$A_{\mu}(x) = V_{\mu}(x) - \frac{\hbar^2}{m^2c^2}\,\partial_{\mu}B(x)$$

where $V^{\mu}(x)$ is the Proca vector field operator while $B(x)$ is the Stückelberg ghost scalar field operator that I have thoroughly discussed in the previous paragraphs. From the normal mode expansions of the massive Proca real vector field and of the ghost scalar field, taking the canonical commutation relations into account, we can readily check that we have

$$\left.\begin{array}{c} [\,V_{\mu}(x)\,,\,V_{\nu}(0)\,] = i\,(\,g_{\mu\nu} + m^{-2}\,\partial_{\mu}\,\partial_{\nu}\,)\,D\,(x\,;m^2) \\ [\,B(x)\,,\,B(0)\,] = im^2 D\,(x\,;\,\xi\,m^2) \\ [\,V_{\mu}(x)\,,\,B(y)\,] = 0 \end{array}\right\} \tag{5.79}$$

the first commutator being due to the closure relation (5.31). Then it is straightforward to check that the following Feynman propagators actually occur: namely,

$$\begin{aligned} &\langle\,0\,|\,T\,V_{\mu}(x)V_{\nu}(y)|\,0\,\rangle = \left[-\,g_{\,\mu\nu} + \frac{\hbar^2}{m^2c^2}\cdot\partial_{\mu\,,\,x}\,\partial_{\nu\,,\,y}\right] D_F(x-y\,;m^2) \\ &\langle\,0\,|\,T\,B(x)B(y)|\,0\,\rangle = -\,m^2\,D_F\left(x-y\,;\xi\,m^2\right) \\ &\langle\,0\,|\,T\,V_{\mu}(x)B(y)|\,0\,\rangle = 0 \end{aligned}$$

`Proof.` The Proca propagator is defined to be

$$\begin{aligned} &\langle\,0\,|\,T\,V_{\mu}(x)\,V_{\nu}(y)|\,0\,\rangle = \\ &\theta(x^0-y^0)\langle\,0\,|\,V_{\mu}(x)\,V_{\nu}(y)|\,0\,\rangle + \theta(y^0-x^0)\langle\,0\,|\,V_{\nu}(y)\,V_{\mu}(x)|\,0\,\rangle \end{aligned}$$

Taking into account the normal modes expansion (5.33) of the Proca field operator, the canonical commutation relations and the closure relations (5.37)

$$\begin{aligned} &V^{\,\nu}(x) = \sum_{\mathbf{k}\,,\,r}\,\left[\,f_{\mathbf{k}\,,\,r}\,u^{\,\nu}_{\mathbf{k}\,,\,r}(x) + f^{\dagger}_{\mathbf{k}\,,\,r}\,u^{\,\nu\,*}_{\mathbf{k}\,,\,r}(x)\,\right] \\ &u^{\nu}_{\mathbf{k}\,,\,r}(x) = [\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{-1/2}\,e^{\,\nu}_{r}(\mathbf{k})\,\exp\{-\,it\,\omega_{\mathbf{k}} + i\,\mathbf{k}\cdot\mathbf{x}\} \\ &[\,f_{\mathbf{k}\,,\,r}\,,\,f^{\dagger}_{\mathbf{p}\,,\,s}\,] = \delta_{rs}\,\delta(\mathbf{k}-\mathbf{p}) \qquad [\,f_{\mathbf{k}\,,\,r}\,,\,f_{\mathbf{p}\,,\,s}\,] = [\,f^{\dagger}_{\mathbf{k}\,,\,r}\,,\,f^{\dagger}_{\mathbf{p}\,,\,s}\,] = 0 \\ &\sum_{\mathbf{k}\,,\,r} u^{\,\mu}_{\mathbf{k}\,,\,r}(x)\,u^{\,\nu\,*}_{\mathbf{k}\,,\,r}(y) \;= i\left(\,g^{\,\mu\nu} - m^{-2}\,\partial^{\,\mu}_{x}\,\partial^{\,\nu}_{y}\,\right)\,D^{\,(-)}(x-y) \end{aligned}$$

we can write

$$\begin{aligned} &\langle\,0\,|\,T\,V^{\,\mu}(x)\,V^{\,\nu}(y)|\,0\,\rangle = \\ = \;&\sum_{\mathbf{k}\,,\,r}\left[\,\theta(x^0-y^0)\,u^{\,\mu}_{\mathbf{k}\,,\,r}(x)\,u^{\,\nu\,*}_{\mathbf{k}\,,\,r}(y) + \theta(y^0-x^0)\,u^{\,\mu\,*}_{\mathbf{k}\,,\,r}(x)\,u^{\,\nu}_{\mathbf{k}\,,\,r}(y)\right] \\ = \;&i\,\theta(x^0-y^0)\,\left(\,g^{\,\mu\nu} - m^{-2}\,\partial^{\,\mu}_{x}\,\partial^{\,\nu}_{y}\,\right)D^{\,(-)}(x-y) \\ - \;&i\,\theta(y^0-x^0)\,\left(\,g^{\,\mu\nu} - m^{-2}\,\partial^{\,\mu}_{x}\,\partial^{\,\nu}_{y}\,\right)D^{\,(+)}(x-y) \end{aligned}$$

Now we get

$$\theta(y^0-x^0)\,\partial^{\,\mu}_{x}\,\partial^{\,\nu}_{y}\,D^{\,(+)}(x-y) - \theta(x^0-y^0)\,\partial^{\,\mu}_{x}\,\partial^{\,\nu}_{y}\,D^{\,(-)}(x-y)$$

$$
\begin{aligned}
&= \partial_x^{\mu}\Big[\,\theta(y^0-x^0)\,\partial_y^{\nu}\,D^{(+)}(x-y)-\theta(x^0-y^0)\,\partial_y^{\nu}\,D^{(-)}(x-y)\Big]\\
&+ g^{\mu 0}\,\delta(x^0-y^0)\,\partial_y^{\nu}\,D(x-y)\\
&= -\,i\,\partial_x^{\mu}\,\partial_y^{\nu}\,D_F(x-y)-g^{0\nu}\,\partial_x^{\mu}\left[\,\delta(x^0-y^0)\,D(x-y)\,\right]\\
&+ g^{0\mu}\,\partial_y^{\nu}\left[\,\delta(x^0-y^0)\,D(x-y)\,\right]+g^{0\mu}\,g^{0\nu}\,\delta\,'(x^0-y^0)\,D(x-y)
\end{aligned}
$$

where $D(x-y)$ is the invariant Pauli-Jordan distribution, while $\delta\,'(x)$ means derivative in the sense of the distributions. Now, owing to the micro-causality property $D(x-y)=0$ for $(x-y)^2<0$, it turns out that solely the first addendum in the right-hand-side of the very last equality does not vanish. Hence we eventually get the Proca causal Green's function

$$
\langle\,0\,|\,T\,V^{\,\mu}(x)\,V^{\,\nu}(y)|\,0\,\rangle=\left[\,-\,g^{\,\mu\nu}+\frac{\hbar^2}{m^2c^2}\cdot\partial_x^{\mu}\,\partial_y^{\nu}\,\right]D_F(x-y\,;m^2)
$$

The propagator for the Stückelberg scalar ghost does immediately follow from the normal modes expansion and the unconventional canonical commutation relations

$$
B\,(x)=m\sum_{\mathbf{k}}\;\Big[\,b_{\mathbf{k}}\,u_{\mathbf{k}}\,(x)+b_{\mathbf{k}}^{\dagger}\,u_{\mathbf{k}}^{*}\,(x)\,\Big]
$$

$$
u_{\mathbf{k}}\,(x)=[\,(2\pi)^3\,2\omega'_{\mathbf{k}}\,]^{-1/2}\,\exp\{-\,i\,\omega'_{\mathbf{k}}\,t+i\,\mathbf{k}\cdot\mathbf{x}\}\qquad\omega'_{\mathbf{k}}\;\equiv\;\sqrt{\mathbf{k}^2+\xi\,m^2}
$$

$$
\left[\,b_{\mathbf{k}}\,,\,b_{\mathbf{p}}^{\dagger}\,\right]=-\,\delta(\mathbf{k}-\mathbf{p})\qquad[\,b_{\mathbf{k}}\,,\,b_{\mathbf{p}}\,]=\Big[\,b_{\mathbf{k}}^{\dagger}\,,\,b_{\mathbf{p}}^{\dagger}\,\Big]=0
$$

that readily yield $\langle\,0\,|\,T\,B\,(x)B\,(y)|\,0\,\rangle=-\,m^2\,D_F\left(x-y\,;\xi\,m^2\right)$. Notice that from the null canonical commutation relations

$$
[\,b_{\mathbf{k}}\,,\,f_{\mathbf{p}\,,\,r}\,]=[\,b_{\mathbf{k}}\,,\,f_{\mathbf{p}\,,\,r}^{\dagger}\,]=[\,b_{\mathbf{k}}^{\dagger}\,,\,f_{\mathbf{p}\,,\,r}\,]=[\,b_{\mathbf{k}}^{\dagger}\,,\,f_{\mathbf{p}\,,\,r}^{\dagger}\,]=0\qquad\forall\,r=1,2,3\,\vee\,\mathbf{k},\mathbf{p}\in\mathbb{R}^3
$$

we definitely obtain $\langle\,0\,|\,T\,B(x)V^{\,\mu}(y)|\,0\,\rangle=0$ which concludes the proof. $\square$

Now it is a very simple exercise to obtain the canonical commutator and the Feynman propagator for the Stückelberg massive vector field operator $A^{\mu}(x)$ taking the basic definition (5.24) into account. As a matter of fact, from the covariant commutation relations (5.79) we immediately obtain

$$
\begin{aligned}
[\,A_{\mu}(x)\,,\,A_{\nu}(0)\,] &= i\,\left(\,g_{\mu\nu}+m^{-2}\,\partial_{\mu}\,\partial_{\nu}\,\right)D(x\,;m^2)\\
&- i\,m^{-2}\,\partial_{\mu}\,\partial_{\nu}\,D\left(x\,;\xi\,m^2\right) &(5.80)\\
[\,A_{\mu}(x)\,,\,B(0)\,] &= -\,i\,\partial_{\mu}\,D\left(x\,;\xi\,m^2\right) &(5.81)
\end{aligned}
$$

Moreover we find

$$
\begin{aligned}
&\langle\,0\,|\,T\,\partial_{\mu}B(x)\,\partial_{\nu}B(y)\,|\,0\,\rangle\\
&= \partial_{\mu\,,\,x}\langle\,0\,|\,T\,B(x)\,\partial_{\nu}B(y)\,|\,0\,\rangle\\
&- \delta(x_0-y_0)\,g_{\mu 0}\,\langle\,0\,|\,[\,B(x)\,,\,\partial_{\nu}B(y)\,]\,|\,0\,\rangle\\
&= \partial_{\mu,\,x}\,\partial_{\nu,\,y}\langle\,0\,|\,T\,B(x)\,B(y)\,|\,0\,\rangle\\
&+ \delta(x_0-y_0)\,g_{\nu 0}\,\partial_{\mu,\,x}\langle\,0\,|\,[\,B(x)\,,\,B(y)\,]\,|\,0\,\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- \;\; \partial_{\nu,\,y}\Big\{\delta(x_0-y_0)\,g_{\mu 0}\,\langle\,0\,|\,[\,B(x)\,,\,B(y)\,]\,|\,0\,\rangle\Big\}\\
&- \;\; \delta'(x_0-y_0)\,g_{\mu 0}\,g_{\nu 0}\,\langle\,0\,|\,[\,B(x)\,,\,B(y)\,]\,|\,0\,\rangle\\
&= \;\; \partial_{\mu,\,x}\,\partial_{\nu,\,y}\langle\,0\,|\,T\,B(x)\,B(y)\,|\,0\,\rangle
\end{aligned}
$$

all the other addenda in the third equality being null for micro-causality. Hence, the Feynman propagator of the Stückelberg vector field becomes

$$
\begin{aligned}
D^{\,F}_{\mu\nu}\left(x\,;m^2,\xi\right) &= \langle\,0\,|\,T\,A_\mu(x)\,A_\nu(0)\,|\,0\,\rangle\\
&= \frac{i}{(2\pi)^4}\int \mathrm{d}^4k\;\exp\{-\,ik\cdot x\}\\
&\times \;\left\{\frac{-\,g_{\mu\nu}+m^{-2}\,k_\mu\,k_\nu}{k^2-m^2+i\varepsilon}-\frac{m^{-2}\,k_\mu\,k_\nu}{k^2-\xi\,m^2+i\varepsilon\,'}\right\}\\
&= \frac{i}{(2\pi)^4}\int \mathrm{d}^4k\,\frac{e^{-\,ik\cdot x}}{k^2-m^2+i\varepsilon}\left\{-\,g_{\mu\nu}+\frac{(1-\xi\,)\,k_\mu\,k_\nu}{k^2-\xi\,m^2+i\varepsilon\,'}\right\}
\end{aligned}
$$

which is the celebrated Stückelberg vector propagator, together with

$$\langle\,0\,|\,T\,A_\mu\,(x)\,B\,(0)|\,0\,\rangle=\partial_\mu\,D_F\,\left(x\,;\,\xi\,m^2\right) \tag{5.82}$$

$$\langle\,0\,|\,T\,B\,(x)\,B\,(0)\,|\,0\,\rangle=-\,m^2\,D_F\,\left(x\,;\,\xi\,m^2\right) \tag{5.83}$$

An important comment is now in order. In the general covariant gauge, for any finite value $\xi\in\mathbb{R}$ of the gauge fixing parameter, the leading asymptotic behavior for large momenta of the momentum space Feynman propagator is provided by

$$
\begin{aligned}
\widetilde{D}^{\,F}_{\mu\nu}\,(k\,;m^2,\xi) &= \frac{i}{k^2-m^2+i\varepsilon}\,\left\{-\,g_{\,\mu\nu}+\frac{(1-\xi\,)\,k_\mu\,k_\nu}{k^2-\xi\,m^2+i\varepsilon\,'}\right\}\\
&\sim\;\; k^{-2}\,d_{\,\mu\nu}\qquad\qquad (\,k_\mu\,\to\,\infty\,)
\end{aligned}
\tag{5.84}
$$

where $d_{\mu\nu}$ is a constant 4×4 matrix, that is a momentum space isotropic and scale homogeneous quadratic decreasing law. On the one hand, this *naïve* power counting property will be one of the crucial necessary though not sufficient hypothesis in all the available proofs of the perturbative order by order renormalizability of any interacting quantum field theory. On the other hand, the price to be paid is the unavoidable introduction of an auxiliary nonphysical $B-$field, that must be eventually excluded from the physical sector of the theory, in such a manner to guarantee the standard orthodox quantum mechanical interpretation. In the free field theory, the subsidiary condition (5.78) is what we need to remove the nonphysical quanta. However, it turns out that an extension of the subsidiary condition to the interacting theories appears to be, in general, highly nontrivial.

Hence, the crucial issue of the decoupling of the auxiliary field from the physical sector of an interacting theory involving massive vector field will result to be one of the the most selective and severe model building criterion for a perturbatively renormalizable interacting quantum field theory. In other words, this means that if we consider the scattering operator S of the theory, then, for any pair of physical states $|\,\mathrm{phys}\rangle$ and $|\,\mathrm{phys}'\rangle$, the unitarity relation

$$\langle\,\mathrm{phys}\,|\,S^{\dagger}S\,|\,\mathrm{phys}'\,\rangle = \sum_{\imath}\,\langle\,\mathrm{phys}\,|\,S^{\dagger}\,|\,\imath\,\rangle\,\langle\,\imath\,|\,S\,|\,\mathrm{phys}'\,\rangle = \langle\,\mathrm{phys}\,|\,\mathrm{phys}'\,\rangle$$

must be saturated by a complete orthogonal set of physical intermediate states or, equivalently, the contribution of the nonphysical states must cancel in the sum over intermediate states. This unitarity criterion will guarantee the existence of a well-defined unitary restriction of the scattering operator to the physical subspace $\mathcal{H}_{\mathrm{phys}} \subset \mathcal{F}$ of the whole Fock space, thus allowing a consistent physical interpretation of the theory.

To this concern, it is important to remark that in the limit $\xi \to \infty$, the auxiliary $B-$field just disappears, so that we are left with the Proca field, the quanta of which do carry just the three physical polarization states. However, the ultraviolet leading behavior of the corresponding Feynman propagator becomes

$$\lim_{\xi\to\infty}\ \widetilde{D}^{\,F}_{\mu\nu}(k\,;m^2,\xi) = \frac{i}{k^2-m^2+i\varepsilon}\ \left\{-\,g_{\,\mu\nu} + m^{-2}\,k_{\,\mu}\,k_{\,\nu}\right\} \tag{5.85}$$

The lack of scale homogeneity and *naïve* power counting property of this expression is the evident obstacle that makes the proof of perturbative order by order renormalizability beyond the present day capabilities. In turn, this is the ultimate reason why the spontaneous gauge symmetry breaking and the Higgs mechanism, to provide the masses for the vector fields which mediate the weak interaction[3], still nowadays appear to be the best buy solution of the above mentioned renormalizability versus unitarity issue.

5.3.4 The Gauge Vector Potential

The quantum theory of the mass-less gauge vector potential can be conveniently obtained from the manifestly covariant formulation. As a matter of fact, from the Lorentz covariance and the bi-linearity and anti-symmetry

[3]The weak interaction is mediated by two charged complex vector fields $W^{\pm}$ with mass $M_{\pm} = 80.425 \pm 0.038$ GeV and a neutral real vector field Z^{0} with a mass $M_0 = 91.1876 \pm 0.0021$ GeV.

properties of the commutators we obtain that the following canonical commutation relations must hold true

$$\left[\,A^{\lambda}(x)\,,\,A^{\nu}(y)\,\right] = i\,g^{\lambda\nu}\,D_0(x-y) + i\;(1-\xi)\;\partial_x^{\lambda}\,\partial_y^{\nu}\,\mathfrak{E}(x-y) \quad (5.86)$$

$$\left[\,F^{\lambda\rho}(x)\,,\,A^{\nu}(y)\,\right] = \left(g^{\nu\rho}\,i\partial_x^{\lambda} - g^{\lambda\nu}\,i\partial_x^{\rho}\right) D_0(x-y) \quad (5.87)$$

$$[\,B(x)\,,\,A^{\nu}(y)\,] \;=\; i\partial_x^{\nu}\,D_0(x-y)$$

$$\left[\,F^{\rho\lambda}(x)\,,\,B(y)\,\right] = 0 \qquad [\,B(x)\,,\,B(y)\,] = 0 \quad (5.88)$$

where the mass-less Pauli-Jordan real and odd distribution is as usual

$$D_0(x) = D_0^{(-)}(x) + D_0^{(+)}(x) = \lim_{m\to 0}\; D(x\,;\,m)$$

$$D_0^{(\pm)}(x) \equiv\; \pm\,\frac{1}{i}\int\frac{\mathrm{d}k}{(2\pi)^3}\;\delta\left(k^2\right)\,\theta\,(k_0)\;\exp\{\pm\,i\,k\cdot x\}$$

$$\lim_{x_0\to 0}\; D_0(x) = 0 \qquad\qquad \lim_{x_0\to 0}\;\partial_0\,D_0(x) = \delta\,(\mathbf{x})$$

$$D_0(x) = D_0^*(x) = \;-\,D_0(-x)$$

whereas $\mathfrak{E}(x)$ is named the mass-less dipole ghost invariant distribution and is defined by the property

$$\Box\,\mathfrak{E}(x) = D_0(x) \quad (5.89)$$

an explicit representation being provided by

$$\begin{aligned}\mathfrak{E}(x) \;&=\; \tfrac{1}{2}\,\left(\boldsymbol{\nabla}^2\right)^{-1}(x_0\,\partial_0 - 1)\;D_0(x)\\ &=\; -\;\lim_{m\to 0}\;\frac{\partial}{\partial m^2}\,D(x\,;m)\end{aligned} \quad (5.90)$$

It is an easy task to prove the following useful formula

$$\partial_x^{\mu}\,\partial_x^{\nu}\,\mathfrak{E}(x-y) = \left(\partial_x^{\,\mu}\,\partial_{*\,x}^{\,\nu} + \partial_y^{\,\nu}\,\partial_{*\,y}^{\,\mu}\right) D_0(x-y)$$

It is a simple and instructive exercise to verify the compatibility between the canonical commutation relations (5.88) and the equations of motion

$$\left\{g_{\mu\nu}\,\Box\;-\;\left(1-\frac{1}{\xi}\right)\,\partial_{\mu}\,\partial_{\nu}\right\}\,A^{\nu}(x) = 0 \quad (5.91)$$

$$\Box\,B(x) = 0 \quad (5.92)$$

$$\partial\cdot A(x) = \xi\,B(x) \quad (5.93)$$

Moreover one can readily check that the initial conditions fulfilled by the canonical commutation relations (5.88) are

$$\left[\,A^{k}(t,\mathbf{x})\,,\,E^{\ell}(t,\mathbf{y})\,\right] = i\hbar\,g^{k\ell}\,\delta\,(\mathbf{x}-\mathbf{y})$$

$$[\,B(t,\mathbf{x})\,,\,\Pi\,(t,\mathbf{y})\,] = i\hbar\,\delta\,(\mathbf{x}-\mathbf{y}) = [\,B(t,\mathbf{x})\,,\,A_0(t,\mathbf{y})\,]$$

all the remaining equal time commutation relations being equal to zero, in full agreement with the Dirac correspondence principle and the classical Poisson brackets (5.22). Hence it follows that the canonical commutation relations (5.88) are the unique solutions of the Euler-Lagrange field equations with the required symmetry properties and that obey the canonical equal time commutation relations which arise from the Dirac correspondence principle. It is worthwhile to remark that from the canonical equal time commutation relations

$$\left[\, A_k\,(t,\mathbf{x})\,,\, E^{\ell}\,(t,\mathbf{y})\,\right] = i\hbar\,\delta^{\,\ell}_{\,k}\,\delta\,(\mathbf{x}-\mathbf{y})$$

one can immediately obtain for $\varepsilon_{\,123} = +1$

$$\begin{aligned}
& \tfrac{1}{2}\,\varepsilon^{\,\imath\jmath k}\left[\,\nabla_{\jmath} A_k\,(t,\mathbf{x}) - \nabla_k\, A_{\jmath}\,(t,\mathbf{x})\,,\, E^{\ell}\,(t,\mathbf{y})\,\right] \\
= \;& \tfrac{1}{2}\,i\hbar\,\varepsilon^{\,\imath\jmath k}\left\{\delta^{\,\ell}_{\,k}\,\nabla_{\jmath}\,\delta\,(\mathbf{x}-\mathbf{y}) - \delta^{\,\ell}_{\,\jmath}\,\nabla_k\,\delta\,(\mathbf{x}-\mathbf{y})\right\} \\
= \;& \tfrac{1}{2}\,i\hbar\left\{\varepsilon^{\,\imath\jmath\ell}\,\nabla_{\jmath}\,\delta\,(\mathbf{x}-\mathbf{y}) - \varepsilon^{\,\imath\ell k}\,\nabla_k\,\delta\,(\mathbf{x}-\mathbf{y})\right\} \\
\Rightarrow \;& \left[\,B^{\imath}\,(t,\mathbf{x})\,,\, E^{\ell}\,(t,\mathbf{y})\,\right] = i\hbar\,\varepsilon^{\,\imath k\ell}\,\nabla_k\,\delta\,(\mathbf{x}-\mathbf{y})
\end{aligned}$$

which shows that the electric and magnetic components of the radiation field do not commute at space-like separations.

5.3.5 Fock Space with Indefinite Metric

From the normal mode expansion (5.59) and the orthogonality and closure relations (5.61–5.63), we readily obtain the canonical commutation relations for the creation and destruction operators, *viz.*,

$$\left[\, g_{\,\mathbf{h}\,,A}\,,\, g^{\dagger}_{\,\mathbf{k}\,,B}\,\right] = \delta\,(\,\mathbf{h}-\mathbf{k}\,)\;\eta_{\,AB} \tag{5.94}$$

all other commutators vanishing.

`Proof.` To simplify things but without loss of generality one can always select the Feynman gauge $\xi = 1$. Then we obtain

$$\begin{aligned}
& \left[\,A^{\lambda}\,(x)\,,\,A^{\nu}\,(y)\,\right] = i\,g^{\,\lambda\nu}\,D_0\,(x-y) = -\,g^{\,\lambda\nu}\sum_{\mathbf{k}}\Big(u_{\,\mathbf{k}}(x)\,u^{*}_{\,\mathbf{k}}(y) - u^{*}_{\,\mathbf{k}}(x)\,u_{\,\mathbf{k}}(y)\Big) \\
= \;& \sum_{\mathbf{k}\,,A}\sum_{\mathbf{k}'\,,A'}\Big[\,g_{\,\mathbf{k}\,,A}\,u^{\lambda}_{\,\mathbf{k}\,,A}\,(x) + g^{\dagger}_{\,\mathbf{k}\,,A}\,u^{\lambda\,*}_{\,\mathbf{k}\,,A}\,(x)\,,\, g_{\,\mathbf{k}'\,,A'}\,u^{\nu}_{\,\mathbf{k}'\,,A'}\,(y) + g^{\dagger}_{\,\mathbf{k}'\,,A'}\,u^{\nu\,*}_{\,\mathbf{k}'\,,A'}\,(y)\Big] \\
= \;& \sum_{\mathbf{k}\,,A}\sum_{\mathbf{k}'\,,A'}\varepsilon^{\,\lambda}_{\,A}(\mathbf{k})\,\varepsilon^{\,\nu}_{\,A'}(\mathbf{k}')\,\Big\{u_{\,\mathbf{k}}(x)\,u_{\,\mathbf{k}'}(y)\,[\,g_{\,\mathbf{k}\,,A}\,,\,g_{\,\mathbf{k}'\,,A'}\,] \quad - \quad \text{h.c.} \\
+ \;& u_{\,\mathbf{k}}(x)\,u^{*}_{\,\mathbf{k}'}(y)\,[\,g_{\,\mathbf{k}\,,A}\,,\,g^{\dagger}_{\,\mathbf{k}'\,,A'}\,] - u^{*}_{\,\mathbf{k}}(x)\,u_{\,\mathbf{k}'}(y)\,[\,g_{\,\mathbf{k}'\,,A'}\,,\,g^{\dagger}_{\,\mathbf{k}\,,A}\,]\,\Big\}
\end{aligned}$$

where

$$u_{\,\mathbf{k}}(x) = \exp\{i\,|\,\mathbf{k}\,|\,(\,\widehat{\mathbf{k}}\cdot\mathbf{x} - ct\,)\}/\sqrt{(2\pi)^3\,2\,|\,\mathbf{k}\,|}$$

From the above expressions it is clear that the matching occurs if and only if the following canonical commutation relations hold true: namely,

$$\left[g_{\mathbf{k}\,,A}\,,\, g^{\dagger}_{\mathbf{k}'\,,A'} \right] = \delta\,(\,\mathbf{k}-\mathbf{k}'\,)\;\eta_{AA'} \qquad\qquad [\, g_{\mathbf{k}\,,A}\,,\, g_{\mathbf{k}'\,,A'}\,] = \left[g^{\dagger}_{\mathbf{k}\,,A}\,,\, g^{\dagger}_{\mathbf{k}'\,,A'} \right] = 0$$

which endorses the claim. □

Then the gauge vector potential operator valued distribution becomes

$$\begin{aligned} A^{\lambda}(x) &= V^{\lambda}(x) + \xi\,\partial^{\,\lambda}\,\mathcal{D} * B\,(x) \\ &= \sum_{\mathbf{k}\,,A} \left[g_{\mathbf{k}\,,A}\, u^{\lambda}_{\mathbf{k}\,,A}(x) + g^{\dagger}_{\mathbf{k}\,,A}\, u^{\lambda\,*}_{\mathbf{k}\,,A}(x) \right] \\ &- (\,1-\xi\,)\,\partial^{\,\lambda}\,\mathcal{D} * B\,(x) \end{aligned} \tag{5.95}$$

with

$$\begin{aligned} B\,(x) &= \partial_{\lambda} \sum_{\mathbf{k}\,,A} \left[g_{\mathbf{k}\,,A}\, u^{\lambda}_{\mathbf{k}\,,A}(x) + g^{\dagger}_{\mathbf{k}\,,A}\, u^{\lambda\,*}_{\mathbf{k}\,,A}(x) \right] \\ &= i \sum_{\mathbf{k}} |\,\mathbf{k}\,| \Big(u^{*}_{\mathbf{k}}\,(x)\, g^{\dagger}_{\mathbf{k}\,,S} - u_{\mathbf{k}}\,(x)\, g_{\mathbf{k}\,,S} \Big) \end{aligned} \tag{5.96}$$

The photon covariant wave function, which is a gauge dependent tool, is most easily expressed in the Feynman gauge $\xi = 1$ that yields

$$\langle\, 0\,|\; A^{\lambda}\,(x)\,|\,\mathbf{k}\;A\,\rangle = \langle\, 0\,|\; A^{\lambda}\,(x)\, g^{\dagger}_{\mathbf{k}\,,A}\;|\,0\,\rangle = u^{\lambda}_{\mathbf{k}\,,A}\,(x)$$

It is worthwhile to remark that the very important canonical commutation relation

$$\left[\, F^{\,\rho\lambda}\,(x)\,,\, B\,(y)\,\right] = 0 \tag{5.97}$$

does tell us that a physical local operator such as the electromagnetic field strength tensor indeed commutes with the nonphysical auxiliary field for any space-time separations. Actually, a weaker condition will specify the concept of gauge invariance in the quantum field theory of the electromagnetism as I will show in the sequel.

Let us now set up the Hilbert space of the physical states. To this concern, I will first define the Fock space $\mathcal{F}$ in the conventional way starting from the cyclic vacuum state

$$g_{\,\mathbf{k}\,,A}\,|\,0\,\rangle = 0 = \langle\,0\,|\,g^{\dagger}_{\,\mathbf{k}\,,A} \qquad\qquad \forall\,\mathbf{k} \in \mathbb{R}^{3}\,,\; A = 1,2,L,S \tag{5.98}$$

a generic polarized $N-$photon energy-momentum eigenstate being given by

$$|\,\mathbf{k}_1\,A_1\,\mathbf{k}_2\,A_2\,\ldots\,\mathbf{k}_N\,A_N\,\rangle \stackrel{\mathrm{def}}{=} \prod_{\jmath=1}^{N}\,g^{\,\dagger}_{\,\mathbf{k}_\jmath\,,\,A_\jmath}\,|\,0\,\rangle$$

It is very important to realize that the Fock space $\mathcal{F}$ for the massless gauge vector particles is of an indefinite metric. As a matter of fact, the inner product 4×4 real symmetric matrix $\eta\equiv\|\,\eta\,\|_{AB}\ \ (\,A,B=1,2,L,S\,)$ does satisfy

$$\eta^2=\mathbb{I} \qquad\qquad \mathrm{tr}\,\eta=2$$

which means that it admits three positive eigenvalues equal to $+1$ and one negative eigenvalue equal to -1 . Hence, negative norm states do indeed exist, for example

$$\frac{1}{\sqrt{2}}\left(g^{\,\dagger}_{\,\mathbf{k}\,,\,L}+g^{\,\dagger}_{\,\mathbf{k}\,,\,S}\right)\,|\,0\,\rangle$$

as well as null norm states just like

$$g^{\,\dagger}_{\,\mathbf{k}\,,\,L}\,|\,0\,\rangle \qquad\qquad g^{\,\dagger}_{\,\mathbf{k}\,,\,S}\,|\,0\,\rangle$$

Actually we readily find

$$\tfrac{1}{2}\langle\,0\,|\ (\,g_{\,\mathbf{h}\,,\,L}+g_{\,\mathbf{h}\,,\,S})\,(\,g^{\,\dagger}_{\,\mathbf{k}\,,\,L}+g^{\,\dagger}_{\,\mathbf{k}\,,\,S})\,|\,0\,\rangle \ = \ -\,\delta\,(\mathbf{h}-\mathbf{k}) \tag{5.99}$$

Then, an arbitrary physical state $|\,\mathrm{phys}\,\rangle\in\mathcal{H}_{\,\mathrm{phys}}\subset\mathcal{F}$ will be defined by the auxiliary condition

$$B^{\,(-)}\,(x)\,|\,\mathrm{phys}\,\rangle \ = \ 0 \tag{5.100}$$

where the positive frequency part of the auxiliary $B-$field is given by the normal mode expansion (5.58)

$$iB^{\,(-)}\,(x) \ = \sum_{\mathbf{k}}\ |\,\mathbf{k}\,|\;u_{\,\mathbf{k}}\,(x)\,g_{\,\mathbf{k}\,,\,S} \tag{5.101}$$

To understand the meaning of the auxiliary condition (5.100) consider the polarized 1-photon energy-momentum eigenstates

$$|\,\mathbf{k}\,A\,\rangle=g^{\,\dagger}_{\,\mathbf{k}\,,\,A}\,|\,0\,\rangle \qquad\qquad \langle\,B\,\mathbf{h}\,|\,\mathbf{k}\,A\,\rangle=\eta_{AB}\,\delta\,(\,\mathbf{h}-\mathbf{k}\,)$$

From the canonical commutation relations (5.94) it follows that the 1-photon states with transverse polarization are physical

$$B^{\,(-)}\,(x)\,|\,\mathbf{k}\,A\,\rangle \ = \ 0 \qquad\quad \forall\,\mathbf{k}\in\mathbb{R}^3 \qquad \forall\,A=1,2$$

as well as the scalar photon 1-particle states

$$B^{(-)}(x)\,|\,\mathbf{k}\,S\,\rangle \;=\; 0 \qquad\qquad \forall\,\mathbf{k}\in\mathbb{R}^3$$

Notice, however, that for any given wave packet $\varphi\,(\mathbf{k})$ normalized to one, *i.e.*

$$\int \mathrm{d}\mathbf{k}\ \ |\varphi\,(\mathbf{k})\,|^{\,2} = 1$$

we find for $A, B = 1, 2$

$$\begin{aligned}
\langle\,\varphi_{\,B}\,|\,\varphi_{\,A}\,\rangle \;&=\; \int \mathrm{d}\mathbf{k}\int d\mathbf{h}\;\varphi\,(\mathbf{h})\,\varphi^{*}\,(\mathbf{k})\;\langle\,B\,\mathbf{k}\,|\,\mathbf{h}\,A\,\rangle = \delta_{\,AB}\\
\langle\,\varphi_{\,S}\,|\,\varphi_{\,S}\,\rangle \;&=\; \int \mathrm{d}\mathbf{k}\int d\mathbf{h}\;\varphi\,(\mathbf{h})\,\varphi^{*}\,(\mathbf{k})\;\langle\,S\,\mathbf{k}\,|\,\mathbf{h}\,S\,\rangle = 0\\
\langle\,\varphi_{\,S}\,|\,\varphi_{\,A}\,\rangle \;&=\; \int \mathrm{d}\mathbf{k}\int d\mathbf{h}\;\varphi\,(\mathbf{h})\,\varphi^{*}\,(\mathbf{k})\;\langle\,S\,\mathbf{k}\,|\,\mathbf{h}\,A\,\rangle = 0
\end{aligned}$$

From the above table of scalar products it follows that the 1-photon states with transverse polarization are physical states with positive norm and are always orthogonal to the 1-particle scalar states, which are the physical states with zero norm. Hence, the 1-particle physical Hilbert space

$$\mathcal{H}_{\,1\,,\,\mathrm{phys}} \;\stackrel{\mathrm{def}}{=}\; \overline{V}_1 \qquad\qquad V_1 \;\equiv\; \left\{\,|\,\mathbf{k}\,A\,\rangle\,|\,\mathbf{k}\in\mathbb{R}^3\;A = 1, 2, S\,\right\}$$

is a Hilbert space with a positive semi-definite metric. It is clear that the very same construction can be generalized in a straightforward way to define the N-particle completely symmetric physical Hilbert space – the closure of the symmetric product of 1-particle physical Hilbert spaces

$$\mathcal{H}_{\,N\,,\,\mathrm{phys}} \equiv \overline{\mathrm{V}}_N \qquad \mathrm{V}_N = \{\underbrace{\mathcal{H}_{\,1\,,\,\mathrm{phys}} \overset{\mathrm{s}}{\otimes} \mathcal{H}_{\,1\,,\,\mathrm{phys}} \overset{\mathrm{s}}{\otimes} \ldots \overset{\mathrm{s}}{\otimes} \mathcal{H}_{\,1\,,\,\mathrm{phys}}}_{\mathrm{N\ \ times}}\} = \mathcal{H}_{\,1\,,\,\mathrm{phys}}^{\overset{\mathrm{s}}{\otimes}\,n}$$

so that

$$\mathcal{H}_{\,\mathrm{phys}} \;\equiv\; \mathbf{C}\oplus\mathcal{H}_{\,1\,,\,\mathrm{phys}}\oplus\mathcal{H}_{\,2\,,\,\mathrm{phys}}\oplus\;\ldots\;\oplus\mathcal{H}_{\,N\,,\,\mathrm{phys}}\;\oplus\;\ldots\;=\;\bigoplus_{n=1}^{\infty}\mathcal{H}_{\,1\,,\,\mathrm{phys}}^{\overset{\mathrm{s}}{\otimes}\,n}$$

By their very construction, we see that the covariant physical photon states are equivalence classes of positive norm photon states with only transverse polarization, up to the addition of any number of zero norm scalar photons.

This fact represents the quantum mechanical counterpart of the classical gauge transformations of the second kind. As a matter of fact, in classical electrodynamics the invariant Lorentz gauge condition $\partial \cdot A\,(x) = 0$ does not fix uniquely the gauge potential, for a gauge transformation $A'_\mu\,(x) = A_\mu\,(x) + \partial_\mu\, f(x)$ with $f(x)$ satisfying the d'Alembert wave equation, is still compatible with the Lorentz condition. Hence, an equivalence class of gauge potentials obeying the invariant Lorentz condition indeed exists, what is known as classical gauge invariance of the second kind. Notice that such an invariance is no longer there for Lorentz non-covariant gauge conditions, like *e.g.* the Coulomb gauge $\boldsymbol{\nabla} \cdot \mathbf{A}(t, \mathbf{x}) = 0\,.$

`Remark`. As a matter of fact, it turns out that the transverse vector potential of the radiation field $A^\nu = (\varphi, \mathbf{A})$ with $\varphi = 0\,,\ \boldsymbol{\nabla} \cdot \mathbf{A} = 0$ is gauge invariant: under a gauge transformation

$$\mathbf{A}(t, \mathbf{x}) \quad \longmapsto \quad \mathbf{A}'(t, \mathbf{x}) = \mathbf{A}(t, \mathbf{x}) + \boldsymbol{\nabla} f(\mathbf{x})$$

by holding true the transverse condition upon the new vector potential entails

$$\boldsymbol{\nabla} \cdot \mathbf{A}'(t, \mathbf{x}) = \boldsymbol{\nabla}^{\,2} f(\mathbf{x}) = 0 \quad \Longrightarrow \quad f(\mathbf{x}) \equiv 0$$

under the usual boundary condition $\lim_{\,|\mathbf{x}|\,\to\,\infty} f(\mathbf{x}) = 0\,.$

Hence, the Coulomb gauge condition for the radiation field does indeed allow to select the physical, transverse and gauge invariant degrees of freedom, *i.e.* the two transverse components of the vector potential and the two transverse components of the electric field satisfying the Gauß law. Owing to this feature, the Coulomb gauge is said to be a manifestly unitary or physical gauge for the radiation field; of course, this can be achieved at the price of giving up the manifest Lorentz covariance.

As a final comment, I'd like to stress that the notion of gauge invariant local observable in the covariant quantum theory of the free radiation field is as follows: a gauge invariant local observable $\mathfrak{O}\,(x)$ is a self-adjoint operator that maps the physical Hilbert space onto itself, *i.e.*

$$\mathfrak{O}\,(x)\,|\,\mathrm{phys}\,\rangle \in \mathcal{H}_{\,\mathrm{phys}} \qquad \forall\,|\,\mathrm{phys}\,\rangle \in \mathcal{H}_{\,\mathrm{phys}} \qquad \mathfrak{O}\,(x) = \mathfrak{O}^{\dagger}\,(x) \tag{5.102}$$

which implies

$$|\,\mathrm{phys}\,\rangle \in \mathcal{H}_{\,\mathrm{phys}} \quad \Leftrightarrow \quad B^{\,(-)}\,(x)\,|\,\mathrm{phys}\,\rangle = 0$$

$$\begin{aligned} B^{\,(-)}\,(x)\,\mathfrak{O}\,(y)\,|\,\mathrm{phys}\,\rangle &= \left[\,B^{\,(-)}\,(x)\,,\,\mathfrak{O}\,(y)\,\right]\,|\,\mathrm{phys}\,\rangle \\ &\propto\ B^{\,(-)}\,(x)\,|\,\mathrm{phys}\,\rangle = 0 \end{aligned}$$

It follows therefrom that the Maxwell field equation as well as the usual form of the energy-momentum tensor for the radiation field hold true solely in a weak sense, *i.e.* as matrix elements between physical states: namely

$$
\begin{aligned}
&\langle\,\mathrm{phys}'\,|\,\partial_\mu F^{\mu\nu}(x) + \partial^\nu B(x)\,|\,\mathrm{phys}\,\rangle \;= \\
&\langle\,\mathrm{phys}'\,|\,\partial_\mu F^{\mu\nu}(x)\,|\,\mathrm{phys}\,\rangle = 0 \\
&\langle\,\mathrm{phys}'\,|\,\Theta_{\mu\nu}(x)\,|\,\mathrm{phys}\,\rangle \;= \\
&\langle\,\mathrm{phys}'\,|\,\tfrac14 g_{\mu\nu}\,F^{\rho\sigma}(x)\,F_{\rho\sigma}(x) - g^{\rho\sigma}\,F_{\mu\rho}(x)\,F_{\nu\sigma}(x)\,|\,\mathrm{phys}\,\rangle \\
&\forall\;|\,\mathrm{phys}\,\rangle\,,\,|\,\mathrm{phys}'\rangle\,\in\,\mathcal{H}_{\mathrm{phys}}
\end{aligned}
$$

In respect to the above definition, the canonical energy-momentum tensor of the mass-less vector gauge field theory is neither symmetric nor observable, as it appears to be evident from its expression

$$
\begin{aligned}
T_{\mu\nu} \;&=\; :\,A_\mu\,\partial_\nu B - \;F_{\mu\lambda}\,\partial_\nu A^\lambda \\
&+\; g_{\mu\nu}\left(\tfrac14 F^{\rho\sigma}F_{\rho\sigma} - A^\lambda\,\partial_\lambda B - \tfrac12\,\xi\,B^2\right)\,:
\end{aligned}
\tag{5.103}
$$

because, from the canonical commutation relations (5.88), we immediately get

$$
\begin{aligned}
[\,B(x)\,,\,T_{\mu\nu}(y)\,] \;&=\; i\,\partial_{\mu\,,\,x}\,D_0(x-y)\,\partial_\nu\,B(y) \\
&-\; i\,g_{\mu\nu}\,\partial_x^\lambda\,D_0(x-y)\,\partial_\lambda\,B(y) \\
&+\; i\,F_{\mu\lambda}(y)\,\partial_{\nu\,,\,x}\,\partial_x^\lambda\,D_0(x-y)
\end{aligned}
$$

which does not fulfill the criterion (5.102) owing to the presence of the very last term. Conversely, the symmetric energy-momentum local operator

$$
\begin{aligned}
\Theta_{\mu\nu} \;&\stackrel{\mathrm{def}}{=}\; :A_\mu\,\partial_\nu B + A_\nu\,\partial_\mu B - g^{\lambda\rho}\,F_{\mu\lambda}\,F_{\nu\rho} - g_{\mu\nu}\,\mathcal{L}_{A\,,\,B}\,: \\
&=\; :A_\mu\,\partial_\nu B + A_\nu\,\partial_\mu B - g_{\mu\nu}\left(A^\lambda\,\partial_\lambda B - \tfrac12\,\xi\,B^2\right): \\
&-\; :F_\mu{}^\rho\,F_{\nu\rho} + \tfrac14\,g_{\mu\nu}F^{\rho\sigma}\,F_{\rho\sigma}\,:
\end{aligned}
\tag{5.104}
$$

as well as, *a fortiori*, the energy-momentum four vector do indeed satisfy the requirements (5.102), *i.e.* they are observable in the quantum mechanical sense, since we have

$$
\begin{aligned}
[\,B(x)\,,\,\Theta_{\mu\nu}(y)\,] \;&=\; i\,\partial_{\mu\,,\,x}\,D_0(x-y)\,\partial_\nu\,B(y) \\
&+\; i\,\partial_{\nu\,,\,x}\,D_0(x-y)\,\partial_\mu\,B(y) \\
&-\; i\,g_{\mu\nu}\,\partial_x^\lambda\,D_0(x-y)\,\partial_\lambda\,B(y)
\end{aligned}
$$

$$
\Theta_{\mu\nu}(y) = \Theta_{\nu\mu}(y) \qquad\qquad \Theta_{\mu\nu}(y) = \Theta^\dagger_{\mu\nu}(y)
$$

For the very same reason, from the obvious canonical commutation relations

$$
\begin{aligned}
\left[\,B(x)\,,\,y^\lambda\,\Theta_{\mu\nu}(y)\,\right] \;&=\; i\,y^\lambda\,\partial_{\mu\,,\,x}\,D_0(x-y)\,\partial_\nu\,B(y) \\
&+\; i\,y^\lambda\,\partial_{\nu\,,\,x}\,D_0(x-y)\,\partial_\mu\,B(y) \\
&-\; i\,y^\lambda\,g_{\mu\nu}\,\partial_x^\lambda\,D_0(x-y)\,\partial_\lambda\,B(y)
\end{aligned}
$$

we immediately conclude that the six components of the angular momentum operator, which is of a purely orbital form, do correspond to truly observable quantities, as expected. In particular we get the three spatial components

$$M^{\imath\jmath} = \int \mathrm{d}\mathbf{x}\ : x^{\imath}\,\Theta^{0\jmath}(t,\mathbf{x}) - x^{\jmath}\,\Theta^{0\imath}(t,\mathbf{x}) : \qquad (5.105)$$

which are related to an orbital angular momentum from the generalized Poynting vector operator, while the spatial-temporal components

$$M^{0\ell} = x^0\,P^\ell - \int x^\ell\ : \Theta_{00}(t,\mathbf{x}) : \mathrm{d}^3x$$

lead again to the definition of the center of the energy operator, *viz.*,

$$\Upsilon_t \equiv \int \frac{\mathbf{x}}{\hbar c} : \Theta_{00}(t,\mathbf{x}) : \mathrm{d}^3x$$

which is dimensionless and satisfies the suggestive particle velocity relation

$$\dot{M}^{0\ell} = 0 \quad \Leftrightarrow \quad \dot{\Upsilon}_t = \frac{c\,\mathbf{P}}{\hbar}$$

5.3.6 Photon Helicity

We have already met the concept of helicity of the massive Dirac field in the previous chapter of the present notes. In particular, its very definition stems on the propagation of the spinor field along a specific spatial direction: in such a circumstance, one can actually define the spin projection along the direction of motion and the related left- and right-handed polarization states for particles and antiparticles.

In the case of massless vector particles like photons, the definition and physical meaning of the helicity concept become even more stringent and mandatory because:

1. it is not possible to set up any inertial reference frame in which any photon is at rest and the projection of its spin vector along any axis can not be safely established;

2. from Wigner's theorem about the irreducible representations of the non-homogeneous Lorentz group – see Chapter 1 of the present notes – we know that for a mass-less and spin-one vector field, the 1-particle states are just labeled by the spatial momentum $\mathbf{k} \in \mathbb{R}$ and the helicity eigenvalues, which are equal to ± 1.

As we shall see here below, it turns out that the notion of projection of the spin vector along some fixed direction of the propagation can be suitably set even for mass-less photons. On the one hand, this can be achieved in the manifestly unitary and physical Coulomb gauge, in which only the physical, transverse and gauge invariant degrees of freedom of the radiation field are involved in the dynamics. On the other hand, this fundamental result can be reached at the expense of the manifest Lorentz and gauge invariance. Here is a first enlightening example of the impossibility of keeping simultaneously into the game gauge invariance, unitarity and Lorentz covariance for the quantum theory of the radiation field.

Consider the creation and destruction operators $g^{\dagger}_{\mathbf{k}\,,A}\,,\,g_{\mathbf{k}\,,A}$ $(A = 1,2)$ which correspond to physical, transverse, linearly polarized photons, and set

$$g^{\dagger}_{\mathbf{k}\,,1} \;\equiv\; \frac{1}{\sqrt{2}}\left[-\,g^{\dagger}_{\mathbf{k}\,,+} +\; g^{\dagger}_{\mathbf{k}\,,-}\right] \tag{5.106}$$

$$g^{\dagger}_{\mathbf{k}\,,2} \;\equiv\; \frac{i}{\sqrt{2}}\left[g^{\dagger}_{\mathbf{k}\,,+} +\; g^{\dagger}_{\mathbf{k}\,,-}\right] \tag{5.107}$$

$$g^{\dagger}_{\mathbf{k}\,,\pm} \;=\; \mp\,\frac{1}{\sqrt{2}}\left[g^{\dagger}_{\mathbf{k}\,,1} \;\pm\; i\,g^{\dagger}_{\mathbf{k}\,,2}\right] \tag{5.108}$$

$$\mathrm{h} \;\equiv\; \hbar\sum_{\mathbf{k}}\left[g^{\dagger}_{\mathbf{k}\,,+}\,g_{\mathbf{k}\,,+} \;-\; g^{\dagger}_{\mathbf{k}\,,-}\,g_{\mathbf{k}\,,-}\right] \tag{5.109}$$

$$\;=\; i\hbar\sum_{\mathbf{k}}\left[g^{\dagger}_{\mathbf{k}\,,2}\,g_{\mathbf{k}\,,1} \;-\; g^{\dagger}_{\mathbf{k}\,,1}\,g_{\mathbf{k}\,,2}\right] \tag{5.110}$$

the latter being named the helicity operator. From the canonical commutation relations (5.94) we immediately get

$$[\,g_{\mathbf{k}\,,+}\,,\,g^{\dagger}_{\mathbf{k}'\,,+}\,] = [\,g_{\mathbf{k}\,,-}\,,\,g^{\dagger}_{\mathbf{k}'\,,-}\,] = \delta(\mathbf{k}-\mathbf{k}') \tag{5.111}$$

all the other commutators being null. It follows that the 1-photon states

$$g^{\dagger}_{\mathbf{k}'\,,\pm}\,|\,0\,\rangle \;\equiv\; |\,\mathbf{k}'\,\pm\,\rangle \tag{5.112}$$

are common eigenstates of the energy, momentum and helicity operators with eigenvalues $\hbar ck, \hbar\mathbf{k}, \pm\hbar$ respectively, with $k = |\mathbf{k}|$. As a matter of fact we have

$$W \;=\; \tfrac{1}{2}\int \mathrm{d}\mathbf{x} : \mathbf{E}^2(t,\mathbf{x}) + \mathbf{B}^2(t,\mathbf{x}) :$$

$$\;=\; \sum_{\mathbf{k}}\hbar ck\left[g^{\dagger}_{\mathbf{k}\,,+}\,g_{\mathbf{k}\,,+} \;+\; g^{\dagger}_{\mathbf{k}\,,-}\,g_{\mathbf{k}\,,-}\right] \tag{5.113}$$

$$= \sum_{\mathbf{k}} \hbar ck \left[g^{\dagger}_{\mathbf{k},1}\, g_{\mathbf{k},1} \;+\; g^{\dagger}_{\mathbf{k},2}\, g_{\mathbf{k},2} \right] \tag{5.114}$$

$$\mathbf{P} = \frac{1}{c} \int \mathrm{d}\mathbf{x} : \mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}) : \tag{5.115}$$

$$= \sum_{\mathbf{k}} \hbar \mathbf{k} \left[g^{\dagger}_{\mathbf{k},+}\, g_{\mathbf{k},+} \;+\; g^{\dagger}_{\mathbf{k},-}\, g_{\mathbf{k},-} \right] \tag{5.116}$$

$$= \sum_{\mathbf{k}} \hbar \mathbf{k} \left[g^{\dagger}_{\mathbf{k},1}\, g_{\mathbf{k},1} \;+\; g^{\dagger}_{\mathbf{k},2}\, g_{\mathbf{k},2} \right] \tag{5.117}$$

and consequently

$$\left.\begin{array}{c} W\,|\,\mathbf{k}'\,\pm\,\rangle = \hbar ck'\,|\,\mathbf{k}'\,\pm\,\rangle \\ \mathbf{P}\,|\,\mathbf{k}'\,\pm\,\rangle = \hbar\mathbf{k}'\,|\,\mathbf{k}'\,\pm\,\rangle \\ \mathrm{h}\,|\,\mathbf{k}'\,\pm\,\rangle = \pm\,\hbar\,|\,\mathbf{k}'\,\pm\,\rangle \end{array}\right\} \tag{5.118}$$

Moreover we can introduce the complex polarization vectors according to

$$g_{\mathbf{k},1}\,\boldsymbol{\varepsilon}_1(\mathbf{k}) + g_{\mathbf{k},2}\,\boldsymbol{\varepsilon}_2(\mathbf{k}) = g_{\mathbf{k},+}\,\boldsymbol{\varepsilon}_+(\mathbf{k}) + g_{\mathbf{k},-}\,\boldsymbol{\varepsilon}_-(\mathbf{k}) \tag{5.119}$$

$$\boldsymbol{\varepsilon}_{\pm}(\mathbf{k}) \;\equiv\; \mp\frac{1}{\sqrt{2}}\,[\,\boldsymbol{\varepsilon}_1(\mathbf{k}) \pm i\boldsymbol{\varepsilon}_2(\mathbf{k})\,] \qquad \boldsymbol{\varepsilon}^{*}_{\pm}(\mathbf{k}) = -\,\boldsymbol{\varepsilon}_{\mp}(\mathbf{k}) \tag{5.120}$$

Notice that we get

$$\boldsymbol{\varepsilon}_1(\mathbf{k}) \times \boldsymbol{\varepsilon}_2(\mathbf{k}) = \widehat{\mathbf{k}} \qquad \widehat{\mathbf{k}} \times \boldsymbol{\varepsilon}_1(\mathbf{k}) = \boldsymbol{\varepsilon}_2(\mathbf{k}) \qquad \boldsymbol{\varepsilon}_2(\mathbf{k}) \times \widehat{\mathbf{k}} = \boldsymbol{\varepsilon}_1(\mathbf{k}) \tag{5.121}$$

$$\boldsymbol{\varepsilon}_1(-\mathbf{k}) = \boldsymbol{\varepsilon}_2(\mathbf{k}) \qquad \boldsymbol{\varepsilon}_2(-\mathbf{k}) = \boldsymbol{\varepsilon}_1(\mathbf{k}) \tag{5.122}$$

$$\boldsymbol{\varepsilon}_-(\mathbf{k}) \times \boldsymbol{\varepsilon}_+(\mathbf{k}) = i\widehat{\mathbf{k}} \qquad \widehat{\mathbf{k}} \times \boldsymbol{\varepsilon}_{\pm}(\mathbf{k}) = \mp\, i\boldsymbol{\varepsilon}_{\pm}(\mathbf{k}) \tag{5.123}$$

$$\boldsymbol{\varepsilon}_{\pm}(-\mathbf{k}) = \pm\, i\boldsymbol{\varepsilon}_{\mp}(\mathbf{k}) \tag{5.124}$$

It turns out that we can expand the physical transverse vector potential and the electric field in the two equivalent forms

$$\mathbf{A}(t,\mathbf{x}) = \sum_{\mathbf{k}} \Big(g_{\mathbf{k},1}\,\boldsymbol{\varepsilon}_1(\mathbf{k}) + g_{\mathbf{k},2}\,\boldsymbol{\varepsilon}_2(\mathbf{k}) \Big) u_{\mathbf{k}}(t,\mathbf{x}) + \mathrm{h.c.} \tag{5.125}$$

$$\mathbf{A}(t,\mathbf{x}) = \sum_{\mathbf{k}} \Big(g_{\mathbf{k},+}\,\boldsymbol{\varepsilon}_+(\mathbf{k}) + g_{\mathbf{k},-}\,\boldsymbol{\varepsilon}_-(\mathbf{k}) \Big) u_{\mathbf{k}}(t,\mathbf{x}) + \mathrm{h.c.} \tag{5.126}$$

$$u_{\mathbf{k}}(t,\mathbf{x}) = \sqrt{\frac{\hbar c}{(2\pi)^3\,2k}}\;\exp\{\,i(\mathbf{k}\cdot\mathbf{x} - ckt)\} \tag{5.127}$$

$$\mathbf{E}(t,\mathbf{x}) = -\frac{1}{c}\,\dot{\mathbf{A}}(t,\mathbf{x}) \;=\; -\frac{1}{c}\,\boldsymbol{\Pi}(t,\mathbf{x}) \tag{5.128}$$

$$= \sum_{\mathbf{k}} \Big(g_{\mathbf{k},1}\,\boldsymbol{\varepsilon}_1(\mathbf{k}) + g_{\mathbf{k},2}\,\boldsymbol{\varepsilon}_2(\mathbf{k}) \Big) ik\,u_{\mathbf{k}}(t,\mathbf{x}) + \mathrm{h.c.} \tag{5.129}$$

$$= \sum_{\mathbf{k}} \Big(g_{\mathbf{k},+}\,\boldsymbol{\varepsilon}_+(\mathbf{k}) + g_{\mathbf{k},-}\,\boldsymbol{\varepsilon}_-(\mathbf{k}) \Big) ik\,u_{\mathbf{k}}(t,\mathbf{x}) + \mathrm{h.c.} \tag{5.130}$$

Notice that the transverse vector potential and the electric field of radiation do satisfy the transverse conditions

$$\nabla\cdot\mathbf{A}(t,\mathbf{x})=\nabla\cdot\mathbf{E}(t,\mathbf{x})=0 \tag{5.131}$$

so that they are physical, gauge invariant and consistently fulfill the canonical commutation relations

$$[\,E^k(t,\mathbf{x})\,,\,A^\ell(t,\mathbf{y})\,]=i\hbar c\,\left(\delta_{k\ell}-\partial_k\,\partial_\ell\,\nabla^{-2}\right)\delta(\mathbf{x}-\mathbf{y}) \tag{5.132}$$

all the other commutators being null.

From the classical Lagrangian $\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ and the Nöther theorem, by taking the normal modes expansions of the vector potential and the electric field operators suitably into account,

$$\mathbf{A}^{(-)}(t,\mathbf{x})=\sum_{\mathbf{k}}\sqrt{\frac{\hbar c}{(2\pi)^3\,2k}}\;\sum_{A=1,2}g_{\mathbf{k}\,,A}\,\boldsymbol{\varepsilon}_A(\mathbf{k})\;\exp\{\,i(\mathbf{k}\cdot\mathbf{x}-ckt)\}$$

$$\mathbf{A}^{(+)}(t,\mathbf{x})=\sum_{\mathbf{k}}\sqrt{\frac{\hbar c}{(2\pi)^3\,2k}}\;\sum_{A=1,2}g^\dagger_{\mathbf{k}\,,A}\,\boldsymbol{\varepsilon}_A(\mathbf{k})\;\exp\{-\,i(\mathbf{k}\cdot\mathbf{x}-ckt)\}$$

$$\mathbf{E}^{(-)}(t,\mathbf{x})=i\sum_{\mathbf{k}'}\sqrt{\frac{\hbar ck'}{2(2\pi)^3}}\;\sum_{A'=1,2}g_{\mathbf{k}'\,,A'}\,\boldsymbol{\varepsilon}_{A'}(\mathbf{k}')\;\exp\{\,i(\mathbf{k}'\cdot\mathbf{x}-ck't)\}$$

$$\mathbf{E}^{(+)}(t,\mathbf{x})=\frac{1}{i}\sum_{\mathbf{k}'}\sqrt{\frac{\hbar ck'}{2(2\pi)^3}}\;\sum_{A'=1,2}g^\dagger_{\mathbf{k}'\,,A'}\,\boldsymbol{\varepsilon}_{A'}(\mathbf{k}')\exp\{-\,i(\mathbf{k}'\cdot\mathbf{x}-ck't)\}$$

one can define the spin angular momentum tensor of the radiation field. As a matter of fact we find

$$\begin{aligned} c\,\boldsymbol{\Sigma}(t,\mathbf{x}) &= \mathbf{E}^{(-)}(t,\mathbf{x})\times\mathbf{A}^{(-)}(t,\mathbf{x})+\mathbf{E}^{(+)}(t,\mathbf{x})\times\mathbf{A}^{(+)}(t,\mathbf{x})\\ &+ \mathbf{E}^{(+)}(t,\mathbf{x})\times\mathbf{A}^{(-)}(t,\mathbf{x})-\mathbf{A}^{(+)}(t,\mathbf{x})\times\mathbf{E}^{(-)}(t,\mathbf{x}) \end{aligned}$$

the very last minus sign being due to the exchange in the vector product. An explicit calculation yields

$$\begin{aligned} &\frac{1}{c}\int\mathrm{d}\mathbf{x}\;\left[\,\mathbf{E}^{(+)}(t,\mathbf{x})\times\mathbf{A}^{(-)}(t,\mathbf{x})-\mathbf{A}^{(+)}(t,\mathbf{x})\times\mathbf{E}^{(-)}(t,\mathbf{x})\,\right]\\ =\;& i\hbar\sum_{\mathbf{k}}\left(g^\dagger_{\mathbf{k}\,,2}\,g_{\mathbf{k}\,,1}-g^\dagger_{\mathbf{k}\,,1}\,g_{\mathbf{k}\,,2}\right)\widehat{\mathbf{k}}\\ =\;& \hbar\sum_{\mathbf{k}}\left(g^\dagger_{\mathbf{k}\,,+}\,g_{\mathbf{k}\,,+}-g^\dagger_{\mathbf{k}\,,-}\,g_{\mathbf{k}\,,-}\right)\widehat{\mathbf{k}} \end{aligned}$$

$$
\begin{aligned}
&\frac{1}{c}\int \mathrm{d}\mathbf{x}\ \left[\,\mathbf{E}^{(+)}(t,\mathbf{x})\times\mathbf{A}^{(+)}(t,\mathbf{x})+\mathbf{E}^{(-)}(t,\mathbf{x})\times\mathbf{A}^{(-)}(t,\mathbf{x})\,\right]\\
=\ &\tfrac{1}{2}\,i\hbar\sum_{\mathbf{k}}\Big(g_{-\mathbf{k},1}\,\boldsymbol{\varepsilon}_1(-\mathbf{k})+g_{-\mathbf{k},2}\,\boldsymbol{\varepsilon}_2(-\mathbf{k})\Big)\\
\times\ &\Big(g_{\mathbf{k},1}\,\boldsymbol{\varepsilon}_1(\mathbf{k})+g_{\mathbf{k},2}\,\boldsymbol{\varepsilon}_2(\mathbf{k})\Big)\,\mathrm{e}^{-\,2ikct}\ +\ \text{h.c.}\\
=\ &\tfrac{1}{2}\,i\hbar\sum_{\mathbf{k}}\Big(g_{-\mathbf{k},1}\,\boldsymbol{\varepsilon}_2(\mathbf{k})+g_{-\mathbf{k},2}\,\boldsymbol{\varepsilon}_1(\mathbf{k})\Big)\\
\times\ &\Big(g_{\mathbf{k},1}\,\boldsymbol{\varepsilon}_1(\mathbf{k})+g_{\mathbf{k},2}\,\boldsymbol{\varepsilon}_2(\mathbf{k})\Big)\,\mathrm{e}^{-\,2ikct}\ +\ \text{h.c.}\\
=\ &\sum_{\mathbf{k}}\tfrac{1}{2}\,i\hbar\,\widehat{\mathbf{k}}\Big(g_{-\mathbf{k},2}\,g_{\mathbf{k},2}-g_{-\mathbf{k},1}\,g_{\mathbf{k},1}\Big)\,\mathrm{e}^{-\,2ikct}\ +\ \text{h.c.}\equiv 0
\end{aligned}
$$

the very last equality being due to symmetric integration. Hence we get the constant spin vector for the radiation field

$$
\begin{aligned}
\boldsymbol{\Sigma}\ &=\ \frac{1}{c}\int \mathrm{d}\mathbf{x} : \mathbf{E}(t,\mathbf{x})\times\mathbf{A}(t,\mathbf{x}) :\\
&=\ \hbar\sum_{\mathbf{k}}\Big(g^{\dagger}_{\mathbf{k},+}\,g_{\mathbf{k},+}-g^{\dagger}_{\mathbf{k},-}\,g_{\mathbf{k},-}\Big)\,\widehat{\mathbf{k}} \qquad (5.133)
\end{aligned}
$$

Suppose now that the motion of the electromagnetic radiation occurs along some given spatial direction. One can always fixes a reference frame with axes chosen so that the photon progressive waves are propagating in the positive OZ direction, while $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are in the positive OX and OY directions, respectively. Then we have the vector potential

$$
\mathbf{A}(t,z)=\int_{-\infty}^{\infty}\mathrm{d}k\Big(g_{k,+}\,\boldsymbol{\varepsilon}_+(k)+g_{k,-}\,\boldsymbol{\varepsilon}_-(k)\Big)u_k(t,z)+\text{h.c.} \qquad (5.134)
$$

$$
u_k(t,z)=\sqrt{\frac{\hbar c}{4\pi|k|}}\ \exp\{\,i(kz-c|k|t)\} \qquad (5.135)
$$

and the electric field

$$
\begin{aligned}
\mathbf{E}(t,z)\ &=\ -\,\frac{1}{c}\,\dot{\mathbf{A}}(t,z)\ =\ -\,\frac{1}{c}\,\boldsymbol{\Pi}(t,z) \qquad (5.136)\\
=\ &\int_{-\infty}^{\infty}\mathrm{d}k\Big(g_{k,+}\,\boldsymbol{\varepsilon}_+(k)+g_{\mathbf{k},-}\,\boldsymbol{\varepsilon}_-(k)\Big)i|k|\,u_k(t,z)+\text{h.c.} \qquad (5.137)
\end{aligned}
$$

At a fixed point in space, the above expressions are such that *e.g.* the normal modes of the electric vector are constant in magnitude, but sweep around in a circle at a frequency $\omega=c|k|$. For any momentum $k_z=k\in\mathbb{R}$

and the positive sign $(\boldsymbol{\varepsilon}_1 + i\boldsymbol{\varepsilon}_2)/\sqrt{2}$, the rotation is counterclockwise when the observer is facing into the oncoming wave: this photon is called left circularly polarized in Optics, or to have positive helicity in Particle Physics jargon. The latter terminology appears to be more appropriate because such a photon has a positive projection of its spin angular momentum along the $OZ-$axis, *i.e.* along the direction and versus of its motion. Conversely, for the negative sign $(\boldsymbol{\varepsilon}_1 - i\boldsymbol{\varepsilon}_2)/\sqrt{2}$, the rotation is clockwise when the observer is facing into the oncoming wave: this photon is called right circularly polarized in Optics, or to have negative helicity in Particle Physics jargon. If we denote as customary by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the unit vectors along the OX, OY, OZ directions respectively, we readily find

$$\begin{aligned} \mathbf{k} \cdot \boldsymbol{\Sigma} &= \frac{\widehat{\mathbf{z}}}{c} \cdot \int_{-\infty}^{\infty} \mathrm{d}z : \mathbf{E}(t,z) \times \mathbf{A}(t,z) : \\ &= \hbar \int_{-\infty}^{\infty} \mathrm{d}k \left(g^{\dagger}_{k,+}\, g_{k,+} - g^{\dagger}_{k,-}\, g_{k,-} \right) = \mathrm{h} \end{aligned} \tag{5.138}$$

Consider therefore the generic state

$$|\,\Upsilon\,\rangle = \Upsilon_+\, g^{\dagger}_{k,+}\,|\,0\,\rangle + \Upsilon_-\, g^{\dagger}_{k,-}\,|\,0\,\rangle$$

where $\Upsilon_\pm$ are complex numbers, which is not an helicity eigenstate because

$$\mathrm{h}\,|\,\Upsilon\,\rangle = \hbar\,\Upsilon_+\, g^{\dagger}_{k,+}\,|\,0\,\rangle - \hbar\,\Upsilon_-\, g^{\dagger}_{k,-}\,|\,0\,\rangle$$

If Υ_+ and Υ_- have different magnitudes but equal phases, then the above state represents an elliptically polarized photon traveling along the OZ axis and with principal axes of the ellipse in the direction of $\boldsymbol{\varepsilon}_1 = \mathbf{i}$ and $\boldsymbol{\varepsilon}_2 = \mathbf{j}$. The ratio of semi-major to semi-minor axis is $(1+r)/(1-r)$ where $\Upsilon_-/\Upsilon_+ = r$. If, instead, the complex amplitudes have a phase difference between them, *i.e.* $\Upsilon_-/\Upsilon_+ = r\,\mathrm{e}^{i\theta}$, then it is easy to show that the ellipse traced out by the electric field of the photon

$$\langle\, 0\,|\,\mathbf{E}(t,z)|\,\Upsilon\,\rangle = \Big(\Upsilon_+\,\boldsymbol{\varepsilon}_+(k) + \Upsilon_-\,\boldsymbol{\varepsilon}_-(k)\Big) i|k|\, u_k(t,z) \tag{5.139}$$

has its axes rotated by an angle $\frac{1}{2}\theta$. For $r = \pm 1$ we get back to linearly polarized photons. Notice that we find the transformation law

$$\mathrm{e}^{i\mathrm{h}\,\theta/\hbar}\, g^{\dagger}_{k,\pm}\, \mathrm{e}^{-\,i\mathrm{h}\,\theta/\hbar} = \mathrm{e}^{\pm\, i\theta}\, g^{\dagger}_{k,\pm} \tag{5.140}$$

showing that the helicity operator is the generator of the rotations around the direction of the photon propagation, just like in the spinor case.

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5.4 Problems

1. **Covariance of the vector field.** Find the transformation laws of the quantum vector wave field under the Poincaré group.

   Solution. We shall treat in detail the mass-less case, the generalization to the massive case being straightforward. We have to recall some standard definitions: namely,

$$\int \mathrm{D}k \stackrel{\text{def}}{=} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}} = \frac{1}{(2\pi)^3}\int \mathrm{d}k\;\theta(k_0)\,\delta\left(k^2\right) \quad (5.141)$$

$$|\,k\,A\,\rangle \stackrel{\text{def}}{=} [\,(2\pi)^3\,2\,|\,\mathbf{k}\,|\,]^{\frac{1}{2}}\,g^{\dagger}_{\mathbf{k}\,,\,A}\,|\,0\,\rangle = g^{\dagger}_{A}\,(k)\,|\,0\,\rangle \quad (5.142)$$

   i) covariant 1-particle states for the mass-less vector field

$$\left\{\,|\,k\,A\,\rangle = [\,(2\pi)^3\,2\,|\,\mathbf{k}\,|\,]^{\frac{1}{2}}\,g^{\dagger}_{\mathbf{k}\,,\,A}\,|\,0\,\rangle\,|\;\;\mathbf{k}\in\mathbb{R}^3\,\right\} \quad (5.143)$$

   ii) orthogonality and closure relations

$$\langle\,h\,A\,|\,k\,B\,\rangle = (2\pi)^3\,2\,|\,\mathbf{k}\,|\,\delta\,(\mathbf{h}-\mathbf{k})\,\eta_{AB}$$
$$\sum_A\int \mathrm{D}k\;|\,k\,A\,\rangle\,\langle\,k\,A\,| = \mathbb{I}_{\mathcal{H}_1}$$

   iii) gauge potential covariant normal mode expansion

$$A^{\lambda}\,(x) = \sum_A\int \mathrm{D}k\;\left[\,\varepsilon^{\lambda}_{A}\,(k)\,g_{A}\,(k)\,\mathrm{e}^{-\,ikx}\;+\;\text{h. c.}\,\right]_{k_0\,=\,|\mathbf{k}|}$$

   iv) wave functions

$$u^{\lambda}_{k,A}\,(x) \equiv \langle\,0\,|\,A^{\lambda}\,(x)\,|\,k\,A\,\rangle = \varepsilon^{\lambda}_{A}\,(k)\;\exp\{i(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)\}$$
$$-\;\;g_{\lambda\nu}\int \mathrm{d}\mathbf{x}\;u^{\nu\,*}_{h,A}\,(x)\,i\,\overset{\leftrightarrow}{\partial}_0\,u^{\lambda}_{k,B}\,(x) = 2\,|\,\mathbf{k}\,|\,(2\pi)^3\,\eta_{AB}\,\delta\,(\,\mathbf{h}-\mathbf{k}\,)$$

   For each element of the non-homogeneous Lorentz group, which is uniquely specified by ten canonical coordinates $(\mathrm{a}^{\lambda},\omega^{\,\mu\nu}) = (c\tau,\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\eta})$ there will correspond a unitary operator $U(\omega,\mathrm{a})\;:\;\mathcal{F}_1\;\rightarrow\;\mathcal{F}_1$ so that

$$\begin{aligned}
|\,k'\,A\,\rangle &= U(\omega,\mathrm{a})\,|\,k\,A\,\rangle = \mathrm{e}^{ik\cdot\mathrm{a}}\;|\,\Lambda\,k\;\;A\,\rangle = \mathrm{e}^{ik\cdot\mathrm{a}}\;g^{\dagger}_{A}(\Lambda\,k)|\,0\,\rangle\\
\langle\,A\;\Lambda\,h\,|\,\Lambda\,k\;\;B\,\rangle &= \langle\,A\,h\,|\,k\,B\,\rangle = \langle\,A\,h\,|\,U^{\dagger}(\omega,\mathrm{a})\,U(\omega,\mathrm{a})\,k\,B\,\rangle\\
&= \eta_{AB}\;\delta\,(\,\mathbf{h}-\mathbf{k}\,)\;(2\pi)^3\,2\,|\,\mathbf{k}\,|
\end{aligned}$$

where we obviously understand *e.g.*

$$
\begin{aligned}
&|\,A\,\Lambda\,k\,\rangle = |\,A\,k'\,\rangle = [\,(2\pi)^3\,2\,k'_0\,]^{\frac{1}{2}}\,g^{\dagger}_{\mathbf{k}'\,,\,A}\,|\,0\,\rangle = g^{\dagger}_{A}\,(\Lambda\,k)\,|\,0\,\rangle \\
&k'_\mu = \Lambda_\mu^{\ \nu}\,k_\nu \qquad k'_0 = |\,\mathbf{k}'\,| \\
&g^{\mu\nu}\,k'_\mu\,k'_\nu = k'^2 = k^2 = 0
\end{aligned}
$$

According to the general principles of the local Quantum Field Theory, any (passive) Poincaré transformation will act upon space-time events, state vectors and vector field operator valued tempered distributions in the form

$$
\begin{aligned}
A^\mu(x'\,)\,|\,\Upsilon'\,\rangle &= A^\mu(x'\,)\,U(\omega,\mathrm{a})\,|\,\Upsilon\,\rangle = \Lambda^\mu{}_\nu\,A^\nu(x)\,|\,\Upsilon\,\rangle \\
A'^\mu(x'\,) &\equiv U^{\dagger}(\omega,\mathrm{a})\,A^\mu(x'\,)\,U(\omega,\mathrm{a}) = \Lambda^\mu{}_\nu\,A^\nu(x)
\end{aligned}
$$

Under a Poincaré transformation the vacuum is invariant

$$
U(\omega)\,|\,0\,\rangle = |\,0\rangle
$$

and the annihilation operators transform according to the law

$$
U(\omega,\mathrm{a})\,g_A(k) = \mathrm{e}^{-\,ik\cdot\mathrm{a}}\;g_A(\Lambda\,k)\,U(\omega,\mathrm{a})
$$

$$
g'_A(k'\,) = U^{-1}(\omega,\mathrm{a})\,g_A(\Lambda\,k)\,U(\omega,\mathrm{a}) = \mathrm{e}^{ik\cdot\mathrm{a}}\;g_A(k)
$$

As a consequence we can write with $k' = \Lambda\cdot k$

$$
\begin{aligned}
& A'^\lambda\,(x'\,) \stackrel{\mathrm{def}}{=} U^{\dagger}(\omega,\mathrm{a})\,A^\lambda\,(x'\,)\,U(\omega,\mathrm{a}) \\
=\;& \sum_A \int \mathrm{D}k'\;\varepsilon'^{\lambda}_A\,(k'\,)\,\Big[\,U^{\dagger}(\omega,\mathrm{a})\,g_A(k'\,)\,U(\omega,\mathrm{a})\,\mathrm{e}^{-\,i\,k'\,x'} \\
+\;& U^{\dagger}(\omega,\mathrm{a})\,g^{\dagger}_A(k'\,)\,U(\omega,\mathrm{a})\,\mathrm{e}^{\,i\,k'\,x'}\,\Big]_{k'_0\,=\,|\mathbf{k}'|} \\
=\;& \Lambda^\lambda{}_\sigma\,\sum_A \int \mathrm{D}k\;\varepsilon^{\sigma}_A\,(k)\,g_A\,(k)\,\Big[\,\exp\{-\,i\,k\cdot x\} \\
+\;& g^{\dagger}_A\,(k)\,\exp\{\,i\,k\cdot x\}\,\Big]_{k_0\,=\,|\,\mathbf{k}\,|} = \Lambda^\lambda{}_\sigma\;A^\sigma(x)
\end{aligned}
$$

in which I have used the standard transformation rule (2.20) for the linear polarization real tetra-vectors

$$
\varepsilon'^{\lambda}_A\,(k'\,) = \Lambda^\lambda{}_\sigma\,\varepsilon^{\sigma}_A\,(k)
$$

the new equivalent basis of the polarization vectors $\{\varepsilon'^{\mu}_A(k')\}$ with $A = 1, 2, L, S$ still obeying the orthogonality and closure relations

$$
-\,g_{\,\mu\nu}\,\varepsilon'^{\mu}_A\,(k'\,)\,\varepsilon'^{\nu}_B\,(k'\,) = \eta_{AB}
$$

$$
\eta_{AB}\,\varepsilon'^{\mu}_A\,(k'\,)\,\varepsilon'^{\nu}_B\,(k'\,) = -\,g^{\,\mu\nu}
$$

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