Field Theory 2.
Intermediate Quantum Field Theory
(A Next-to-Basic Course for Primary Education)

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Contents

Tastiere

Nel campo di concentramento
si eseguivano soltanto ordini:
così hanno riferito nei processi.

Il pilota di Hiroshima, di lassù,
ha soltanto schiacciato quel bottone.

Quanti tasti hai premuto oggi
senza saperlo, nella tua tastiera?

Tienine conto, sempre

Può non essere vero
ciò in cui più fermamente credi.
Può non essere stato
ciò che pensi essere il tuo passato.
Potrà benissimo non arrivare
quell'obiettivo che hai
per tanto tempo accarezzato.

Maurizio Bacchilega

Chapter 1

Generating Functionals

In this chapter I aim to introduce the functional integral approach to the quantum field theory. Historically the so called Path Integral Approach to Quantum mechanics has been first developed by Richard Phillips Feynman, Who actually implemented a former idea by Paul Adrian Maurice Dirac. Here below, however, I will follow a further approach first developed by Kurt Symanzik and Julian Schwinger in the fifties. According to this point of view, the functional integrals arise as the formal solutions of the functional equations satisfied by the generating functional of the Green's functions.

George Green (Nottingham, 14 July 1793 – Nottingham, 31 May 1841) *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* (1828)

It turns out that for free field theories the latter objects can be precisely defined in terms of generalized Gaussian integration and by means of the $\zeta-$function regularization technique invented by Stephen Hawking in the late seventies. As a final consequence, the functional integrals for interacting quantum field theories appears to be eventually defined in terms of formal perturbation expansions, starting from the generating functional for free field theories. This leads to the Feynman rules of the perturbation expansion for quantum field theories, that we shall analyze in details in the next chapter.

1.1 The Scalar Generating Functional

The simplest construction concerns of course the derivation of the generating functional of the Green's functions for a real scalar free field satisfying the Klein-Gordon equation.

1.1.1 The Symanzik Functional Equation

Consider the vacuum expectation value

$$\begin{aligned} Z_0[\,J\,] &= \left\langle T \exp\left\{ i\int_{-\infty}^{\infty} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\, \phi(t,\mathbf{x})\, J(t,\mathbf{x}) \right\} \right\rangle_0 \\ &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathrm{d}^4 x_1\, J(x_1) \cdots \int \mathrm{d}^4 x_n\, J(x_n) \\ &\times \ \langle 0 |\, T\,\phi\,(x_1) \cdots \phi\,(x_n)\, | 0 \rangle \end{aligned} \tag{1.1}$$

the suffix zero denoting the free field theory, where $J(x)$ is the so called source density function or classical external source, with the canonical engineering dimension $[\,J\,] = \mathrm{eV}^{-\frac{1}{2}}\ \mathrm{cm}^{-\frac{7}{2}}$ in the physical C. G. S. system of units. The vacuum expectation values of the chronological ordered products of n Klein-Gordon Hermitean field operators at different space-time points are named the $n-$point Green functions of the non-interacting or free field theory. By construction, the latter ones can be expressed as functional derivatives of the generating functional: namely,

$$\begin{aligned} G_0^{(n)}(x_1, \cdots, x_n) &\equiv \langle 0 |\, T\,\phi\,(x_1) \cdots \phi\,(x_n)\, | 0 \rangle \\ &= (-\,i\,)^{\,n}\, \delta^{\,(n)}\, Z_0[\,J\,] \,/\, \delta\, J(x_1) \cdots \delta\, J(x_n)\, \big\rfloor_{J=0} \end{aligned} \tag{1.2}$$

Taking one functional derivative of the generating functional (1.1) we find

$$(-\,i\,)\, \frac{\delta}{\delta\, J(x)}\ Z_0[\,J\,] = \left\langle T\,\phi(x)\ \exp\left\{ i\int \mathrm{d}\,y\ \phi(y)\ J(y) \right\} \right\rangle_0 \tag{1.3}$$

In order to evaluate the above quantity it is convenient to introduce the operator

$$E(\,t^{\,\prime}\,,\,t\,) \equiv T\ \exp\left\{ i\int_t^{t^{\,\prime}} \mathrm{d}y_0 \int \mathrm{d}\,\mathbf{y}\ \phi(y)\, J(y) \right\} \tag{1.4}$$

so that we can write

$$(-\,i\,)\, \frac{\delta}{\delta\, J(x)}\ Z_0[\,J\,] = \langle 0 |\, E(\infty\,,\, x_0\,)\, \phi(x)\, E(x_0\,,\, -\infty\,)\, | 0 \rangle \tag{1.5}$$

Taking a derivative with respect to x_0 we find

$$\begin{aligned} &\frac{\partial}{\partial\, x_0}\, \langle 0 |\, E(\infty\,,\, x_0\,)\, \phi(x)\, E(x_0\,,\, -\,\infty\,)\, | 0 \rangle \ = \\ &\langle 0 |\, E(\infty\,,\, x_0\,)\, \Pi(x)\, E(x_0\,,\, -\,\infty\,)\, | 0 \rangle \ - \\ &i\int \mathrm{d}\mathbf{y}\ J(x_0,\mathbf{y}) \langle 0 |\, E(\infty\,,\, x_0\,)\, [\,\phi(x_0,\mathbf{y})\,,\, \phi(x_0,\mathbf{x})\,]\, E(x_0\,,\, -\,\infty\,)\, | 0 \rangle \\ &= \ \langle 0 |\, E(\infty\,,\, x_0\,)\, \Pi(x)\, E(x_0\,,\, -\,\infty\,)\, | 0 \rangle \end{aligned}$$

owing to micro-causality. One more derivative evidently yields

$$\frac{\partial}{\partial x_0}\langle 0|\,E(\infty\,,\,x_0)\,\Pi(x)\,E(x_0\,,\,-\infty)\,|0\rangle\ =$$
$$\langle 0|\,E(\infty\,,\,x_0)\,\dot\Pi(x)\,E(x_0\,,\,-\infty)\,|0\rangle\ +\ \hbar\,J(x)\,Z_0[\,J\,]$$

whence we eventually obtain the functional differential equation for the free spin-less neutral field generating functional, that is

$$\Big[\,(\,\Box_x + m^2c^2/\hbar^2\,)\,\frac{i\delta}{\delta J(x)} + \hbar c\,J(x)\Big]\,Z_0[\,J\,]\ =\ 0 \tag{1.6}$$

where we used physical units and the fact that the free scalar field operator valued distribution has canonical dimensions $[\,\phi\,] = \mathrm{eV}^{\frac12}\ \mathrm{cm}^{-\frac12}$ and satisfy the Klein-Gordon wave equation. The above functional equation has been first obtained by

> Kurt Symanzik
> *Über das Schwingersche Funktional in der Feldtheorie*
> Zeitschrift für Naturforschung **9A** (1954) 809-824

and will thereby named the Symanzik functional equation. This functional differential equation (1.6) has a unique solution that fulfills causality, *viz.*

$$Z_0[\,J\,] = \exp\left\{-\tfrac12\int \mathrm{d}^4x\int \mathrm{d}^4y\ J(x)\,D_F\,(x-y)\,J(y)\right\} \tag{1.7}$$

as it can be verified by direct inspection.

The classical Action for the real scalar field in the presence of an external source density can be rewritten as

$$\begin{aligned} S_J[\,\phi\,] &= \tfrac12\int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}\mathbf{x}\,\Big[\,\partial_\mu\phi(x)\partial^\mu\phi(x) - (mc/\hbar)^2\phi^2(x) + 2\hbar c\,\phi(x)J(x)\Big] \\ &\doteq \int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}\mathbf{x}\,\Big[\,-\,\tfrac12\,\phi(x)\,\big(\Box + m^2c^2/\hbar^2\big)\,\phi(x) + \hbar c\,\phi(x)\,J(x)\Big] \\ &\equiv S_0[\,\phi\,] + \hbar\int \mathrm{d}^4x\,\phi(x)\,J(x) \end{aligned} \tag{1.8}$$

where the symbol $\doteq$ indicates that the total four divergence term

$$\tfrac12\,g^{\mu\nu}\int \mathrm{d}^4x\ \partial_\mu\Big(\phi(x)\partial_\nu\phi(x)\Big)$$

has been dropped as it does not contribute to the equations of motion

$$\frac{\delta S_J[\,\phi\,]}{\delta\,\phi\,(x)} = 0 \iff (\,\Box + m^2c^2/\hbar^2\,)\,\phi(x) = \hbar c\,J(x) \tag{1.9}$$

1.1.2 The Functional Integrals for Bosons

The Symanzik functional equation (1.6) is a **linear functional differential equation**. Hence the most customary method to solve it is by means of some functional Fourier transform. To reconstruct the solution (1.6) *via* Fourier methods we formally write $Z_0[\,J\,]$ as a functional integral: namely,

$$\begin{aligned} Z_0[\,J\,] &= \left\langle \mathrm{T}\exp\left\{i\textstyle\int \mathrm{d}y\;\phi(y)J(y)\right\}\right\rangle_0 \\ &= \exp\left\{-\tfrac{1}{2}\textstyle\int \mathrm{d}x \int \mathrm{d}^4y\;J(x)\,D_F(x-y)\,J(y)\right\} \\ &= \int \mathfrak{D}\phi\;\widetilde{Z}_0[\,\phi\,]\;\exp\left\{i\,\langle\,\phi_y\,J_y\,\rangle\right\} \end{aligned} \tag{1.10}$$

where $\mathfrak{D}\phi$ formally denotes integration over an infinite dimensional functional space of Lorentz scalar real functions on the Minkowski space $\phi : \mathbb{R}^{1,3} \to \mathbb{R}$, while I have used the discrete index like notation

$$\langle\,\phi_y\,J_y\,\rangle \equiv \textstyle\int \mathrm{d}^4y\;\phi(y)J(y)$$

It is worthwhile to remark that, as it is customary in the Literature, the very same symbol $\phi(x)$ is employed to denote either the Klein-Gordon Hermitean quantum field, *i.e.* an operator valued tempered distribution belonging to $\mathsf{S}'(\mathbb{R}^{1,3})$, or the classical field in $\mathsf{S}(\mathbb{R}^{1,3})$, which appears to be the argument of the classical Action as well as the integration variable in the functional integral (1.10). The precise role and meaning of the symbol $\phi(x)$ will be henceforth readily extracted from the context without ambiguities.

Heuristically, a preliminary although quite suggestive way to understand the functional measure $\mathfrak{D}\phi$ is in terms of

$$\int \mathfrak{D}\phi \;:=:\; \prod_{x\in\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}\,\phi_x$$

Here :=: denotes a formal equality, the precise mathematical meaning of which has to be further specified, while the space-time events x of Minkowski space $\mathcal{M} = \mathbb{R}^{1,3}$ are treated as discrete indexes, in such a manner that the functional integral could be formally understood as an infinitely continuous generalization of a multiple Lebesgue integral. Taking the functional derivative operator $i\delta/\delta J(x)$ through the functional Fourier integral (1.10) equation (1.6) becomes

$$\begin{aligned} 0 &= \left[\,(\,\Box_x + m^2\,)\,\frac{i\,\delta}{\delta J(x)} + J(x)\right] Z_0[\,J\,] \\ &= \int \mathfrak{D}\phi\,\left[\,-\,(\,\Box_x + m^2\,)\phi(x) + J(x)\right]\widetilde{Z}_0[\,\phi\,]\;\exp\left\{i\,\langle\,\phi_x\,J_x\,\rangle\right\} \end{aligned}$$

Incidentally, this means that we require linearity for the Fourier-like functional representation: namely,

$$\int \mathfrak{D}\phi \ \left\{ f(x)\,\widetilde{F}[\phi] + g(x)\,\widetilde{G}[\phi] \right\} = f(x) \int \mathfrak{D}\phi \ \widetilde{F}[\phi] + g(x) \int \mathfrak{D}\phi \ \widetilde{G}[\phi]$$

for any pair of functional $\widetilde{F}, \widetilde{G}$ and any pair of complex functions f, g. Now, since we have the equality

$$-\,(\,\Box_x + m^2c^2/\hbar^2\,)\phi(x) \doteq \frac{\delta\, S_0}{\delta\,\phi\,(x)}$$

up to an irrelevant boundary term, we are led to eventually identify

$$\widetilde{Z}_0\,[\,\phi\,] = \mathcal{N}\ \exp\,\{(\,i/\hbar\,)\,S_0\,[\,\phi\,]\} \tag{1.11}$$

$\mathcal{N}$ being any arbitrary classical external source independent quantity.

Proof. As a matter of fact we have in natural units

$$\begin{aligned} & \int \mathfrak{D}\phi \,\Big[\,\frac{\delta\, S_0}{\delta\,\phi\,(x)} \ +\ J(x)\Big]\,\widetilde{Z}_0\,[\,\phi\,]\ \exp\,\{\,i\,\langle\,\phi_y\,J_y\,\rangle\} \\ =\ & \mathcal{N}\int \mathfrak{D}\phi \,\Big[\,\frac{\delta\, S_0}{\delta\,\phi\,(x)} \ +\ J(x)\Big]\ \exp\,\{\,i\,S_0\,[\,\phi\,] + i\,\langle\,\phi_y\,J_y\,\rangle\} \\ =\ & \mathcal{N}\int \mathfrak{D}\phi\ \frac{-\,i\,\delta}{\delta\,\phi\,(x)}\ \exp\,\{\,i\,S_0\,[\,\phi\,] + i\,\langle\,\phi_y\,J_y\,\rangle\} \end{aligned}$$

so that, if we assume the validity of the functional integration by parts, the very last expression formally yields

$$\begin{aligned} & \mathcal{N}\ \exp\big\{-\,i\,\big[\,\tfrac12\,\phi_x\,\big(\Box_x + m^2 - i\varepsilon\big)\ \phi_x - \phi_x\,J_x\,\big]\big\}\ \Big|\,^{\phi_x\,=\,+\infty}_{\phi_x\,=\,-\infty} \\ \times\ & \prod_{y\,\in\,\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}\,\phi_y\ \exp\,\big\{-\,\tfrac12\,i\,\phi_y\,\big(\Box_y + m^2 - i\varepsilon\big)\,\phi_y + i\,\phi_y\,J_y\big\}\,\Big]_{\,y\,\neq\,x} = 0 \end{aligned}$$

the convergence factor being provided by the causal $+i\varepsilon$ prescription, where we have taken into account that the boundary values for the co-domain of the scalar field functional space are just

$$-\,\infty < \phi_x < \infty \qquad\qquad \forall\,x \in \mathcal{M}$$

A comparison with equation (1.7) leads to the formal equality

$$\begin{aligned} Z_0\,[\,J\,] \quad &=\quad \exp\,\big\{-\int \mathrm{d}\,x \int \mathrm{d}\,y\ \tfrac12\,J(x)\,D_F\,(x-y)\,J(y)\big\} \\ &:=:\quad \mathcal{N} \int \mathfrak{D}\phi\ \exp\,\{\,i\,S_0\,[\,\phi\,] + i\,\langle\,\phi_x\,J_x\,\rangle\} \\ &:=:\quad \mathcal{N} \prod_{x\,\in\,\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}\,\phi_x\ \exp\,\big\{-\,\tfrac12\,i\,\phi_x\,\mathcal{K}_x\,\phi_x + i\,\phi_x\,J_x\big\} \end{aligned} \tag{1.12}$$

where as usual

$$S_0[\phi] = -\int \mathrm{d}x\ \tfrac{1}{2}\,\phi(x)\left(\Box + m^2 - i\varepsilon\right)\phi(x) \equiv \langle -\tfrac{1}{2}\,\phi_x\,\mathcal{K}_x\,\phi_x\,\rangle$$

This completes the proof of the above identity. □

The functional measure $\mathfrak{D}\phi$, which is a formal entity until now, can be implemented by the requirement of **invariance under field translations**

$$\phi(x) \mapsto \phi'(x) = \phi(x) + f(x)$$

Once it is assumed, after the change of variable

$$\begin{aligned} \phi(x) \mapsto \phi'(x) &= \phi(x) - (\Box + m^2 - i\varepsilon)^{-1} J(x) \\ &= \phi(x) - i\int \mathrm{d}y\ D_F(x-y)\,J(y) \\ &\equiv \phi_x - \langle\, iD_{xy}\,J_y\,\rangle \end{aligned} \tag{1.13}$$

we find

$$\begin{aligned} Z_0[J] \;&:=:\; \exp\left\{-\int \mathrm{d}x \int \mathrm{d}y\ \tfrac{1}{2}\,J(x)\,D_F(x-y)\,J(y)\right\} \\ &\times\ \mathcal{N}\int \mathfrak{D}\phi\ \exp\left\{-i\int \mathrm{d}x\ \tfrac{1}{2}\,\phi(x)\,(\Box + m^2 - i\varepsilon)\,\phi(x)\right\} \\ &=\ Z_0[0]\,\exp\left\{-\int \mathrm{d}x \int \mathrm{d}y\ \tfrac{1}{2}\,J(x)\,D_F(x-y)\,J(y)\right\} \end{aligned}$$

`Proof` : from the change of the integration variable

$$\phi_x \mapsto \phi'_x = \phi_x - \langle\, iD_{xy}\,J_y\,\rangle \qquad \mathrm{d}\phi_x = \mathrm{d}\phi'_x$$

equation (1.12) yields

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}\phi_x\ \exp\left\{-\tfrac{1}{2}\,i\phi_x\,\mathcal{K}_x\,\phi_x + i\phi_x\,J_x\right\} \\ =\ &\int_{-\infty}^{\infty} \mathrm{d}\phi'_x\ \exp\left\{-\tfrac{1}{2}\,i\,(\phi'_x + \langle\, iD_{xy}\,J_y\,\rangle)\,\mathcal{K}_x\,(\phi'_x + \langle\, iD_{xz}\,J_z\,\rangle)\right\} \\ \times\ &\exp\{\, i\phi'_x\,J_x - J_x\,\langle\, D_{xy}\,J_y\,\rangle\} \end{aligned}$$

Now, from the equality

$$\mathcal{K}_x\,\langle\, iD_{xz}\,J_z\,\rangle = J_x$$

and the further equality

$$\langle\, iD_{xy}\,J_y\,\rangle\,\mathcal{K}_x\,\phi'_x \doteq \phi'_x\,\mathcal{K}_x\,\langle\, iD_{xy}\,J_y\,\rangle = J_x\,\phi'_x$$

which is true by neglecting twice a boundary term, we can finally write

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}\phi_x\ \exp\left\{-\tfrac{1}{2}\,i\phi_x\,\mathcal{K}_x\,\phi_x + i\phi_x\,J_x\right\} \\ \doteq\ &\exp\left\{-\tfrac{1}{2}\,J_x\,\langle\, D_{xy}\,J_y\,\rangle\right\}\int_{-\infty}^{\infty} \mathrm{d}\phi'_x\ \exp\left\{-\tfrac{1}{2}\,i\phi'_x\,\mathcal{K}_x\,\phi'_x\right\} \qquad (\forall\, x \in \mathcal{M}) \end{aligned}$$

and thereby

$$
\begin{aligned}
Z_0[\,J\,] &:=: \mathcal{N} \prod_{x \in \mathcal{M}} \textstyle\int_{-\infty}^{\infty} \mathrm{d}\phi_x \;\; \exp\left\{-\tfrac{1}{2}\, i\phi_x \mathcal{K}_x \phi_x + i\phi_x J_x\right\} \\
&\doteq \prod_{x \in \mathcal{M}} \exp\left\{-\tfrac{1}{2}\, J_x \langle D_{xy} J_y \rangle\right\} \mathcal{N} \prod_{z \in \mathcal{M}} \textstyle\int_{-\infty}^{\infty} \mathrm{d}\phi'_z \;\; \exp\left\{-\tfrac{1}{2}\, i\phi'_z \mathcal{K}_z \phi'_z\right\} \\
&= \exp\left\{\langle -\tfrac{1}{2}\, J_x D_{xy} J_y \rangle\right\} \mathcal{N} \int \mathfrak{D}\phi' \;\; \exp\{iS_0[\,\phi'\,]\} \\
&= \exp\left\{-\textstyle\int \mathrm{d}x \int \mathrm{d}y \; \tfrac{1}{2}\, J(x)\, D_F(x-y)\, J(y)\right\} \; Z_0[\,0\,]
\end{aligned}
$$

which completes the proof. □

Hence, self-consistency actually entails the formal identification

$$
Z_0[\,0\,] :=: \mathcal{N} \int \mathfrak{D}\phi \;\; \exp\{(i/\hbar) S_0[\,\phi\,]\} = 1
$$

As a matter of fact, from the very definition (1.1) it appears quite evident that $Z_0[0]$ is nothing but the vacuum to vacuum amplitude $\langle\, 0 \,|\, 0 \,\rangle$ that we suppose to be normalized to one. Thus, at this point, the strategy should be clear: if we were able to give a precise mathematical meaning to the above formal quantity $Z_0[0]$, then we will be able in turn to set up a mathematically sound and precise definition of the functional integral (1.10).

To this aim, let us first consider the Euclidean formulation. Then we have to make the replacements

$$
iS_0[\,\phi\,] \;\longmapsto\; -S_E^{(0)}[\,\phi_E\,] \qquad x_{E\mu} = (\mathbf{x}\,,\; x_4 = ix_0) \tag{1.14}
$$

$$
\begin{aligned}
S_E^{(0)}[\,\phi_E\,] &= \int \mathrm{d}x_E \;\tfrac{1}{2}\Big(\partial_\mu \phi_E(x_E)\,\partial_\mu \phi_E(x_E) + m^2\, \phi_E^2(x_E)\Big) \\
&\doteq \int \mathrm{d}x_E \;\phi_E(x_E)\,\tfrac{1}{2}(m^2 - \partial_\mu \partial_\mu)\,\phi_E(x_E) \\
Z_E^{(0)}[\,0\,] &:=: \mathcal{N} \int \mathfrak{D}\phi_E \;\; \exp\Big\{-S_E^{(0)}[\,\phi_E\,]\Big\}
\end{aligned} \tag{1.15}
$$

The above quantity is, formally, an absolutely convergent Gaussian integral.

1.1.3 The $\zeta-$Function Regularization

According to the previously suggested heuristic interpretation, we could now understand the latter as

$$
Z_E^{(0)}[\,0\,] \;:=:\; \mathcal{N} \prod_x \int_{-\infty}^{\infty} \mathrm{d}\phi_x \;\; \exp\left\{-\tfrac{1}{2}\,\phi_x K_E \phi_x\right\} \qquad K_E \equiv m^2 - \partial_\mu \partial_\mu
$$

and if we assume that the functional integration variable can be changed through the scale transformation

$$\phi_x \;\mapsto\; \phi'_x = \mu\,\phi_x$$

where μ is an arbitrary mass scale which does not influence the relevant $J(x)$ dependence of the generating functional, then we come to the expression

$$Z_E^{(0)}[\,0\,] \;:=:\; \mathcal{N}'\prod_x \int_{-\infty}^{\infty} \mathrm{d}\phi'_x\; \exp\left\{-\tfrac{1}{2}\,\phi'_x\,\mu^{-2}\,K_E\,\phi'_x\right\} \tag{1.16}$$

in which the dimensionless, positive, second order and symmetric differential operator $\mu^{-2}\;(\;m^2 - \partial_\mu\partial_\mu\;)$ is involved, the spectrum of which is purely continuous and given by the positive eigenvalues $\mu^{-2}\;(\;m^2 + k_\mu k_\mu\;)$ with $k_\mu \in \mathbb{R}\;(\,\mu = 1,2,3,4\,)$.

Now, it is worthwhile to recall that, for any finite dimensional positive definite symmetric matrix $A = A^\top$ of rank $n \in \mathbb{N}$, a real orthogonal matrix $R \in SO(n,\mathbb{R})$ always exists and such that $R^\top A\,R = \mathrm{diag}(\lambda_1,\cdots,\lambda_n)$ where $\lambda_i > 0\;(\forall\, i = 1,2,\ldots,n\,)$ are the positive eigenvalues of the symmetric matrix. Hence we immediately obtain as a result of the Gaussian integration

$$\begin{aligned}
I &= (2\pi)^{-n/2}\int_{-\infty}^{\infty}\mathrm{d}x_1\cdots\int_{-\infty}^{\infty}\mathrm{d}x_n\;\exp\{-\tfrac{1}{2}\,x^\top A\,x\}\\
&= (2\pi)^{-n/2}\int_{-\infty}^{\infty}\mathrm{d}y_1\cdots\int_{-\infty}^{\infty}\mathrm{d}y_n\;\exp\{-\tfrac{1}{2}\,(Ry)^\top A\,Ry\}\\
&= (2\pi)^{-n/2}\int_{-\infty}^{\infty}\mathrm{d}y_1\cdots\int_{-\infty}^{\infty}\mathrm{d}y_n\;\exp\left\{-\tfrac{1}{2}\sum_{i=1}^{n}\lambda_i\,y_i^2\right\}\\
&= (2\pi)^{-n/2}\prod_{i=1}^{n}\int_{-\infty}^{\infty}\mathrm{d}y_i\;\exp\left\{-\tfrac{1}{2}\,\lambda_i\,y_i^2\right\}\\
&= (2\pi)^{-n/2}\prod_{i=1}^{n}\sqrt{\frac{2\pi}{\lambda_i}} \;=\; (\det A\,)^{-1/2}
\end{aligned} \tag{1.17}$$

In view of this simple result we shall attempt to **define** the formal quantity $Z_E^{(0)}[\,0\,]$ to be precisely given by

$$\begin{aligned}
Z_E^{(0)}[\,0\,] \;&:=:\; \mathcal{N}'\prod_x \int_{-\infty}^{\infty} \mathrm{d}\phi'_x\; \exp\left\{-\tfrac{1}{2}\,\phi'_x\,\mu^{-2}\,K_E\,\phi'_x\right\}\\
&\equiv\; \mathcal{N}'\,\det\|\,\mu^{-2}\,(m^2 - \partial_\mu\,\partial_\mu\,)\,\|^{-1/2}
\end{aligned} \tag{1.18}$$

where the determinant of a positive symmetric differential operator can be suitably defined by means of the so called $\zeta-$function regularization:

Steven W. Hawking (1977)
Zeta function regularization of path integrals in curved space time
Communication of Mathematical Physics, Vol. **55**, p. 133

The idea beyond this method is as simple as powerful and is based upon the analytic continuation tool. Consider for example a compact positive operator $A > 0$ with the spectral decomposition

$$A = \sum_{k=1}^{\infty} \lambda_k P_k \qquad \lambda_k > 0 \quad \mathrm{tr}\, P_k = d_k < \infty \quad \forall\, k \in \mathbb{N}$$

The complex powers of the positive operator $A > 0$ can be easily obtained in terms of its spectral resolution

$$A^{-s} = \sum_{k=1}^{\infty} \lambda_k^{-s} P_k = \sum_{k=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\ t^{\,s-1}\ \exp\{-t\lambda_k\}\, P_k \qquad (\,\Re\mathrm{e}\, s > 0\,)$$

where use has been made of the Mellin's integral transform. Let us now further suppose the compact operator A^{-s} to be of the trace class in a strip of the complex $s-$plane. Hence, from the spectral decomposition theorem we can write the integral kernel, or Green function,

$$\langle x \,|\, A^{-s} \,|\, y \rangle = \sum_{k=1}^{\infty} \lambda_k^{-s} \langle x \,|\, P_k \,|\, y \rangle = \sum_{k=1}^{\infty} \lambda_k^{-s}\, \psi_k(x)\, \psi_k^*(y) \tag{1.19}$$

$$A^{-s}\, \psi_k(x) = \lambda_k^{-s}\, \psi_k(x) \qquad \int \mathrm{d}x\ \psi_k(x)\, \psi_n^*(x) = \delta_{kn} \tag{1.20}$$

$$\begin{aligned} \mathrm{Tr}\, A^{-s} &= \int \mathrm{d}x\ \langle x \,|\, A^{-s} \,|\, x \rangle = \sum_{k=1}^{\infty} \lambda_k^{-s}\, d_k \\ &= \sum_{k=1}^{\infty} d_k\ \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\ t^{\,s-1}\ \exp\{-t\lambda_k\} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\ t^{\,s-1} \sum_{k=1}^{\infty} d_k\ \exp\{-t\lambda_k\} < \infty \end{aligned} \tag{1.21}$$

always in the half-plane $\Re\mathrm{e}\, s > 0$. Now we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \mathrm{Tr}\, A^{-s} = \sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s}\, \lambda_k^{-s}\, d_k = \sum_{k=1}^{\infty} (-\, \ln \lambda_k)\, \lambda_k^{-s}\, d_k \tag{1.22}$$

and thereby we obtain the zeta function regularization of the determinant of a positive and compact operator of the trace class: namely,

$$\ln\det A = \sum_{k=1}^{\infty} d_k \, \ln\lambda_k \;\stackrel{\rm def}{=}\; - \left.\frac{\rm d}{{\rm d}s}\,{\rm Tr}\,A^{-s}\right]_{s=0} \tag{1.23}$$

provided the analytic continuation is possible to include the imaginary axis $\Re{\rm e}\,s = 0$, *via* some suitable deformation of the integration path in complex plane, just like in the original case of the Riemann's Zeta function [1]

$$\zeta(s,q) = -\,\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{(0+)} {\rm d}\theta\;\frac{(-\theta)^{\,s-1}\,e^{\,-q\theta}}{1-e^{\,-\theta}} \tag{1.24}$$

in which it is assumed that the path of integration does not pass through the points $2n\pi i$, where n is a natural number, and of course $s \neq 1,2,3,\ldots$.

In order to apply the above treatment to the case of interest, one is faced with the problem that the Euclidean Klein-Gordon operator is neither compact nor of the trace class. To overcome this difficulty, it is expedient to introduce a very large box, *e.g.* a symmetric hyper-cube of side $2L$, to impose periodic boundary conditions on its faces and to make eventually the transition to the infinite volume continuum limit. In the presence of a symmetric hyper-cube with periodic boundary conditions, the spectrum of the Euclidean Klein-Gordon operator is purely discrete and non-degenerate

$$\lambda_n = m^2 + \frac{\pi^{\,2}}{L^2}\,n_\mu n_\mu \qquad n_\mu\in\mathbb{Z} \quad \mu = 1,2,3,4 \tag{1.25}$$

and in the limit of $L\to\infty$ we can safely replace

$$\sum_{n_\mu=-\infty}^{\infty} \quad\longmapsto\quad 2L\int_{-\infty}^{\infty}\frac{{\rm d}\,k_\mu}{2\pi} \qquad (\,\mu = 1,2,3,4\,)$$

$$\sum_n \quad\longmapsto\quad V\int\frac{{\rm d}^4\,k}{(2\pi)^4}$$

Then we find for $A = \mu^{-2}\,(\,m^2 - \partial_\mu\partial_\mu\,)$

$$\begin{aligned} {\rm Tr}\,A^{-s} &\doteq V\,\mu^{2s}\int\frac{{\rm d}^4\,k}{(2\pi)^4}\,(\,m^2+k^{\,2}\,)^{\,-s} \\ &= \frac{V\,\mu^{2s}}{16\pi^{\,2}\,\Gamma(s)}\int_0^\infty {\rm d}t\;t^{\,s-3}\,\exp\{-tm^2\} \end{aligned}$$

[1]Gradshteyn and Ryzhik [14] § **9.5** pp.1100-1103 ; Bruno Pini (1979) *Lezioni sulle distribuzioni, 1. Distribuzioni temperate*, § **4** Appendice, cap. **3.** p. 276.

$$
\begin{aligned}
&= \frac{Vm^4}{16\pi^2}\left(\frac{\mu}{m}\right)^{2s}\frac{\Gamma(s-2)}{\Gamma(s)} \\
&= \frac{Vm^4}{16\pi^2}\left(\frac{\mu}{m}\right)^{2s}(s^2-3s+2)^{-1} \qquad (1.26)
\end{aligned}
$$

where $\doteq$ means that the transition to the continuum limit is understood. Hence

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Tr}\,A^{-s} &= (s^2-3s+2)^{-2}\,\mathrm{Tr}\,A^{-s} \\
&\times \left[2(s^2-3s+2)\,\ln\frac{\mu}{m}-2s+3\right] \qquad (1.27)
\end{aligned}
$$

and thereby

$$
\det \|\left(m^2-\partial_\mu\partial_\mu\right)/\mu^2\| = \exp\left\{\frac{Vm^4}{16\pi^2}\left(\ln\frac{m}{\mu}-\frac{3}{4}\right)\right\} \qquad (1.28)
$$

Turning back to equation (1.18) we see that

$$
Z_E^{(0)}[\,0\,]=1 \quad\Longleftrightarrow\quad \mathcal{N}'\equiv\exp\left\{\frac{Vm^4}{32\pi^2}\left(\ln\frac{m}{\mu}-\frac{3}{4}\right)\right\}
$$

and the transition to the Minkowski space can be immediately done by simply replacing the volume factor $V_E \leftrightarrow i\,V_{\mathcal{M}}$, the very meaning of which is that of an infrared regulator.

The conclusion of all the above formal reasoning is as follows: we are enabled to define the functional integral for a free scalar field theory by the equality

$$
\begin{aligned}
Z_0[\,J\,] &= \left\langle \mathrm{T}\,\exp\left\{i\textstyle\int \mathrm{d}^4y\;\phi(y)J(y)\right\}\right\rangle_0 \\
&= \exp\left\{-\tfrac{1}{2}\textstyle\int \mathrm{d}^4x\int \mathrm{d}^4y\;J(x)\,D_F(x-y)\,J(y)\right\} \\
&\stackrel{\mathrm{def}}{=} \mathcal{N}\textstyle\int \mathfrak{D}\phi\;\exp\left\{(i/\hbar)\,S_0[\,\phi\,]+i\int \mathrm{d}^4x\;\phi(x)\,J(x)\right\} \\
S_0[\,\phi\,] &= -\int_{-\infty}^{\infty}\mathrm{d}t\int \mathrm{d}\mathbf{x}\;\tfrac{1}{2}\,\phi(x)\left[\Box+(mc/\hbar)^2-i\varepsilon\right]\phi(x) \\
\mathcal{N} &= \text{constant}\times\sqrt{\det\|\Box+m^2c^2/\hbar^2\|} \\
&\stackrel{\mathrm{def}}{=} \exp\left\{\frac{iVm^4c^4}{32\pi^2\hbar^4}\left(\ln\frac{m}{\mu}-\frac{3}{4}\right)\right\} \qquad \text{(Zeta regularization)} \\
Z_0[\,0\,] &= \mathcal{N}\textstyle\int \mathfrak{D}\phi\;\exp\{(i/\hbar)\,S_0[\,\phi\,]\}=1
\end{aligned}
$$

Notice that the integral kernel $D_F\,(x-y)$ which appears in the exponent of the right hand side of eq. (1.7) is just the opposite of the inverse for the kinetic operator $-i(\Box + m^2)$ that specifies the exponent $(i/\hbar)S_0[\phi]$ of the functional integral.

To summarize the above long discussion concerning the meaning and the construction of the functional integration, I can list a number of key features. The functional integral does fulfill by construction the following properties:

1. LINEARITY

$$\int \mathfrak{D}\phi \; (f\,F[\phi] + g\,G[\phi]) = f \int \mathfrak{D}\phi \; F[\phi] + g \int \mathfrak{D}\phi \; G[\phi]$$

   for any pair of complex functions $f, g \,:\, \mathcal{M} \to \mathbb{C}$

2. TRANSLATION INVARIANCE

$$\int \mathfrak{D}\phi \; F[\phi + f] = \int \mathfrak{D}\phi \; F[\phi] \qquad \forall f \,:\, \mathcal{M} \to \mathbb{C}$$

3. change under a SCALE TRANSFORMATION

$$\int \mathfrak{D}\phi \; F[(A\phi)(x)] = (\det A)^{-1} \int \mathfrak{D}\phi \; F[\phi]$$

   where A is any non-singular integral-differential operator

4. INTEGRATION BY PARTS

$$0 = \int \mathfrak{D}\phi \; \frac{\delta F[\phi]}{\delta\phi(x)}\, G[\phi] + \int \mathfrak{D}\phi \; F[\phi]\, \frac{\delta G[\phi]}{\delta\phi(x)}$$

The above properties 1. − 4. are valid for any functional pair F, G of the Gaussian kind

$$\mathcal{P}[\phi] \exp\left\{ \frac{i}{\hbar} S_0[\phi] + i \int \mathrm{d}x \; \phi(x)\, J(x) \right\}$$

with $\mathcal{P}[\phi]$ any polynomial functional of the scalar field and its derivatives.

1.2 The Spinor Generating Functional

The construction of a generating functional for the free fermionic fields, such as the quantized Dirac bispinor fields $\psi(x)$ and $\overline{\psi}(y)$ that fulfill the canonical anti-commutation relations

$$\{\psi(x)\,,\,\psi(y)\} = \left\{\overline{\psi}(x)\,,\,\overline{\psi}(y)\right\} = 0 \qquad \left\{\psi(x)\,,\,\overline{\psi}(y)\right\} = S(x-y)$$

does require the latter ones to be coupled to a suitable pair of sources. The Green's functions for the free Dirac bispinor quantum field are defined to be

$$S_0^{(n)}(x_1,\ldots,x_n;y_1,\ldots,y_n) = \langle\, 0\,|\,T\,\psi(x_1)\cdots\psi(x_n)\,\overline{\psi}(y_1)\cdots\overline{\psi}(y_n)\,|\,0\,\rangle$$

in which the $2n$ bispinor indexes of the kind $1L, 2L, 1R, 2R$ are understood for the sake of brevity. The above expression for the Dirac spinor Green's functions does exhibit the full anti-symmetry property

$$\begin{aligned} & S_0^{(n)}(x_1,\ldots,x_\imath,\ldots,x_\jmath,\ldots,x_n;y_1,\ldots,y_k,\ldots,y_\ell,\ldots,y_n) \\ = & -S_0^{(n)}(x_1,\ldots,x_\jmath,\ldots,x_\imath,\ldots,x_n;y_1,\ldots,y_k,\ldots,y_\ell,\ldots,y_n) \\ = & -S_0^{(n)}(x_1,\ldots,x_\imath,\ldots,x_\jmath,\ldots,x_n;y_1,\ldots,y_\ell,\ldots,y_k,\ldots,y_n) \\ \forall & \quad \imath,\jmath = 1,\ldots,n\,,\ \imath\neq\jmath \qquad \forall \quad k,\ell = 1,\ldots,n\,,\ k\neq\ell \end{aligned} \tag{1.29}$$

which follows from the canonical anti-commutation relations, just endorsing the Fermi-Dirac statistics for the many-particle states. Notice that for $n=1$ we just obtain the Feynman's spinor propagator

$$S_0^{(1)}(x_1;y_1) = S_F(x_1-y_1)$$

It turns out that the above complete anti-symmetry property of the spinor Green's functions can be effected by constructing the generating functional over a Graßmann algebra of anti-commuting classical sources $\overline{\zeta}(x)\,,\,\zeta(y)$ as we shall see here below.

1.2.1 Symanzik Equations for Fermions

Consider the vacuum expectation value

$$\begin{aligned} Z_0[\,\overline{\zeta}\,,\,\zeta\,] \;&\equiv\; \left\langle 0\right|T\,\exp\left\{i\int \mathrm{d}x\,\bar{\zeta}(x)\,\psi(x) + i\int \mathrm{d}y\,\bar{\psi}(y)\,\zeta(y)\right\}\left|0\right\rangle \\ &=\; 1-\int \mathrm{d}x\,\bar{\zeta}_\alpha(x)\int \mathrm{d}y\,\zeta_\beta(y)\,\left\langle 0\right|T\,\psi_\alpha(x)\,\bar{\psi}_\beta(y)\left|0\right\rangle \\ &+\; \cdots\cdots \end{aligned} \tag{1.30}$$

the suffix zero denoting the free field theory, where $\zeta(x)$ and $\bar{\zeta}(x)$ are the so called classical fermion external sources, which turn out to be Graßmann valued functions, with the canonical dimensions $[\,\zeta\,] = \mathrm{cm}^{-5/2}$ in natural units, which satisfy

$$\{\zeta(x)\,,\,\zeta(y)\} = \{\bar\zeta(x)\,,\,\bar\zeta(y)\} = 0 \tag{1.31}$$
$$\{\zeta(x)\,,\,\bar\zeta(y)\} = \{\bar\zeta(x)\,,\,\zeta(y)\} = 0 \tag{1.32}$$
$$\{\zeta(x)\,,\,\psi\,(y)\} = \{\bar\zeta(x)\,,\,\bar\psi\,(y)\} = 0 \tag{1.33}$$
$$\{\zeta(x)\,,\,\bar\psi\,(y)\} = \{\bar\zeta(x)\,,\,\psi\,(y)\} = 0 \tag{1.34}$$

Notice that, just owing to (1.34), the products $\bar\zeta(x)\,\psi\,(x)$ and $\bar\psi\,(y)\,\zeta(y)$ do not take up signs under time ordering, *i.e.*

$$T\Big(\bar\zeta(x)\,\psi\,(x)\,\bar\psi\,(y)\,\zeta(y)\Big) = \left\{\begin{array}{ll} \bar\zeta(x)\,\psi\,(x)\,\bar\psi\,(y)\,\zeta(y) & \text{if } x_0 > y_0 \\ \bar\psi\,(y)\,\zeta(y)\,\bar\zeta(x)\,\psi\,(x) & \text{if } x_0 < y_0 \end{array}\right.$$

In fact

$$\begin{array}{rcll} \bar\zeta(x)\,\psi\,(x)\,\bar\psi\,(y)\,\zeta(y) &=& \bar\zeta(x)\,\zeta(y)\,\psi\,(x)\,\bar\psi\,(y) & \qquad (x_0 > y_0) \\ \bar\psi\,(y)\,\zeta(y)\,\bar\zeta(x)\,\psi\,(x) &=& \bar\zeta(x)\,\zeta(y)\,\big(-\,\bar\psi\,(y)\,\psi\,(x)\big) & \qquad (x_0 < y_0) \end{array}$$

so that

$$\begin{array}{rcl} \Big\langle T\Big(\bar\zeta(x)\,\psi\,(x)\,\bar\psi\,(y)\,\zeta(y)\Big)\Big\rangle_0 &=& \bar\zeta(x)\,\zeta(y)\,\left\langle T\,\psi\,(x)\,\bar\psi\,(y)\right\rangle_0 \\ &=& \bar\zeta(x)\,\zeta(y)\,S_F(x-y) \\ &=& \Big\langle T\Big(\bar\psi\,(y)\,\zeta(y)\,\bar\zeta(x)\,\psi\,(x)\Big)\Big\rangle_0 \end{array}$$

The functional differentiation with respect to the classical Graßmann valued sources is defined by

$$\begin{array}{c} \{\delta/\delta\bar\zeta(x)\,,\,\bar\zeta(y)\} = \delta(x-y) = \{\delta/\delta\zeta(x)\,,\,\zeta(y)\} \\ \{\delta/\delta\bar\zeta(x)\,,\,\zeta(y)\} = 0 = \{\delta/\delta\zeta(x)\,,\,\bar\zeta(y)\} \\ \{\delta/\delta\bar\zeta(x)\,,\,\delta/\delta\bar\zeta(y)\} = 0 = \{\delta/\delta\zeta(x)\,,\,\delta/\delta\zeta(y)\} \\ \{\delta/\delta\bar\zeta(x)\,,\,\delta/\delta\zeta(y)\} = 0 = \{\delta/\delta\zeta(x)\,,\,\delta/\delta\bar\zeta(y)\} \end{array} \tag{1.35}$$

where all operators act on their right. It follows that

$$\begin{array}{l} -\,i\,\delta Z_0[\,\overline{\zeta}\,,\,\zeta\,]/\delta\bar\zeta(x) \;= \\ \Big\langle T\,\psi\,(x)\exp\Big\{i\int \mathrm{d}y\,\Big[\bar\zeta(y)\,\psi\,(y) + \bar\psi\,(y)\,\zeta(y)\Big]\Big\}\Big\rangle_0 \\ i\,\delta Z_0[\,\overline{\zeta}\,,\,\zeta\,]/\delta\zeta(x) \;= \\ \Big\langle T\,\bar\psi\,(x)\exp\Big\{i\int \mathrm{d}y\,\Big[\bar\zeta(y)\,\psi\,(y) + \bar\psi\,(y)\,\zeta(y)\Big]\Big\}\Big\rangle_0 \end{array} \tag{1.36}$$

where the plus sign in the second equality is because

$$\begin{aligned}\frac{\delta}{\delta\zeta(x)}\int \mathrm{d}\,y\,\bar{\psi}\,(y)\,\zeta(y) &= \int \mathrm{d}\,y\,\frac{\delta}{\delta\zeta(x)}\Big(\bar{\psi}\,(y)\,\zeta(y)\Big)\\ &= -\int \mathrm{d}\,y\,\bar{\psi}\,(y)\,\frac{\delta\zeta(y)}{\delta\zeta(x)}\\ &= -\int \mathrm{d}\,y\,\bar{\psi}\,(y)\,\delta(x-y)\\ &= -\bar{\psi}\,(x)\end{aligned} \tag{1.37}$$

Taking one more functional derivatives of the generating functional (1.30) we find

$$\begin{aligned}&[\,\delta/\delta\bar{\zeta}(x)\,]\,\delta Z_0\,[\,\overline{\zeta}\,,\,\zeta\,]\,/\,\delta\zeta\,(y)\ = \\ &\big\langle T\,\psi\,(x)\,\bar{\psi}\,(y)\ \exp\,\big\{i{\textstyle\int}\,\mathrm{d}\,z\ \ [\,\bar{\zeta}(z)\,\psi\,(z)+\bar{\psi}\,(z)\,\zeta(z)\,]\,\big\}\big\rangle_0\end{aligned} \tag{1.38}$$

The vacuum expectation values of the chronological ordered products of n pairs of free spinor field and its adjoint operators at different space-time points are named the $n-$point fermion Green's functions of the (free) Dirac spinor quantum field theory. By construction, the latter ones can be expressed as functional derivatives of the generating functional: namely,

$$\begin{aligned}&S_0^{(n)}(\,x_1,\cdots,x_n\,;\,y_1,\cdots,y_n\,)\\ =\ &\langle\,0\,|\,T\,\psi\,(x_1)\,\cdots\,\psi\,(x_n)\,\overline{\psi}\,(y_1)\,\cdots\,\overline{\psi}\,(y_n)\,|\,0\,\rangle\\ =\ &\delta^{\,(2n)}\,Z_0\,[\,\overline{\zeta}\,,\,\zeta\,]\,/\,\delta\bar{\zeta}(x_1)\cdots\delta\bar{\zeta}(x_n)\,\delta\zeta(y_1)\cdots\delta\zeta(y_n)\,\big\rfloor_{\zeta=\bar{\zeta}=0}\end{aligned}$$

Now it should be evident that the very same steps, which have led to establish the functional equation (1.6) for the free scalar field generating functional, can be repeated in a straightforward manner. To this purpose, let me introduce the finite times chronologically ordered exponential operator for the spinor fields that reads

$$E(t'\,,\,t)\equiv T\,\exp\left\{i\int_t^{t'}\mathrm{d}y_0\int\mathrm{d}\,\mathbf{y}\ [\,\bar{\psi}(y)\,\zeta(y)+\bar{\zeta}(y)\,\psi(y)\,]\right\} \tag{1.39}$$

Then we can derive the Symanzik functional equation for the generating functional of the Dirac field: namely,

$$\Big[\ (\,i\,\partial\!\!\!/_x-M\,)\ (\,i\,\delta/\delta\,\bar{\zeta}_x)-\zeta_x\,\Big]\,Z_0\,[\,\zeta\,,\,\bar{\zeta}\,]\ =0 \tag{1.40}$$

`Proof.` First we obtain

$$\begin{aligned}&-i\,\delta Z_0\,[\,\overline{\zeta}\,,\,\zeta\,]\,/\,\delta\,\bar{\zeta}\,(x)=\\ &\langle\,0\,|\,E\,(\infty,x_0)\,\psi(x_0,\mathbf{x})\,E\,(x_0,-\infty)\,|\,0\,\rangle\end{aligned}$$

so that the application of the Dirac differential operator yields

$$
\begin{aligned}
& (i\,\partial\!\!\!/_x - M\,)\Big(-i\,\delta Z_0[\,\bar\zeta\,,\,\zeta\,]\,/\,\delta\bar\zeta\,(x)\Big) \;= \\
& i\,\gamma^0 \textstyle\int \mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)\,\{\,\psi\,(x)[\,\bar\psi\,(x^0,\mathbf{y})\,\zeta\,(x^0,\mathbf{y}) + \bar\zeta\,(x^0,\mathbf{y})\,\psi\,(x^0,\mathbf{y})\,] \\
& -\,[\,\bar\psi\,(x^0,\mathbf{y})\,\zeta\,(x^0,\mathbf{y}) + \bar\zeta\,(x^0,\mathbf{y})\,\psi\,(x^0,\mathbf{y})\,]\,\psi\,(x)\,\}\,E\,(x^0,-\infty)\,|\,0\,\rangle \\
& +\,\langle\,0\,|\,E\,(\infty,x_0)\;(\,i\,\partial\!\!\!/_x - M\,)\,\psi(x_0,\mathbf{x})\,E\,(x_0,-\infty)\,|\,0\,\rangle \\
& =\; i\,\gamma^0\int \mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)\,\{\psi\,(x)\,,\,\bar\psi\,(x^0,\mathbf{y})\}\,\zeta\,(x^0,\mathbf{y})\,E\,(x^0,-\infty)\,|\,0\,\rangle \\
& -\,i\,\gamma^0\int \mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)\,\bar\zeta\,(x^0,\mathbf{y})\,\{\psi\,(x)\,,\,\psi\,(x^0,\mathbf{y})\}\,E\,(x^0,-\infty)\,|\,0\,\rangle \\
& =\; -\,\zeta\,(x)\,Z_0[\,\bar\zeta\,,\,\zeta\,]
\end{aligned}
$$

in which I have made use of the canonical anti-commutation relations $\{\psi(x), \psi(y)\} = 0$, as well as of the Dirac equation $(i\,\partial\!\!\!/_x - M)\psi(x)$. This proves the Symanzik equation (1.40).

□

In a complete analogous way we find the functional differential formula

$$
\left[\,\frac{i\delta}{\delta\zeta_x}\left(\,i\,\overleftarrow{\partial\!\!\!/}_x + M\right) - \bar\zeta_x\,\right] Z_0[\,\bar\zeta\,,\,\zeta\,] \;= 0 \tag{1.41}
$$

`Proof.` First we obtain

$$
\begin{aligned}
& i\,\delta Z_0[\,\bar\zeta\,,\,\zeta\,]\,/\,\delta\,\zeta\,(x) = \\
& \langle\,0\,|\,E\,(\infty,x_0)\,\bar\psi(x_0,\mathbf{x})\,E\,(x_0,-\infty)\,|\,0\,\rangle
\end{aligned}
$$

and taking left time derivative

$$
\begin{aligned}
& \langle\,0\,|\,E\,(\infty,x_0)\,\bar\psi(x_0,\mathbf{x})\,E\,(x_0,-\infty)\,|\,0\,\rangle\,\frac{\overleftarrow{\partial}}{\partial x^0} \\
= \;& i\int \mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)\,\bar\psi\,(x)\,[\,\bar\psi\,(x^0,\mathbf{y})\,\zeta\,(x^0,\mathbf{y}) + \bar\zeta\,(x^0,\mathbf{y})\,\psi\,(x^0,\mathbf{y})\,]E\,(x^0,-\infty)\,|\,0\,\rangle \\
- \;& i\int \mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)[\,\bar\psi\,(x^0,\mathbf{y})\,\zeta\,(x^0,\mathbf{y}) + \bar\zeta\,(x^0,\mathbf{y})\,\psi\,(x^0,\mathbf{y})\,]\,\bar\psi\,(x)\,E\,(x^0,-\infty)\,|\,0\,\rangle \\
+ \;& \left\langle T\,\partial_0\bar\psi\,(x)\exp\left\{i\int \mathrm{d}y\,\left[\,\bar\zeta(y)\,\psi\,(y) + \bar\psi\,(y)\,\zeta(y)\,\right]\right\}\right\rangle_0
\end{aligned}
$$

Now, using the anti-commutation relations (1.33), (1.34) and the equal time canonical anti-commutation relations for the Dirac spinor fields we find

$$
\begin{aligned}
& \langle\,0\,|\,E\,(\infty,x_0)\,\bar\psi(x_0,\mathbf{x})\,E\,(x_0,-\infty)\,|\,0\,\rangle\,(\,i\,\overleftarrow{\partial\!\!\!/}_x + M\,) \\
= \;& \gamma^0\int \mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x_0)\,\bar\zeta\,(x_0,\mathbf{y})\{\bar\psi\,(x_0,\mathbf{x})\,,\,\psi\,(x^0,\mathbf{y})\}E\,(x_0,-\infty)\,|\,0\,\rangle \\
- \;& \gamma^0\int \mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x_0)\{\bar\psi\,(x_0,\mathbf{x})\,,\,\bar\psi\,(x_0,\mathbf{y})\}\,\zeta\,(x_0,\mathbf{y})\,E\,(x_0,-\infty)\,|\,0\,\rangle \\
+ \;& \left\langle T\,\bar\psi\,(x)(\,i\,\overleftarrow{\partial\!\!\!/}_x + M\,)\exp\left\{\mathrm{i}\int \mathrm{d}y\,\left[\,\bar\zeta(y)\,\psi\,(y) + \bar\psi\,(y)\,\zeta(y)\,\right]\right\}\right\rangle_0 \\
= \;& \int \mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x_0)\,\bar\zeta\,(x_0,\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y})\,E\,(x_0,-\infty)\,|\,0\,\rangle = \bar\zeta\,(x)\,Z_0[\,\bar\zeta\,,\,\zeta\,]
\end{aligned}
$$

where use has been made of the adjoint Dirac equation $\bar{\psi}(x)(i\overleftarrow{\not\partial}_x + M) = 0$. This proves the Symanzik equation (1.41). $\square$

The solution of the above couple of functional differential equations that satisfies causality is

$$Z_0[\bar{\zeta}, \zeta] = \exp\left\{ - \int \mathrm{d}x \int \mathrm{d}y\ \bar{\zeta}(x)\, S_F(x-y)\, \zeta(y)\right\} \tag{1.42}$$

It is worthwhile to remark that, in close analogy with the case of the free scalar field generating functional, the integral kernel which appears in the exponent of the right hand side of eq. (1.42) does exactly coincide with the inverse of the Dirac operator $(i\not\partial - M)$ that specify the classical Action, since $(i\not\partial_x - M)\, S_F(x-y) = i\delta(x-y)$. The inversion of the classical kinetic Dirac operator is precisely provided by the Feynman propagator or causal 2-point Green's function, which indeed encodes:

1. the covariance properties under the action of the Poincaré group

2. the canonical anti-commutation relations, the related property of the micro-causality and the Fermi-Dirac statistics obeyed by all the many-particle states

3. the causality requirement, that means the possibility to perform the transition to the Euclidean formulation.

Hence the generating functional (1.42) does truly contain all the mutually tied up key features of the relativistic quantum field theory.

1.2.2 The Integration over Graßmann Variables

In order to find a functional integral representation for $Z_0[\zeta, \bar{\zeta}]$ I need to primary define the concept of integration with respect to Graßmann valued functions on the Minkowski space. This latter tool has been constructed by

F.A. Berezin, *The Method of Second Quantization*
Academic Press, New York (1966)

Hereafter I shall follow

Sidney R. Coleman, *The Uses of Instantons*
Proceedings of the 1977 International School of Subnuclear Physics
Erice, Antonino Zichichi Editor, Academic Press, New York (1979)

Consider first a real function $f : \mathfrak{G} \to \mathbb{R}$ of a Graßmann variable $a \in \mathfrak{G}$ so that $a^2 = 0$ and suppose we want to define $\int \mathrm{d}a\, f(a)$. We require this to have the usual linearity property, *viz.*,

$$\begin{aligned}&\int \mathrm{d}a\ [\,\alpha f(a) + \beta\, g\,(a)\,]\\ =\ &\alpha \int \mathrm{d}a\ f(a) + \beta \int \mathrm{d}a\ g\,(a) \qquad (\forall\, \alpha, \beta \in \mathbb{R}\,)\end{aligned} \tag{1.43}$$

and in addition, we would like the integral to be translation invariant

$$\int \mathrm{d}a\ f(a+b) = \int \mathrm{d}a\ f(a) \qquad (\forall\, b \in \mathfrak{G}\,) \tag{1.44}$$

It is easy to show that those conditions determine the integral, up to a normalization constant. The reason is very simple: there are only two linearly independent functions of $a \in \mathfrak{G}$, that is 1 and a , all higher powers being null. As a matter of fact we have $\forall\, a, b \in \mathfrak{G}$ with $\{a, a\} = \{a, b\} = \{b, b\} = 0$

$$f(a) = f_0 + a\, f_1 \qquad f(a+b) = f_0 + (a+b)\, f_1 \tag{1.45}$$

Here, if $f : \mathfrak{G} \longrightarrow \mathbb{R}$ then f_0 is an ordinary real number while f_1 is a real Graßmann number[2] such that $\{a, f_1\} = \{f_1, f_1\} = 0$. Conversely, if $f(a)$ is Graßmann valued, then $f_0 \in \mathfrak{G}$ while f_1 is an ordinary real number. It follows that the linearity property (1.43) indeed entails for *e.g.* $f \in \mathbb{R}$

$$\begin{aligned}\int \mathrm{d}a\ f(a) &= \int \mathrm{d}a\ f_0 + \int \mathrm{d}a\ a\, f_1\\ &= f_0 \int \mathrm{d}a\ 1 + f_1 \int \mathrm{d}a\ a\\ \int \mathrm{d}a\ f(a+b) &= \int \mathrm{d}a\ f_0 + \int \mathrm{d}a\ (a+b)\, f_1\\ &= f_0 \int \mathrm{d}a\ 1 + f_1 \int \mathrm{d}a\ a + b\, f_1 \int \mathrm{d}a\ 1\end{aligned}$$

and from the translation invariance requirement (1.44)

$$\int \mathrm{d}a\ a = N \qquad b\, f_1 \int \mathrm{d}a\ 1 \equiv 0 \qquad (\,\forall\, b \in \mathfrak{G}\,) \tag{1.46}$$

One can always choose the normalization constant N such that

$$\int \mathrm{d}a\ a = 1$$

[2]Of course complex Graßmann numbers are understood to be of the kind $a + ib$ with $a, b \in \mathfrak{G}$.

But then, translation invariance just requires

$$\int \mathrm{d}a \; 1 = 0$$

Hence

$$\int \mathrm{d}a \; f(a) = f_1 = \frac{\mathrm{d}}{\mathrm{d}a} f(a) \qquad \forall f \,:\, \mathfrak{G} \to \mathbb{R}$$

which manifestly shows that for a real functions of a Graßmann variable the integration just coincides with right differentiation.

As a straightforward generalization for any real function of many Graßmann variables, it is natural to define multiple integrals by iteration. Thus, *e.g.* , a complete integration table for the four linearly independent functions of two anti-commuting variables a and $\bar{a}$ is provided by

$$\int \mathrm{d}a \int \mathrm{d}\bar{a} \begin{cases} \bar{a}\, a & = & 1 \\ \bar{a} & = & 0 \\ a & = & 0 \\ 1 & = & 0 \end{cases} \tag{1.47}$$

with $\{a, a\} = \{\bar{a}, \bar{a}\} = \{a, \bar{a}\} = 0$. As an application of this table we can calculate

$$\int \mathrm{d}a \int \mathrm{d}\bar{a} \;\exp\{\,\lambda\, \bar{a}\, a\} = \int \mathrm{d}a \int \mathrm{d}\bar{a}\; (1 + \lambda\, \bar{a}\, a) = \lambda \tag{1.48}$$

The generalization to higher dimensional spaces is easy. Consider for example some $n \times n$ Hermitean matrix $A = A^{\dagger}$ and two collections of n of Graßmann variables

$$\{\theta_i\,,\, \theta_j\} = \{\theta_i\,,\, \bar{\theta}_j\} = \{\bar{\theta}_i\,,\, \bar{\theta}_j\} = 0 \qquad (\,i, j = 1, 2, \ldots, n\,)$$

Then a unitary $n \times n$ matrix always exists such that

$$U^{\dagger} A U = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \qquad \lambda_j \in \mathbb{R} \quad (\,j = 1, 2, \ldots, n\,)$$

As a consequence, if we set

$$a = U\,\theta \qquad \bar{a} = \bar{\theta}\, U^{\dagger}$$

so that

$$\{a_i\,,\, a_j\} = \{a_i\,,\, \bar{a}_j\} = \{\bar{a}_i\,,\, \bar{a}_j\} = 0 \qquad (\,i, j = 1, 2, \ldots, n\,) \tag{1.49}$$

then we can write

$$
\begin{aligned}
I &= \int \mathrm{d}^n a \int \mathrm{d}^n \bar{a}\ \exp\{\bar{a}\, A\, a\} \\
&= \det(U^\dagger U) \int \mathrm{d}^n \theta \int \mathrm{d}^n \bar{\theta}\ \exp\{\bar{\theta}\, U^\dagger A U\, \theta\} \\
&\equiv \prod_{j=1}^{n} \int \mathrm{d}\theta_j \int \mathrm{d}\bar{\theta}_j\ \exp\{\lambda_j\, \bar{\theta}_j\, \theta_j\} \\
&= \prod_{j=1}^{n} \int \mathrm{d}\theta_j \int \mathrm{d}\bar{\theta}_j\ \left(1 + \lambda_j\, \bar{\theta}_j\, \theta_j\right) \\
&= \prod_{j=1}^{n} \lambda_j \ = \ \det A
\end{aligned}
\tag{1.50}
$$

As a further application, consider the Taylor's expansion for a function of the two collections of n of Graßmann variables $(\bar{a}, a)$ satisfying (1.49): namely,

$$
f(\bar{a}, a) \ = \ f_0 + \sum_{\jmath=1}^{n} \bar{a}_\jmath\, \bar{f}_{1\jmath} + \sum_{\imath=1}^{n} a_\imath\, f_{1\imath} + \sum_{\imath=1}^{n} \sum_{\jmath=1}^{n} \bar{a}_\jmath\, a_\imath\, f_{2\imath\jmath} \tag{1.51}
$$

where

$$
\begin{aligned}
\bar{f}_{1\jmath} &= \left(\frac{\partial f}{\partial \bar{a}_\jmath}\right)_0 \equiv \frac{\partial}{\partial \bar{a}_\jmath} f(\bar{a}, a)\bigg\rfloor_{\bar{a}=a=0} \\
f_{1\imath} &= \left(\frac{\partial f}{\partial a_\imath}\right)_0 \equiv \frac{\partial}{\partial a_\imath} f(\bar{a}, a)\bigg\rfloor_{\bar{a}=a=0} \\
f_{2\imath\jmath} &= \left(\frac{\partial^2 f}{\partial a_\imath \partial \bar{a}_\jmath}\right)_0 = \frac{\partial^2 f}{\partial a_\imath \partial \bar{a}_\jmath}\bigg\rfloor_{\bar{a}=a=0}
\end{aligned}
$$

Notice that, for example, if $f \in \mathbb{R}$ then we have $f_0, f_{2\imath\jmath} \in \mathbb{R}$ $(\imath, \jmath = 1, \ldots, n)$, while $\bar{f}_{1\jmath}, f_{1\imath} \in \mathfrak{G}$ $(\imath, \jmath = 1, \ldots, n)$. Then we obtain for instance

$$
\int \mathrm{d}a_\jmath \int \mathrm{d}\bar{a}_\jmath \frac{\partial}{\partial \bar{a}_\jmath} f(\bar{a}, a) = \int \mathrm{d}a_\jmath \left\{ \bar{f}_{1\jmath} - \sum_{\imath=1}^{n} a_\imath\, f_{2\imath\jmath} \right\} \int \mathrm{d}\bar{a}_\jmath \equiv 0
$$

and in general

$$
\int \mathrm{d}a \int \mathrm{d}\bar{a}\, [\, \partial f(\bar{a}, a)/\partial \bar{a}_\jmath\,] = 0 \qquad (\forall\, \jmath = 1, 2, \ldots, n\,) \tag{1.52}
$$

$$
\int \mathrm{d}a \int \mathrm{d}\bar{a}\, [\, \partial f(\bar{a}, a)/\partial a_\imath\,] = 0 \qquad (\forall\, \imath = 1, 2, \ldots, n\,) \tag{1.53}
$$

1.2.3 The Functional Integral for Fermions

Consider now the straightforward generalization of the functional integral representation (1.10) to the fermion case: namely,

$$\begin{aligned} Z_0[\,\overline{\zeta}\,,\,\zeta\,] &= \int \mathfrak{D}\psi \int \mathfrak{D}\overline{\psi}\ \widetilde{Z}_0\,[\,\overline{\psi}\,,\,\psi\,] \\ &\times\ \exp\left\{ i\textstyle\int \mathrm{d}x\,[\,\bar{\zeta}(x)\,\psi\,(x) + \bar{\psi}\,(x)\,\zeta(x)\,]\right\} \end{aligned} \tag{1.54}$$

where we understand once again formally

$$\int \mathfrak{D}\psi \int \mathfrak{D}\bar{\psi}\ =: \prod_{x\in\mathcal{M}} \int \mathrm{d}\,\psi_x \int \mathrm{d}\,\bar{\psi}_x$$

Inserting into (1.40) and (1.41) and making use of (1.52) and (1.53) we are allowed to identify

$$\widetilde{Z}_0\,[\,\overline{\psi}\,,\,\psi\,] = \mathcal{N}\ \exp\left\{\,(i/\hbar)\,S_0[\,\bar{\psi}\,,\,\psi\,]\right\} \tag{1.55}$$

and by comparison with (1.42) we find

$$\begin{aligned} Z_0\,[\,\overline{\zeta}\,,\,\zeta\,] &= \exp\left\{\,-\textstyle\int \mathrm{d}\,x \int \mathrm{d}\,y\ \bar{\zeta}\,(x)\,S_F\,(x-y)\,\zeta\,(y)\right\} \\ &=: \mathcal{N}\int \mathfrak{D}\psi \int \mathfrak{D}\bar{\psi}\ \exp\left\{\,(i/\hbar)\,S_0[\,\bar{\psi}\,,\,\psi\,]\right\} \\ &\times\ \exp\left\{ i\textstyle\int \mathrm{d}\,x\,[\,\bar{\zeta}(x)\,\psi\,(x) + \bar{\psi}\,(x)\,\zeta(x)\,]\right\} \end{aligned} \tag{1.56}$$

$$S_0[\,\bar{\psi}\,,\,\psi\,] = \textstyle\int \mathrm{d}\,x\ \bar{\psi}\,(x)\,(i\,\partial\!\!\!/ - M)\ \psi\,(x)$$

`Proof.` As a matter of fact, from eq. (1.40) we obtain

$$\begin{aligned} & [\,\zeta\,(x) + (\,i\,\partial\!\!\!/_x - M\,)\,\delta/i\,\delta\,\bar{\zeta}\,(x)\,]\ Z_0\,[\,\overline{\zeta}\,,\,\zeta\,] \\ =\ & \mathcal{N}\int \mathfrak{D}\psi \int \mathfrak{D}\bar{\psi}\,[\,\zeta\,(x) + (\,i\,\partial\!\!\!/_x - M)\ \psi\,(x)\,] \\ \times\ & \exp\left\{\,(i/\hbar)\,S_0[\,\bar{\psi}\,,\,\psi\,] + i\textstyle\int \mathrm{d}y\,[\,\bar{\zeta}(y)\,\psi\,(y) + \bar{\psi}\,(y)\,\zeta(y)\,]\right\} \\ =\ & \mathcal{N}\int \mathfrak{D}\psi \int \mathfrak{D}\bar{\psi}\,\left(\delta/i\,\delta\,\bar{\psi}(x)\right) \\ \times\ & \exp\left\{(i/\hbar)\textstyle\int \mathrm{d}y\,[\,\bar{\psi}(y)\,(i\,\partial\!\!\!/_y - M)\,\psi(y) + \hbar\,\bar{\zeta}(y)\psi(y) + \hbar\,\bar{\psi}(y)\zeta(y)\,]\right\} = 0 \end{aligned}$$

owing to the general identity (1.52). Quite analogously we still recover a null result from the application of (1.53) to the adjoint Symanzik functional differential equation (1.41). This completes the proof of the above statements. □

As I did in the case of the scalar field, the functional measure $\mathfrak{D}\psi\,\mathfrak{D}\bar{\psi}$, which is a formal entity until now, can be implemented by the requirement of invariance under field translations [3] $\psi(x)\mapsto\psi(x)+\theta(x)$, $\bar\psi(x)\mapsto\bar\psi(x)+\bar\theta(x)$. Once translation invariance is assumed, let us change the integration variables according to

$$\begin{aligned}\psi(x)\mapsto\psi'(x) &= \psi(x)+(i\partial\!\!\!/-M)^{-1}\zeta(x)\\ &= \psi(x)-i\int dy\,S_F(x-y)\,\zeta(y)\\ &= \psi_x-\langle\, i\,S_{xy}\,\zeta_y\,\rangle \end{aligned}\tag{1.57}$$

$$\begin{aligned}\bar\psi(x)\mapsto\bar\psi'(x) &= \bar\psi(x)-\bar\zeta(x)(i\overleftarrow{\partial\!\!\!/}+M)^{-1}\\ &= \bar\psi(x)+i\int dy\,\bar\zeta(y)\,\bar S_F(y-x)\\ &= \bar\psi_x+\langle\, i\,\bar\zeta_y\,\bar S_{yx}\,\rangle \end{aligned}\tag{1.58}$$

where use has been made of the adjoint Dirac equation. Then we get

$$\begin{aligned} Z_0\,[\,\bar\zeta\,,\,\zeta\,] &=: Z_0\,[\,0\,,\,0\,]\ \exp\left\{\,-i\int dx\int dy\ \bar\zeta(x)\,(i\partial\!\!\!/-M)^{-1}\,\zeta(y)\right\}\\ &= Z_0\,[\,0\,,\,0\,]\ \exp\left\{\,-\int dx\int dy\ \bar\zeta(x)\,S_F\,(x-y)\,\zeta(y)\right\}\end{aligned}$$

and consistently $Z_0[0,0]=\langle 0|0\rangle :=: \mathcal{N}\int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi\exp\left\{(i/\hbar)\,S_0[\bar\psi,\psi]\right\}=1.$

`Proof` : the formal expression (1.56) can be suitably rewritten as

$$Z_0\,[\,\bar\zeta\,,\,\zeta\,] :=: \mathcal{N}\prod_{x\in\mathcal{M}}\int d\psi_x\int d\bar\psi_x\ \exp\left\{(i/\hbar)\,\bar\psi_x\,(i\partial\!\!\!/_x-M)\,\psi_x+i\,\bar\psi_x\,\zeta_x+i\,\bar\zeta_x\,\psi_x\right\}$$

Then, once again, it is very convenient to perform the change of variables

$$\psi_x=\psi'_x+\langle\, i\,S_{xy}\,\zeta_y\,\rangle \qquad d\psi_x = d\psi'_x$$

$$\bar\psi_x=\bar\psi'_x-\langle\, i\,\bar\zeta_y\,\bar S_{yx}\,\rangle \qquad d\bar\psi_x = d\bar\psi'_x$$

in such a manner that we have

$$\begin{aligned} &\int d\psi_x\int d\bar\psi_x\ \exp\left\{(i/\hbar)\,\bar\psi_x\,(i\partial\!\!\!/_x-M)\,\psi_x+i\,\bar\psi_x\,\zeta_x+i\,\bar\zeta_x\,\psi_x\right\}\\ = &\int d\psi_x\int d\bar\psi_x\ \exp\left\{(i/\hbar)\,(\bar\psi'_x-\langle\, i\,\bar\zeta_y\,\bar S_{yx}\rangle)\ (i\partial\!\!\!/_x-M)\ (\psi'_x+\langle\, i\,S_{xz}\,\zeta_z\,\rangle)\right\}\\ \times\ &\exp\left\{\,i\,\bar\psi'_x\,\zeta_x+i\,\bar\zeta_x\,\psi'_x+\langle\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,\zeta_x-\bar\zeta_x\,\langle\, S_{xy}\,\zeta_y\,\rangle\,\right\}\end{aligned}$$

and from the first equality

$$\bar\psi'_x\,(i\partial\!\!\!/_x-M)\,\langle\, i\,S_{xz}\,\zeta_z\,\rangle\ =\ -\bar\psi'_x\,\zeta_x$$

together with the second conditioned equality

$$\begin{aligned} &\langle\, i\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,(i\partial\!\!\!/_x-M)\,\psi'_x\\ = &-(\partial/\partial x^\mu)\ \left[\,\langle\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,\gamma^\mu\,\psi'_x\,\right]-\langle\, i\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,(i\overleftarrow{\partial\!\!\!/}_x+M\,)\,\psi'_x\\ \doteq &-\langle\, i\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,(i\overleftarrow{\partial\!\!\!/}_x+M\,)\,\psi'_x\ =\ \bar\zeta_x\,\psi'_x\end{aligned}$$

[3] It is worthwhile to recall that translation invariance is one of the axioms which define the integration over the Graßmann variables.

which is true by neglecting the four divergence term, we can eventually write

$$
\begin{aligned}
& \exp\left\{(i/\hbar)\left(\bar{\psi}'_x - \langle\, i\,\bar{\zeta}_y\,\bar{S}_{yx}\,\rangle\right)\,(i\,\partial\!\!\!/_x - M)\,(\psi'_x + \langle\, i\,S_{xz}\,\zeta_z\,\rangle)\right\} \\
\doteq\ & \exp\left\{\,(i/\hbar)\,\bar{\psi}'_x\,(\,i\,\partial\!\!\!/_x - M\,)\,\psi'_x - i\,\bar{\psi}'_x\,\zeta_x - \,i\,\bar{\zeta}_x\,\psi'_x + \langle\,\bar{\zeta}_y\,\bar{S}_{yx}\,\rangle\,\zeta_x\,\right\}
\end{aligned}
$$

Hence, collecting altogether we come to the double integral

$$
\begin{aligned}
& \int \mathrm{d}\psi_x \int \mathrm{d}\bar{\psi}_x\ \exp\left\{(i/\hbar)\,\bar{\psi}_x\,(i\,\partial\!\!\!/_x - M)\,\psi_x + i\,\bar{\psi}_x\,\zeta_x + i\,\bar{\zeta}_x\,\psi_x\right\} \\
\doteq\ & \exp\left\{\,-\,\bar{\zeta}_x\,\langle\,S_{xy}\,\zeta_y\,\rangle\,\right\} \int \mathrm{d}\psi'_x \int \mathrm{d}\bar{\psi}'_x\ \exp\left\{(i/\hbar)\,\bar{\psi}'_x\,(i\,\partial\!\!\!/_x - M)\,\psi'_x\right\} \quad (\,\forall x \in \mathcal{M}\,)
\end{aligned}
$$

and thereby

$$
\begin{aligned}
& \mathcal{N} \prod_{x\,\in\,\mathcal{M}} \int \mathrm{d}\psi_x \int \mathrm{d}\bar{\psi}_x\ \exp\left\{(i/\hbar)\,\bar{\psi}_x\,(i\,\partial\!\!\!/_x - M)\,\psi_x + i\,\bar{\psi}_x\,\zeta_x + i\,\bar{\zeta}_x\,\psi_x\right\} \\
\doteq\ & \prod_{x\,\in\,\mathcal{M}} \exp\left\{\,-\,\bar{\zeta}_x\,\langle\,S_{xy}\,\zeta_y\,\rangle\,\right\}\ \mathcal{N} \prod_{z\,\in\,\mathcal{M}} \int \mathrm{d}\psi'_z \int \mathrm{d}\bar{\psi}'_z\ \exp\left\{(i/\hbar)\,\bar{\psi}'_z\,(i\,\partial\!\!\!/_z - M)\,\psi'_z\right\} \\
=\ & \prod_{x\,\in\,\mathcal{M}} \exp\left\{-\,\bar{\zeta}_x\,\langle\,S_{xy}\,\zeta_y\,\rangle\right\}\ \mathcal{N} \int \mathfrak{D}\psi' \int \mathfrak{D}\bar{\psi}'\ \exp\left\{(i/\hbar)\,S_0[\,\bar{\psi}'\,,\,\psi'\,]\right\} \\
=\ & \exp\left\{-\int \mathrm{d}\,x \int \mathrm{d}\,y\ \bar{\zeta}(x)\,S_F\,(x-y)\,\zeta(y)\right\}\ Z_0[\,0\,,\,0\,] = Z_0[\,\zeta\,,\,\bar{\zeta}\,]
\end{aligned}
$$

which completes the proof. □

To the aim of giving some precise mathematical meaning to the above expression, it is convenient to turn to the Euclidean formulation just like I did in the case of the real scalar field – see formula (1.18). Then, taking the Dirac Euclidean Action into account, we can consider

$$
Z_E^{(0)}[\,0\,,\,0\,] = \mathcal{N} \int \mathfrak{D}\psi_E \int \mathfrak{D}\bar{\psi}_E \exp\left\{i\int \mathrm{d}\,x_E\ \bar{\psi}_E\,(i\,\partial\!\!\!/_E + iM)\ \psi_E\right\}
$$

After rescaling

$$
\psi'_{E\,,\,x} = i\mu\,\psi_{E\,,\,x}
$$

where μ is some arbitrary mass scale, the we can write

$$
\begin{aligned}
& Z_E^{(0)}[\,0\,,\,0\,]\ = \\
& \mathcal{N}'\ \prod_{x_E\,\in\,\mathbb{R}^4} \int \mathrm{d}\psi'_{Ex} \int \mathrm{d}\bar{\psi}_{Ex}\ \exp\left\{\bar{\psi}_{Ex}\,(i\,\partial\!\!\!/_{Ex} + iM)\,\mu^{-1}\psi'_{Ex}\right\} \\
& \stackrel{\mathrm{def}}{=} \mathcal{N}'\,\det\ \|\ (i\,\partial\!\!\!/_E + iM)/\mu\ \|
\end{aligned}
\tag{1.59}
$$

which can be understood as a formal continuous generalization of the formula (1.50) that represents a determinant. Nevertheless, it should be noticed that the Euclidean Dirac operator $(i\,\partial\!\!\!/_E + iM)$ is not Hermitean but only normal, *i.e.* , it commutes with its adjoint $[\,i\,\partial\!\!\!/_E - iM\,,\,i\,\partial\!\!\!/_E + iM\,] = 0\,.$

This fact in turn implies that the Euclidean Dirac operator $i\,\partial\!\!\!/_E + iM$ also commutes with the positive definite diagonal operator

$$(i\,\partial\!\!\!/_E - \mathrm{i}M)(i\,\partial\!\!\!/_E + iM) = \left(M^2 - \partial_{E\mu}\,\partial_{E\mu}\right)\,\mathbb{I} = (i\,\partial\!\!\!/_E + iM)(i\,\partial\!\!\!/_E - iM)$$

that means

$$[\,i\,\partial\!\!\!/_E \pm iM\,,\;M^2 - \partial_{E\mu}\,\partial_{E\mu}\,] = 0$$

where the eigenfunctions and the eigenvalues of the Euclidean Klein-Gordon operator are respectively given by

$$\phi_{\,E\,,p}\,(x_E) = (2\pi)^{-2}\,\exp\{\,i p_{E\mu} x_{E\mu}\} \qquad \lambda\,(p_E) = p_E^2 + M^{\,2} > 0$$

Notice that the normal operator $i\,\partial\!\!\!/_E + iM$ is also non-singular, since we have

$$(i\,\partial\!\!\!/_E + iM\,)^{-1} = (i\,\partial\!\!\!/_E - iM)\left(M^2 - \partial_{E\mu}\,\partial_{E\mu}\right)^{-1}$$

From the explicit form

$$i\,\partial\!\!\!/_E + iM = \begin{pmatrix} iM & 0 & i\partial_4 + \partial_3 & \partial_1 - i\partial_2 \\ 0 & iM & \partial_1 + i\partial_2 & i\partial_4 - \partial_3 \\ i\partial_4 - \partial_3 & -\partial_1 + i\partial_2 & iM & 0 \\ -\partial_1 - i\partial_2 & i\partial_4 + \partial_3 & 0 & iM \end{pmatrix}$$

it can be readily checked by direct inspection that

$$\det\ \|\ (i\,\partial\!\!\!/_E + iM)/\mu\ \| = \left(\det\ \|\ \left(M^2 - \partial_E^2\right)/\mu^2\ \|\ \right)^2$$

so that from the Zeta function definition (1.28) we eventually obtain

$$\det\ \|\ (i\,\partial\!\!\!/_E + iM\,)\,/\mu\ \| = \ \exp\left\{\frac{VM^4}{8\pi^{\,2}}\left(\ln\frac{M}{\mu} - \frac{3}{4}\right)\right\} \tag{1.60}$$

and again the transition to the Minkowski space can be immediately done by simply replacing $V_{\text{euclidean}} \leftrightarrow i\,V_{\text{minkowskian}}$ so that

$$\det\ \|\ (i\,\partial\!\!\!/ - M\,)\,/\mu\ \| = \ \exp\left\{\frac{iVM^4}{8\pi^{\,2}}\left(\ln\frac{M}{\mu} - \frac{3}{4}\right)\right\} \tag{1.61}$$

Turning back to (1.59) we eventually have

$$Z_E^{(0)}\,[\,0\,,\,0\,] = 1 \quad \Longleftrightarrow \quad \mathcal{N}^{\,\prime} \equiv \exp\left\{-\,\frac{VM^4}{8\pi^{\,2}}\left(\ln\frac{M}{\mu} - \frac{3}{4}\right)\right\}$$

The conclusion of all the above formal reasoning is as follows: we are allowed to define the functional integral for a free Dirac spinor quantum field theory by the equality

$$\begin{aligned} Z_0\,[\,\bar\zeta\,,\,\zeta\,] &= \exp\left\{-\int \mathrm{d}x \int \mathrm{d}y\ \bar\zeta\,(x)\,S_F\,(x-y)\,\zeta\,(y)\right\} \\ &\stackrel{\mathrm{def}}{=} \mathcal{N}\int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi\ \exp\left\{(i/\hbar)\,S_0\,[\,\bar\psi\,,\,\psi\,]\right\} \\ &\times\ \exp\left\{i\int \mathrm{d}x\ \left[\,\bar\zeta(x)\,\psi\,(x) + \bar\psi\,(x)\,\zeta(x)\,\right]\right\} \end{aligned} \tag{1.62}$$

where

$$S_0\,[\,\bar\psi\,,\,\psi\,] = \int \mathrm{d}x\ \bar\psi\,(x)\,(i\,\partial\!\!\!/ - M)\ \psi\,(x)$$

$$\begin{aligned} \mathcal{N} &= \text{constant} \times \det\ \|\, i\,\partial\!\!\!/ - M\,\|^{-1} \\ &\stackrel{\mathrm{def}}{=} \exp\left\{\frac{VM^4}{8i\pi^2}\left(\ln\frac{M}{\mu} - \frac{3}{4}\right)\right\} \qquad \text{(Zeta regularization)} \end{aligned}$$

$$Z_0\,[\,0\,,\,0\,] = \mathcal{N}\int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi\ \exp\left\{(i/\hbar)\,S_0\,[\,\bar\psi\,,\,\psi\,]\right\} = 1 = \langle\,0\,|\,0\,\rangle$$

It is clear that, by construction, the functional integral for a free Dirac spinor quantum field theory does satisfy all the requirements of linearity, translation invariance, rescaling and integration by parts which I have discussed in the case of the real scalar field.

1.3 The Vector Generating Functional

Finally I want to set up the generating functional for the massive real vector free field theories. This can be done by a straightforward generalization of the real scalar field case. We recall that the most general Lagrangian in a linear covariant gauge reads

$$\begin{aligned} \mathcal{L}_{\,A\,,\,B} &= -\,\frac{1}{4}\,F^{\,\mu\nu}(x)\,F_{\,\mu\nu}(x) + \frac{m^2c^2}{2\hbar^2}\,A^{\nu}(x)\,A_{\nu}(x) \\ &+\ A^{\mu}(x)\,\partial_{\mu}B(x) + \frac{\xi}{2}\,B^{\,2}(x) \end{aligned}$$

where the vector field $A^{\mu}(x)$ has canonical dimensions $[\,A^{\mu}\,] = \sqrt{\mathrm{eV/cm}}$ in physical units, whereas $B(x)$ is an auxiliary nonphysical scalar field of canonical engineering dimension $[\,B\,] = \mathrm{erg}^{\frac{1}{2}}\ \mathrm{cm}^{-\frac{3}{2}}$, while the dimensionless parameter $\xi \in \mathbb{R}$ is named the gauge fixing parameter, the Abelian field strength being as usual $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, in such a manner that the Action $S = \int \mathrm{d}t \int \mathrm{d}\mathbf{x}\,\mathcal{L}$ turns out to be Poincaré invariant. In particular, we can conveniently define

$$u^A(x) \stackrel{\mathrm{def}}{=} (\, A^\mu(x)\,,\, B(x)\,) \qquad A = (\,\mu\,,\,\bullet\,)$$

$$j_A(x) \stackrel{\mathrm{def}}{=} (\, J_\mu(x)\,,\, K(x)\,)$$

$$\int \mathrm{d}x\; u^A(x)\, j_A(x) = \int \mathrm{d}\,\Big[\, A^\mu(x)\, J_\mu(x) + B(x)\, K(x)\,\Big]$$

so that we have a five dimensional diagonal metric tensor

$$g^{AB} = g_{AB} = \mathrm{diag}\,(+1, -1, -1, -1, +1)$$

The classical Action can be written in the form

$$S_0[\,u\,] \;=\; -\,\frac{1}{2}\int \mathrm{d}x\, u^A(x)\, K_{AB}\, u^B(x)$$

with

$$K_{\mu\nu} = -\,g_{\mu\nu}\,\left(\Box + m^2\right) + \partial_\mu\,\partial_\nu \qquad K_{\mu\bullet} = -\,\partial_\mu \qquad K_{\bullet\bullet} = -\,\xi$$

The kinetic operator K_{AB} can be uniquely inverted by means of the causal $+\,i\varepsilon$ prescription leading to the causal Green's function

$$D^{(c)}_{AB}\,(\,x - y\,;\, m\,,\,\xi\,) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\; \widetilde{D}^{(c)}_{AB}\,(\,k\,;\, m\,,\,\xi\,)\; e^{-\,ik\cdot(x-y)}$$

$$\widetilde{D}^{F}_{\mu\nu}\,(\,k\,;\, m\,,\,\xi\,) = \frac{i}{k^2 - m^2 + i\,\varepsilon}\left\{ -\,g_{\mu\nu} + \frac{(1-\xi)\,k_\mu\,k_\nu}{k^2 - \xi\,m^2 + i\,\varepsilon'}\right\}$$

$$\widetilde{D}^{F}_{\nu\bullet}\,(k\,;\, m\,,\,\xi) = \frac{-\,k_\nu}{k^2 - \xi\,m^2 + i\,\varepsilon'}$$

$$\widetilde{D}^{F}_{\bullet\bullet}\,(k\,;\, m\,,\,\xi) = \frac{-\,im^2}{k^2 - \xi\,m^2 + i\,\varepsilon'}$$

In fact we have

$$\begin{aligned} K_\mu{}^\rho\, D^F_{\rho\sigma}(x) + K_{\mu\bullet}\, D^F_{\sigma\bullet}(x) &= -\,i\,g_{\mu\sigma}\,\delta(x) \\ g^{\mu\nu}\, K_{\mu\bullet}\, D^F_{\nu\bullet}(x) - \xi\, D^F_{\bullet\bullet}(x) &= -\,i\,\delta(x) \end{aligned}$$

and in the Fourier space

$$
\begin{aligned}
& \left\{\delta_{\mu}^{\rho}-\frac{k_{\mu}\,k^{\rho}}{k^{2}-m^{2}+i\,\varepsilon}\right\}\left\{-\,g_{\rho\sigma}+\frac{(1-\xi)\,k_{\rho}\,k_{\sigma}}{k^{2}-\xi\,m^{2}+i\,\varepsilon\,'}\right\}-\frac{k_{\mu}\,k_{\sigma}}{k^{2}-\xi\,m^{2}+i\,\varepsilon\,'} \\
= \;& -\,g_{\mu\sigma}+\frac{(1-\xi)\,k_{\mu}\,k_{\sigma}}{k^{2}-\xi\,m^{2}+i\,\varepsilon\,'}+\frac{k_{\mu}\,k_{\sigma}}{k^{2}-m^{2}+i\,\varepsilon}-\frac{k^{2}}{k^{2}-m^{2}+i\,\varepsilon}\cdot\frac{(1-\xi)\,k_{\mu}\,k_{\sigma}}{k^{2}-\xi\,m^{2}+i\,\varepsilon\,'} \\
- \;& \frac{k_{\mu}\,k_{\sigma}}{k^{2}-\xi\,m^{2}+i\,\varepsilon\,'}=-\,g_{\mu\sigma}+k_{\mu}\,k_{\sigma}\left[\,\left(k^{2}-m^{2}+i\,0\right)\left(k^{2}-\xi\,m^{2}+i\,0\right)\,\right]^{-1} \\
\times \;& \left\{(1-\xi)(k^{2}-m^{2})+k^{2}-\xi\,m^{2}-(1-\xi)\,k^{2}-k^{2}+m^{2}\right\}=-\,g_{\mu\sigma}
\end{aligned}
$$

$$
\frac{-\,i\,k^{2}}{k^{2}-\xi\,m^{2}+i\,\varepsilon\,'}+\frac{i\,\xi\,m^{2}}{k^{2}-\xi\,m^{2}+i\,\varepsilon\,'}=-\,i
$$

This means that we can write

$$
g^{\,BC}\,K_{\,AB}\,iD_{\,CD}^{\,(c)}\,(x-y\,;m,\xi\,)=g_{\,AD}\,\delta^{\,(4)}(x-y)
$$

and thereby

$$
\begin{aligned}
Z_{0}[\,j\,] &= \left\langle T\ \exp\left\{i\int \mathrm{d}^{4}x\,u_{\,A}(x)\,j^{\,A}(x)\right\}\right\rangle_{0} \\
&= \exp\left\{-\,\tfrac{1}{2}\int \mathrm{d}^{4}x\int \mathrm{d}^{4}y\,D_{\,AB}^{\,(c)}(x-y)\,j^{\,A}(x)\,j^{\,B}(y)\right\} \\
&= \mathcal{N}\int \mathfrak{D}\,u_{\,A}\ \exp\left\{(i/\hbar)\,S_{0}[\,u\,]+i\int \mathrm{d}^{4}x\ u^{\,A}(x)\,j_{\,A}(x)\right\}
\end{aligned}
$$

with

$$
Z_{0}[\,0\,]=\mathcal{N}\int \mathfrak{D}\,u_{\,A}\ \exp\left\{(i/\hbar)\,S_{0}[\,u\,]\right\}=1
$$

Turning to the Euclidean formulation in the usual way

$$
x_{4}=ix_{0}\qquad\qquad A_{4}(x_{E})=-\,i\,A_{0}\,(-\,ix_{4},\mathbf{x})
$$

$$
-\,g_{\mu\nu}\,x^{\,\mu}x^{\,\nu}=x_{E}^{\,2}\qquad\qquad -\ g_{\mu\nu}\,A^{\mu}(x)\,A^{\nu}(x)=A_{E\mu}(x_{E})\,A_{E\mu}(x_{E})
$$

et cetera, we find

$$
Z_{E}^{(0)}[\,0\,]=\mathcal{N}\int \mathfrak{D}\,A_{E\mu}\int \mathfrak{D}\,B_{E}\ \exp\left\{-\,S_{E}^{(0)}[\,A_{E\mu}\,,\,B_{E}\,]\right\}=1
$$

where

$$
\begin{aligned}
S_{E}^{(0)}[\,A_{E\mu}\,,\,B_{E}\,] &= \int \mathrm{d}^{4}x_{E}\left\{A_{E\mu}\,\partial_{E\mu}B_{E}+\tfrac{1}{2}\,\xi\,B_{E}^{\,2}\right. \\
&+ \left.\tfrac{1}{2}\,A_{E\mu}\left[\,(m^{2}-\partial_{E}^{2})\,\delta_{\mu\nu}+\partial_{E\mu}\,\partial_{E\nu}\,\right]A_{E\nu}\right\}
\end{aligned}
$$

After re-scaling with an arbitrary mass scale μ

$$A_{E\mu} \;\mapsto\; A'_{E\mu} = \mu\, A_{E\mu}$$

we come to fully dimension-less kinetic operator

$$Z_E^{(0)}[\,0\,] = \exp\left\{-\,\frac{1}{2}\int \mathrm{d}^4x_E\, u_A(x_E)\, K_E^{AB}\, u_B(x_E)\right\} = \mathcal{N}'\,\det\;\|\;K_E\;\|^{-\,1/2}$$

$$K_E \;=\; \mu^{-2}\;\begin{pmatrix} (\,m^2 - \,\partial_\lambda\partial_\lambda)\;\delta_{\lambda\nu} + \partial_\lambda\,\partial_\nu & \mu\,\partial_\rho \\ \mu\,\partial_\rho & \mu^2\xi \end{pmatrix}$$

Hence, for $\xi\,\mu^2 + 4m^2 > 0$, from the $\zeta-$regularisation technique we obtain

$$\begin{aligned}
\mathrm{Tr}\,K_E^{-\,s} \;&\stackrel{\mathrm{def}}{=}\; V_E\,\mu^{2s}\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\;(\mathrm{tr}\,K_E)^{-\,s} \\
&=\; V_E\mu^{2s}\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\;\left(\,4m^2 + 3k^2 + \xi\,\mu^2\,\right)^{-\,s} \\
&=\; \frac{V_E\mu^{2s}}{144\pi^2\Gamma(s)}\int_0^\infty \mathrm{d}t\,t^{\,s-3}\;\exp\left\{-t\left(\,4m^2 + \xi\,\mu^2\,\right)\right\} \\
&=\; \frac{m^4V_E}{9\pi^2}\left(1 + \frac{\xi\mu^2}{4m^2}\right)^2\left(\frac{4m^2}{\mu^2} + \xi\right)^{-\,s}\;\frac{\Gamma(s-2)}{\Gamma(s)} \\
&=\; \frac{m^4V_E}{9\pi^2}\left(1 + \frac{\xi\mu^2}{4m^2}\right)^2\left(\frac{4m^2}{\mu^2} + \xi\right)^{-\,s}\;(s^2 - 3s + 2)^{-1}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Tr}\,K_{\mathrm{E}}^{-\,s} \;&=\; (s^2 - 3s + 2)^{-1}\,\mathrm{Tr}\,K_{\mathrm{E}}^{-\,s} \\
&\times\;\left[\,3 - 2s - \,(s^2 - 3s + 2)\;\ln\left(\frac{4m^2}{\mu^2} + \xi\right)\right]
\end{aligned}$$

and thereby

$$\ln\det\|\;K_{\mathrm{E}}\;\| = \frac{m^4V_E}{9\pi^2}\,\left(1 + \frac{\xi\mu^2}{4m^2}\right)^2\left[\,\frac{1}{2}\;\ln\left(\frac{4m^2}{\mu^2} + \xi\right) - \frac{3}{4}\right]$$

so that the normalization condition $Z_E^{(0)}[\,0\,] = 1$ yields

$$\mathcal{N}' \equiv \exp\left\{\frac{m^4V_E}{18\pi^2}\,\left(1 + \frac{\xi\mu^2}{4m^2}\right)^2\left[\,\frac{1}{2}\;\ln\left(\frac{4m^2}{\mu^2} + \xi\right) - \frac{3}{4}\right]\right\}$$

Finally, the transition to the Minkowski space can be immediately done by simply replacing $V_{\text{euclidean}} \leftrightarrow i\,V_{\text{minkowskian}}$ which leads to the final result

$$
\begin{aligned}
Z_0[\,j\,] &= \left\langle T\,\exp\left\{ i\int \mathrm{d}^4x\, u_A(x)\, j^{\,A}(x) \right\} \right\rangle_0 \\
&= \exp\left\{ -\tfrac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y\, D^{(c)}_{AB}(x-y)\, j^{\,A}(x)\, j^{\,B}(y) \right\} \\
&= \mathcal{N}\int \mathfrak{D}\,u_A\,\exp\left\{ (i/\hbar)\,S_0[\,u\,] + i\int \mathrm{d}^4x\, u^A(x)\, j_A(x) \right\} \\
\mathcal{N}(m,\xi;\mu) &= \text{constant} \times (\,\det K_{AB}\,)^{\frac{1}{2}} \\
&\overset{\text{def}}{=} \exp\left\{ \frac{im^4\,V}{18\pi^2}\left(1+\frac{\xi\mu^2}{4m^2}\right)^2 \left[\frac{1}{2}\ln\left(\frac{4m^2}{\mu^2}+\xi\right)-\frac{3}{4}\right] \right\}
\end{aligned}
$$

Notice that the limit for $m \to 0$ is regular and yields

$$
\lim_{m\to 0} \mathcal{N}(m,\xi;\mu) = \exp\left\{ \frac{i\,\mu^4\,\xi^2}{288\pi^2}\left(\frac{1}{2}\ln\xi - \frac{3}{4}\right) V \right\}
$$

and in the Feynman gauge $\xi = 1$ for the massless theory

$$
\lim_{\xi\to 1^+}\lim_{m\to 0} \mathcal{N}(m,\xi;\mu) = \exp\left\{ -\frac{i\,\mu^4 V}{384\pi^2} \right\}
$$

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Chapter 2

The Feynman Rules

We are now in order to develop perturbation theory, which will provide the fundamental tool to calculate the probability amplitudes for all the physical processes involving relativistic quantized fields in mutual interaction. This is precisely the ambitious final task of the quantum theory of relativistic wave fields. In this aim, I will consider the paradigmatic and simplest cases of the self-interacting real scalar field, together with the Dirac field interacting *á la* Yukawa, the generalizations to any other set of mutually interacting fields of any mass, spin and charges being admittedly straightforward.

2.1 Connected Green's Functions

Let me start from the simplest model, *i.e.* the $\mathbb{Z}_2$ symmetric self-interacting real scalar field theory, which is known as the $\lambda\phi_4^4$ theory. We recall the classical Action for the real scalar relativistic wave field, that is

$$S\,[\,\phi\,] = S_0\,[\,\phi\,] - V\,[\,\phi\,]$$

$$S_0\,[\,\phi\,] = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}\mathbf{x}\left\{\partial_\mu\phi(x)\partial^\mu\phi(x) - \left(\frac{mc}{\hbar}\right)^2\,\phi^2(x)\right\}$$

$$V\,[\phi] = \frac{1}{4!}\cdot\frac{\lambda}{\hbar c}\int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}\mathbf{x}\,\phi^4(t,\mathbf{x})$$

where we have to pay attention to the fact that, owing to dimensional reasons, the reduced Planck constant $\hbar$ already appears to the power minus one in the classical potential $V[\phi]$. Then the Euler-Lagrange field equations read

$$\left[\,\Box + (mc/\hbar)^2\,\right]\phi(x) + (\lambda/6\hbar c)\phi^3(x) = 0 \tag{2.1}$$

which are non linear second order differential equations. It turns out that still nowadays nobody knows any nontrivial solution of the above classical self-interacting field equation. This complete ignorance about the dynamics of a self-interacting classical field has profound consequences on the quantum theory. As a matter of fact, there is no idea on how an interacting quantum field operator could be truly built up. One might assume that

1. the self-interacting quantum scalar field $\phi(x)$ is an operator valued tempered distribution which satisfies (2.1)

2. a conjugate momentum operator valued tempered distribution exists

$$c\,\Pi(x) = \partial_0 \phi(x)$$

   such that the customary canonical equal time commutation relation are still fulfilled: namely,

$$[\,\phi(x), \phi(y)\,]_{x_0=y_0} = 0 \qquad [\,\Pi(x), \Pi(y)\,]_{x_0=y_0} = 0$$

$$[\,\phi(x), \Pi(y)\,]_{x_0=y_0} = i\hbar\,\delta(\mathbf{x}-\mathbf{y})$$

3. a unique non-degenerate cyclic vacuum state $|\,\underline{\mathrm{O}}\,\rangle$ exists such that $\langle\,\underline{\mathrm{O}}\,|\;\phi(x)\,|\,\underline{\mathrm{O}}\,\rangle = 0$ with $\langle\,\underline{\mathrm{O}}\,|\,\underline{\mathrm{O}}\,\rangle = 1$

4. the Green's functions

$$G^{(n)}(x_1, x_2, \ldots, x_n) \equiv \langle\,\underline{\mathrm{O}}\,|\,T\phi(x_1)\phi(x_2)\cdots\phi(x_n)\,|\,\underline{\mathrm{O}}\,\rangle$$

   are well defined tempered distributions

Once again, in spite of many ingenious attempts, ranging from the so called constructive quantum field theory of the early sixties until the lattice Wilsonian formulation of quantum field theory of the late eighties of the 20th century, yet nobody knows if and to what extent the above simple basic assumptions can be actually realized in a concrete operative and non-perturbative framework on the four dimensional Minkowski space.

None the less, what we aim to evaluate is the generating functional for the self-interacting real scalar field theory, which is defined to be

$$\begin{aligned} Z[\,J\,] &= \langle\,\underline{\mathrm{O}}\,|\,T\exp\left\{i\textstyle\int \mathrm{d}x\;\phi(x)\,J(x)\right\}\,|\,\underline{\mathrm{O}}\,\rangle \\ &\stackrel{\mathrm{def}}{=} \sum_{n=0}^{\infty} \frac{i^{\,n}}{n!}\int \mathrm{d}x_1\,J(x_1)\cdots\int \mathrm{d}x_n\,J(x_n) \\ &\times\; \langle\,\underline{\mathrm{O}}\,|\;T\,\phi(x_1)\cdots\phi(x_n)\;|\,\underline{\mathrm{O}}\,\rangle \end{aligned} \qquad (2.2)$$

where $J(x)$ is the classical source with engineering dimensions $[\,J\,] = \mathrm{eV}^3$ in natural units. The vacuum expectation values of chronological ordered products of n scalar field operators at different space-time points are named the $n-$point Green functions of the field theory. By construction, the latter ones can be expressed as functional derivatives of the generating functional

$$\begin{aligned} G^{(n)}(x_1,\cdots,x_n) &\equiv \langle\,\underline{\Omega}\,|\; T\,\phi\,(x_1)\cdots\phi\,(x_n)\;|\,\underline{\Omega}\,\rangle \\ &= (\,-\,i\,)^{\,n}\,\delta^{\,(n)}\,Z\,[\,J\,]\,/\,\delta\,J(x_1)\cdots\delta\,J(x_n)\,\big\rfloor_{J=0} \end{aligned} \tag{2.3}$$

We remind that taking one functional derivative of the generating functional (1.1) we find

$$\frac{\delta\,Z\,[\,J\,]}{i\,\delta\,J(x)} = \big\langle\,T\,\phi(x)\,\exp\,\big\{i\textstyle\int \mathrm{d}y\;\phi(y)\,J(y)\big\}\,\big\rangle_{\underline{\Omega}} \tag{2.4}$$

The generating functional for the Klein-Gordon quantum field – which corresponds to $\lambda = 0$ – has been explicitly computed in the first part of these notes. Moreover, its functional integral representation has been obtained after transition to the Euclidean formulation and the use of the $\zeta-$function regularization: namely,

$$\begin{aligned} Z_0\,[\,J\,] &= \langle\,0\,|\;T\exp\big\{i\textstyle\int \mathrm{d}x\;\phi(x)\,J(x)\big\}\;|\,0\,\rangle \\ &= \exp\big\{-\,\tfrac{1}{2}\textstyle\int \mathrm{d}x\int \mathrm{d}y\;J(x)\,D_F\,(x-y)\,J(y)\big\} \\ &\stackrel{\mathrm{def}}{=} \mathcal{N}_0\int\mathfrak{D}\phi\;\exp\left\{\frac{i}{\hbar}\,S_0\,[\,\phi\,] + i\int \mathrm{d}x\;\phi(x)\,J(x)\right\} \\ S_0\,[\,\phi\,] &= -\,\tfrac{1}{2}\textstyle\int \mathrm{d}x\;\phi(x)\,(\,\Box + m^2 - i\varepsilon\,)\;\phi(x) \\ \mathcal{N}_0 &= \sqrt{\det\,\|\,\{(\hbar/c)^2\,\Box + m^2\}/\mu^2\,\|} \\ &\stackrel{\mathrm{def}}{=} \exp\left\{(\,\mathrm{Volume}\,)\;\frac{i\,m^4c^4}{32\pi^2\hbar^4}\left(\ln\frac{m}{\mu} - \frac{3}{4}\right)\right\} \\ Z_0\,[\,0\,] &= \mathcal{N}_0\int\mathfrak{D}\phi\;\exp\{\,(i/\hbar)\,S_0\,[\,\phi\,]\,\} = 1 \end{aligned}$$

It is immediate to gather that

$$V\,[\,\delta\,/\,i\,\delta\,J\,]\,Z_0\,[\,J\,] = \mathcal{N}_0\int\mathfrak{D}\phi\;V\,[\phi]\,\exp\left\{\frac{i}{\hbar}\,S_0\,[\,\phi\,] + i\int \mathrm{d}x\;\phi(x)\,J(x)\right\}$$

in such a manner that I can formally define the generating functional for the real self-interacting scalar field theory as follows

$$Z\,[\,J\,] \;=\; \langle\,\underline{\Omega}\,|\,T\exp\big\{i\textstyle\int \mathrm{d}x\;\phi(x)\,J(x)\big\}\;|\,\underline{\Omega}\,\rangle$$

$$
\begin{aligned}
&= \mathcal{N}\int \mathfrak{D}\phi\ \exp\left\{\frac{i}{\hbar}\,S[\,\phi\,] + i\int \mathrm{d}x\ \phi(x)\,J(x)\right\} \\
&= \mathcal{N}\int \mathfrak{D}\phi\ \exp\left\{-\ \frac{i}{\hbar}\,V[\phi]\right\}\ \exp\left\{\frac{i}{\hbar}\,S_0[\,\phi\,] + i\textstyle\int \mathrm{d}x\,\phi(x)\,J(x)\right\} \\
&\stackrel{\mathrm{def}}{=} \nu\exp\left\{-\ \frac{i}{\hbar}\,V[\,\delta/i\,\delta\,J\,]\right\} Z_0[\,J\,] \qquad (2.5)
\end{aligned}
$$

where $\nu \equiv \mathcal{N}/\mathcal{N}_0$. Notice that since

$$
Z[\,0\,] = \langle\,\underline{\mathrm{O}}\,|\,\underline{\mathrm{O}}\,\rangle = 1 = Z_0[\,0\,] = \langle\,0\,|\,0\,\rangle
$$

it follows that also the ratio ν must be a function of λ such that

$$
\lim_{\lambda\to 0}\nu(\lambda) = 1
$$

Notice *en passant* that the Fock vacuum state of the Klein-Gordom quantum field and the corresponding unique cyclic state of the self-interacting quantum scalar field are expected to be connected by some unitary transformation

$$
|\,\underline{\mathrm{O}}\,\rangle = \mathcal{U}\,|\,0\,\rangle \qquad \mathcal{U}\mathcal{U}^{\dagger} = \mathbb{I} = \mathcal{U}^{\dagger}\mathcal{U}
$$

To go one step further it is convenient to define

$$
Z[\,J\,] = \exp\{(i/\hbar)\,W[\,J\,]\} \qquad Z_0[\,J\,] = \exp\{(i/\hbar)\,W_0[\,J\,]\}
$$

and thereby

$$
\begin{aligned}
Z[\,J\,] &= Z_0[\,J\,]\ \exp\{-(i/\hbar)\,W_0[\,J\,]\}\ \nu\exp\{(-i/\hbar)V[\,\delta/i\,\delta\,J]\}\,Z_0[\,J\,] \\
&= \nu\ \exp\{(i/\hbar)\,W_0[\,J\,]\}\ \Big[1\ +\ \exp\{-(i/\hbar)\,W_0[\,J\,]\} \\
&\times\ \Big(\exp\{(-i/\hbar)V[\,\delta/i\,\delta\,J]\} - 1\Big)\exp\{\,i\,W_0[\,J\,]\}\ \Big] \\
&= \exp\{(i/\hbar)\,W[\,J\,]\}
\end{aligned}
$$

Taking the logarithm of the above relation we find

$$
\begin{aligned}
W[\,J\,] &= W_0[\,J\,] - i\hbar\ln\nu - i\,\hbar\,\ln\Big(1 + X[\,J\,]\Big) \qquad (2.6) \\
X[\,J\,] &= \mathrm{e}^{-(i/\hbar)W_0[\,J\,]}\left[\mathrm{e}^{-(i/\hbar)V[\,\delta/i\,\delta\,J]} - 1\right]\mathrm{e}^{(i/\hbar)W_0[\,J\,]} \qquad (2.7)
\end{aligned}
$$

By formally expanding $\ln(1+X)$ in Taylor's series, on the one side we obtain

$$
\begin{aligned}
\frac{i}{\hbar}\,W &= \frac{i}{\hbar}\,W_0 + \ln\nu + X - \frac{1}{2}\,X^2 + \frac{1}{3}\,X^3 - \ \cdots \\
&= \frac{i}{\hbar}\,W_0 + \ln\nu + \sum_{k=1}^{\infty}\,(-1)^{k+1}\ \frac{X^k}{k} \qquad (2.8)
\end{aligned}
$$

On the other side we can in turn expand the dimensionless quantity X as a power series of the dimensionless small coupling parameter $0 \leq \lambda < 1$ so that we can write

$$X = \lambda\, X_1 + \lambda^2\, X_2 + \lambda^3\, X_3 + \ \cdots \tag{2.9}$$

in such a manner that we finally come to the formal expansion

$$\begin{aligned}
\frac{i}{\hbar}\, W &= \frac{i}{\hbar}\, W_0 + \ln \nu + \Big(\lambda\, X_1 + \lambda^2\, X_2 + \lambda^3\, X_3\ \cdots\Big) \\
&- \tfrac{1}{2}\Big(\lambda\, X_1 + \lambda^2\, X_2 +\ \cdots\Big)^2 + \tfrac{1}{3}\Big(\lambda\, X_1 + \lambda^2\, X_2 + \ldots\Big)^3 +\ \cdots \\
&= \frac{i}{\hbar}\, W_0 + \lambda\, X_1 + \lambda^2\left(X_2 - \frac{1}{2}\, X_1^2\right) + \lambda^3\left(X_3 - X_2\, X_1 + \frac{1}{3}\, X_1^3\right) \\
&+\ \cdots\cdots\ = \frac{i}{\hbar}\, W_0 + \ln \nu + \sum_{n=1}^{\infty}\ \lambda^n\, Y_n \qquad (2.10) \\
Y_1 &= X_1 \\
Y_2 &= X_2 - \frac{1}{2}\, X_1^2 \\
Y_3 &= X_3 - X_2\, X_1 + \frac{1}{3}\, X_1^3 \\
&\ \vdots
\end{aligned}$$

the dimensionless coefficients Y_n being the so called connected parts of the related quantities X_n $(n \in \mathbb{N})$. Hence, within the perturbative approach, the quantity $Z[\,J\,]$ will provide the generating functional for the full Green's function of the interacting theory, while the functional $W[\,J\,] = -\,i\,\ln Z[\,J\,]$ will generate the connected Green's functions. The attentive reader should certainly catch the analogy with statistical thermodynamics. As a matter of fact, the partition function Z and the Helmoltz free energy F do fulfill a very close relation in units of $k_{\mathrm{B}}T$, k_{B} being the Boltzmann constant and T the (absolute) temperature: namely, $\beta\, F = -\ \ln\, Z\ (\,\beta = 1/k_{\mathrm{B}}T = 1\,)$. This means that, after transition to the Euclidean formulation, we can identify the generating functional $Z_E[\,J_E\,]$ with the canonical partition function and the functional $W_E[\,J_E\,] = -\ \ln\, Z_E[\,J_E\,]$ with the Helmoltz free energy $F(T,V)$ at some given equilibrium temperature, in natural units $\beta = 1$. Hence the Euclidean Green's functions will correspond to the correlation function of the corresponding mechanical system in thermodynamic equilibrium with a heat reservoir at unit temperature. The above analogy does provide the bridge to formulate and develop the statistical theory of the phase transitions in terms of the very same conceptual and mathematical tools which lie on the ground of relativistic quantum field theory.

Turning back to natural units and to the definitions (2.7), (2.8) and (2.9) it is convenient to introduce the following shorter notations: namely,

$$\begin{aligned} X_1[\,J\,] &= -\frac{i}{4!}\,\mathrm{e}^{-iW_0[\,J\,]}\int \mathrm{d}x\,\frac{\delta^4}{\delta J_x^4}\,Z_0[\,J\,] \;=\; Y_1[\,J\,] \\ X_2[\,J\,] &= \frac{1}{2}\left(-\frac{i}{4!}\right)\,\mathrm{e}^{-iW_0[\,J\,]}\int \mathrm{d}y\,\frac{\delta^4}{\delta J_y^4}\,\Big(Z_0[\,J\,]\,X_1[\,J\,]\Big) \\ &\;\vdots \end{aligned} \tag{2.11}$$

$$W_0[\,J\,] = \frac{i}{2}\int \mathrm{d}x\int \mathrm{d}y\;J(x)\,D_F\,(x-y)\,J(y) \equiv \frac{i}{2}\,\langle\,J_x\,D_{xy}\,J_y\,\rangle$$

Hence we can write

$$X_1 = Y_1 = \left(-\frac{i}{4!}\right)\,\mathrm{e}^{-iW_0[\,J\,]}\int \mathrm{d}z\,\frac{\delta^4}{\delta J_z^4}\,\exp\Big\{-\tfrac{1}{2}\,\langle\,J_x\,D_{xy}\,J_y\,\rangle\Big\}$$

Let me evaluate this expression: first we find

$$\mathrm{e}^{-iW_0[\,J\,]}\,\frac{\delta}{\delta\,J_z}\,Z_0[\,J\,] = -\,\langle\,D_{zx}\,J_x\,\rangle$$

$$\mathrm{e}^{-iW_0[\,J\,]}\,\frac{\delta^2}{\delta\,J_z^2}\,Z_0[\,J\,] = -\,D_F\,(0) + \langle\,D_{zx}\,J_x\,\rangle^2$$

$$\mathrm{e}^{-iW_0[\,J\,]}\,\frac{\delta^3}{\delta\,J_z^3}\,Z_0[\,J\,] = 3\,D_F\,(0)\,\langle\,D_{zx}\,J_x\,\rangle - \langle\,D_{zx}\,J_x\,\rangle^3$$

so that we finally obtain

$$\begin{aligned} iY_1[\,J\,] &= \frac{1}{4!}\,\mathrm{e}^{-iW_0[J]}\int \mathrm{d}x\,\frac{\delta^4}{\delta\,J_x^4}\,Z_0[\,J\,] = \frac{1}{8}\int \mathrm{d}x\;D_F^2\,(0) \\ &- \frac{1}{4}\int \mathrm{d}x\,D_F\,(0)\int \mathrm{d}x_1\int \mathrm{d}x_2\,D_F(x-x_1)D_F\,(x-x_2)J(x_1)J(x_2) \\ &+ \frac{1}{24}\int \mathrm{d}x\,\prod_{k=1}^{4}\int \mathrm{d}x_k\;D_F\,(x-x_k)\,J(x_k) \end{aligned}$$

or equivalently but in terms of the shorthand notation

$$\begin{aligned} Y_1[\,J\,] &= \left(-\frac{i}{4!}\right)\int \mathrm{d}x\,\Big(3\,D_F^2\,(0) - 6\,D_F\,(0)\,\langle\,D_{x1}\,D_{x2}\,J_1\,J_2\,\rangle \\ &+ \langle\,D_{x1}\,D_{x2}\,D_{x3}\,D_{x4}\,J_1\,J_2\,J_3\,J_4\,\rangle\Big) \end{aligned} \tag{2.12}$$

which leads to the first order $O(\lambda)$ correction to the generating functional of the connected Green's functions

$$\begin{aligned} W[\,J\,] &\approx -\,i\ln\nu + W_0[\,J\,] - \,i\lambda\,Y_1[\,J\,] = i\ln\mathcal{N}_0 - i\ln\mathcal{N} \\ &+ \; W_0[\,J\,] - \frac{\lambda}{4!}\int \mathrm{d}x\,\left(3\,D_F^2\,(0) - 6\,D_F\,(0)\,\langle\,D_{x1}\,D_{x2}\,J_1\,J_2\,\rangle\right. \\ &+ \; \left.\langle\,D_{x1}\,D_{x2}\,D_{x3}\,D_{x4}\,J_1\,J_2\,J_3\,J_4\,\rangle\right) \end{aligned} \tag{2.13}$$

It follows that the lowest order result for the source-independent quantity

$$\ln Z[\,0\,] = \frac{i}{\hbar}\,W[\,0\,] = 0 = \ln\nu + \lambda\,Y_1\,[\,0\,] + \cdots$$

that drives to

$$\ln\mathcal{N} = \ln\mathcal{N}_0 - \frac{i\lambda}{8}\,D_F^2\,(0)\int \mathrm{d}^4x + \cdots$$

which appears to be a divergent (both infrared and ultraviolet) correction to the normalization constant

$$\mathcal{N}^{\,-1} = \int \mathfrak{D}\phi\;\exp\left\{\frac{i}{\hbar}\,S\,[\,\phi\,]\right\} \tag{2.14}$$

due to the real scalar field self-interaction: namely,

$$\mathcal{N} = \exp\left\{\frac{iVm^{\,4}}{8\pi^{\,2}}\left[\frac{1}{4}\left(\ln\,\frac{m}{\mu} - \frac{3}{4}\right) + \lambda\,\frac{\pi^{\,2}}{m^{\,4}}\,D_F^2\,(0) + \ldots\right]\right\} \tag{2.15}$$

where V stands for the measure of a very large box in Minkowski space - an infrared divergent quantity which is already present in $\mathcal{N}_0$ - while

$$D_F(0) = \frac{i}{(2\pi)^4}\int \frac{\mathrm{d}^4k}{k^2 - m^2 + i\varepsilon}$$

is an ill-defined ultraviolet divergent quantity, as we shall closely analyze later on. Hence, it turns out that even in the perturbative calculation of the normalization constant of the generating functional, even to the lowest order approximation, one has to face the problem of dealing with the so called ultraviolet divergences.

Next we have to calculate X_2. The straightforward generalization of the symbolic Leibnitz chain rule to the functional differentiation reads

$$\begin{aligned} \frac{\delta^{\,n}}{\delta J^{\,n}}\,(\,Z_0\,X_1\,) &= (\,Z_0 + X_1\,)^{(n)} \\ &= Z_0\,\frac{\delta^{\,n}}{\delta J^{\,n}}\,X_1 + \binom{n}{1}\left(\frac{\delta}{\delta J}\,Z_0\right)\,\frac{\delta^{\,n-1}}{\delta J^{\,n-1}}\,X_1 \\ &+ \binom{n}{2}\left(\frac{\delta^2}{\delta J^{\,2}}\,Z_0\right)\,\frac{\delta^{\,n-2}}{\delta J^{\,n-2}}\,X_1 \;+\cdots\;+\left(\frac{\delta^{\,n}}{\delta J^{\,n}}\,Z_0\right)\,X_1 \end{aligned}$$

and in particular

$$\frac{\delta^4}{\delta J^4}\,(\,Z_0\,X_1\,) = Z_0\,\frac{\delta^4}{\delta J^4}\,X_1 + 4\left(\frac{\delta}{\delta J}\,Z_0\right)\,\frac{\delta^3}{\delta J^3}\,X_1$$
$$+\,6\left(\frac{\delta^2}{\delta J^2}\,Z_0\right)\,\frac{\delta^2}{\delta J^2}\,X_1 + 4\left(\frac{\delta^3}{\delta J^3}\,Z_0\right)\,\frac{\delta}{\delta J}\,X_1 + \left(\frac{\delta^4}{\delta J^4}\,Z_0\right)\,X_1$$

Then, from equation (2.11) we readily get

$$X_2 = \frac{1}{2}\,Y_1^{\,2} + Y_2 \qquad\qquad X_2 - \frac{1}{2}\,X_1^{\,2} = Y_2$$

$$\begin{aligned} Y_2 \;&=\; \frac{1}{2}\left(-\,\frac{i}{4!}\right)\,\mathrm{e}^{-\,iW_0}\int \mathrm{d}z\,\left\{\,Z_0\,\frac{\delta^4}{\delta J_z^4} + 4\left(\frac{\delta}{\delta J_z}\,Z_0\right)\,\frac{\delta^3}{\delta J_z^3}\right.\\ &+\;\left. 6\left(\frac{\delta^2}{\delta J_z^2}\,Z_0\right)\,\frac{\delta^2}{\delta J_z^2} + 4\left(\frac{\delta^3}{\delta J_z^3}\,Z_0\right)\,\frac{\delta}{\delta J_z}\,\right\}\,Y_1[\,J\,] \end{aligned} \tag{2.16}$$

Taking the functional derivatives of equation (2.12) we find

$$\begin{aligned} \frac{\delta}{\delta J_z}\,Y_1[\,J\,] \;&=\; -\,\frac{i}{4!}\int \mathrm{d}x\,\Big(-\,12\,D_F\,(0)\,D_F\,(x-z)\,\langle\;D_{x1}\,J_1\,\rangle\\ &+\;4\,D_F\,(x-z)\,\langle\,D_{x1}\,D_{x2}\,D_{x3}\,J_1\,J_2\,J_3\,\rangle\Big)\\ \frac{\delta^2}{\delta J_z^2}\,Y_1[\,J\,] \;&=\; \frac{1}{2i}\int \mathrm{d}x\;D_F^2\,(x-z)\,\Big(-\;D_F\,(0) + \langle\,D_{x1}\,D_{x2}\,J_1\,J_2\,\rangle\Big)\\ \frac{\delta^3}{\delta J_z^3}\,Y_1[\,J\,] \;&=\; -\,i\int \mathrm{d}x\;D_F^3\,(x-z)\,\langle\,D_{x1}\,J_1\,\rangle\\ \frac{\delta^4}{\delta J_z^4}\,Y_1[\,J\,] \;&=\; -\,i\int \mathrm{d}x\;D_F^4\,(x-z) \end{aligned} \tag{2.17}$$

so that explicit term-by-term evaluation yields

$$\frac{1}{2}\left(-\,\frac{i}{4!}\right)\int \mathrm{d}z\;\frac{\delta^4}{\delta J_z^4}\,Y_1[\,J\,] = \;\frac{-\,1}{2(4!)}\int \mathrm{d}x\int \mathrm{d}y\;D_F^4\,(x-y)$$

$$\left(-\,\frac{i}{4!}\right)\frac{2}{Z_0}\int \mathrm{d}z\;\left(\frac{\delta}{\delta J_z}\,Z_0\right)\,\frac{\delta^3}{\delta J_z^3}\,Y_1 \;=$$
$$\frac{4}{2\,(4!)}\int \mathrm{d}x\int \mathrm{d}y\;D_F^3\,(x-y)\,\langle\,D_{x1}\,D_{y2}\,J_1\,J_2\,\rangle$$

$$\frac{1}{2}\left(-\frac{i}{4!}\right)\frac{6}{Z_0}\int \mathrm{d}z\ \left(\frac{\delta^2}{\delta J_z^2}Z_0\right)\ \frac{\delta^2}{\delta J_z^2}Y_1\ =$$
$$\frac{3}{2}\left(-\frac{1}{4!}\right)\int \mathrm{d}x\int \mathrm{d}y\ D_F^2(x-y)\ \times$$
$$\Big(D_F^2(0)-2\,D_F(0)\,\langle\,D_{x1}\,D_{x2}\,J_1\,J_2\,\rangle+\langle\,D_{x1}\,D_{x2}\,D_{y3}\,D_{y4}\,J_1\,J_2\,J_3\,J_4\,\rangle\Big)$$

$$\begin{aligned}
&\frac{1}{2}\left(-\frac{i}{4!}\right)\frac{4}{Z_0}\int \mathrm{d}z\ \left(\frac{\delta^3}{\delta J_z^3}Z_0\right)\ \frac{\delta Y_1}{\delta J_z}\\
=\ &\frac{1}{2}\left(-\frac{i}{4!}\right)^2\cdot 4\cdot 4\int \mathrm{d}x\int \mathrm{d}y\ D_F(x-y)\ \times\\
&\Big(-\ 3\,D_F(0)\,\langle\,D_{x1}\,J_1\,\rangle+\langle\,D_{x1}\,D_{x2}\,D_{x3}\,J_1\,J_2\,J_3\,\rangle\Big)\\
&\times\ \Big(3\,D_F(0)\,\langle\,D_{y2}\,J_2\,\rangle-\langle\,D_{y4}\,D_{y5}\,D_{y6}\,J_4\,J_5\,J_6\,\rangle\Big)
\end{aligned}$$

Thus, the source dependent lowest order contribution take the form

$$\begin{aligned}
Y_2[\,J\,]\ &=\ \frac{1}{12}\int \mathrm{d}x\int \mathrm{d}y\ D_F^3(x-y)\,\langle\,D_{x1}\,D_{y2}\,J_1\,J_2\,\rangle\\
&+\ \frac{1}{8}\int \mathrm{d}x\int \mathrm{d}y\ D_F^2(x-y)D_F(0)\,\langle\,D_{x1}\,D_{x2}\,J_1\,J_2\,\rangle\\
&+\ \frac{1}{8}\int \mathrm{d}x\int \mathrm{d}y\ D_F(x-y)\,D_F^2(0)\,\langle\,D_{x1}\,D_{y2}\,J_1\,J_2\,\rangle\\
&-\ \frac{3}{2\,(4!)}\int \mathrm{d}x\int \mathrm{d}y\ D_F^2(x-y)\ \langle\,D_{x1}\,D_{x2}\,D_{y3}\,D_{y4}\,J_1\,J_2\,J_3\,J_4\,\rangle\\
&-\ \frac{2}{4!}\int \mathrm{d}x\int \mathrm{d}y\ D_F(x-y)\,D_F(0)\ \langle\,D_{x1}\,D_{y2}\,D_{y3}\,D_{y4}\,J_1\,J_2\,J_3\,J_4\,\rangle\\
&+\ \frac{1}{2\,(3!)^2}\int \mathrm{d}x\int \mathrm{d}y\ D_F(x-y)\\
&\times\ \langle\,D_{x1}\,D_{x2}\,D_{x3}\,D_{y4}\,D_{y5}\,D_{y6}\,J_1\,J_2\,J_3\,J_4\,J_5\,J_6\,\rangle
\end{aligned}$$

The resulting connected Green's functions follow from the definitions

$$\begin{aligned}
W[\,J\,]\ &\stackrel{\text{def}}{=}\ \hbar\sum_{n=1}^{\infty}\frac{i^{\,n-1}}{n!}\int \mathrm{d}x_1\,J(x_1)\cdots\int \mathrm{d}x_n\,J(x_n)\\
&\times\ \ G_c^{(n)}(x_1,\cdots,x_n)
\end{aligned}\tag{2.18}$$

$$G_c^{(n)}(x_1,\cdots,x_n)\ \stackrel{\text{def}}{=}\ \frac{i}{\hbar}\,(-\,i\,)^{\,n}\,\delta^{\,(n)}W[\,J\,]/\,\delta\,J(x_1)\cdots\delta\,J(x_n)\,\Big|_{J=0}$$

The perturbative expansion of the $n-$point connected Green's function does follow directly from the series expansion (2.10): namely,

$$G_c^{(2)}(x_1 - x_2) = G_0^{(2)}(x_1 - x_2) - \sum_{k=1}^{\infty} \frac{\lambda^k \, \delta^{(2)} Y_k}{\delta J(x_1) \, \delta J(x_2)} \Bigg]_{J=0} \tag{2.19}$$

while for $n > 2$ we have

$$G_c^{(n)}(x_1, \cdots, x_n) = (-i)^n \sum_{k=1}^{\infty} \lambda^k \, \delta^{(n)} Y_k[J] / \delta J(x_1) \cdots \delta J(x_n) \Bigg]_{J=0} \tag{2.20}$$

Here below we list the lowest order 2-point connected Green's function, which is usually named the full propagator, as well as the 4-point and 6-point connected Green's functions, *viz.*,

$$\begin{aligned}
G_c^{(2)}(x_1 - x_2) &= G_0^{(2)}(x_1 - x_2) - \sum_{n=1}^{\infty} \frac{\lambda^n \, \delta^{(2)} Y_n}{\delta J(x_1) \, \delta J(x_2)} \Bigg]_{J=0} \\
&= D_F(x_1 - x_2) \\
&- \frac{i\lambda}{2} \int \mathrm{d}y \; D_F(x_1 - y) D_F(0) D_F(y - x_2) \\
&- \frac{\lambda^2}{6} \int \mathrm{d}x \int \mathrm{d}y \; D_F(x_1 - x) D_F^3(x - y) D_F(y - x_2) \\
&- \frac{\lambda^2}{4} \int \mathrm{d}x \int \mathrm{d}y \; D_F(x_1 - x) D_F(0) D_F^2(x - y) D_F(x - x_2) \\
&- \frac{\lambda^2}{4} \int \mathrm{d}x \int \mathrm{d}y \; D_F(x_1 - x) D_F^2(0) D_F(x - y) D_F(y - x_2) \\
&+ O(\lambda^3)
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
G_c^{(4)}(x_1, x_2, x_3, x_4) \;&=\; \sum_{n=1}^{\infty} \frac{\lambda^n \, \delta^{(4)} Y_n}{\delta J(x_1) \, \delta J(x_2) \, \delta J(x_3) \, \delta J(x_4)} \Bigg]_{J=0} \\
& -i\lambda \int \mathrm{d}x \; D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - x) D_F(x_4 - x) \\
& - \frac{\lambda^2}{2} \int \mathrm{d}x \int \mathrm{d}y \; D_F^2(x - y) \; \times \\
& \Big[D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - y) D_F(x_4 - y) \\
& \quad + D_F(x_1 - x) D_F(x_3 - x) D_F(x_2 - y) D_F(x_4 - y) \\
& \quad + D_F(x_1 - x) D_F(x_4 - x) D_F(x_2 - y) D_F(x_3 - y) \Big]
\end{aligned}$$

$$- \frac{\lambda^2}{2} \int \mathrm{d}x \int \mathrm{d}y \; D_F\,(x-y) D_F\,(0) \;\times$$
$$\Big[D_F\,(x_1 - x) D_F\,(x_2 - x) D_F\,(x_3 - x) D_F\,(x_4 - y)$$
$$+ \text{ cyclic permutations} \Big] \;+\; O(\lambda^3) \tag{2.22}$$

and finally

$$G_c^{(6)}(x_1, \ldots, x_6) \;=\; -\lambda^2 \int \mathrm{d}x \int \mathrm{d}y \; D_F\,(x-y) \;\times$$
$$\sum_{(\imath\, \jmath\, \kappa)} D_F\,(x_\imath - x)\, D_F\,(x_\jmath - x)\, D_F\,(x_\kappa - x)$$
$$\times \; D_F\,(x_\ell - x)\, D_F\,(x_m - x)\, D_F\,(x_n - x) + \; O(\lambda^3) \tag{2.23}$$

where the sum in the last expression runs over the triples $(\,\imath\,\jmath\,\kappa\,)$ in which $\imath < \jmath < \kappa$, with $\imath,\, \jmath,\, \kappa \,=\, 1, 2, \ldots, 6$, while the triples $(\,\ell\, m\, n\,)$ take the complementary values, *i.e.* , $(\,\ell\, m\, n\,) = (456)$ when $(\,\imath\,\jmath\,\kappa\,) = (123)$ *et cetera.* The remaining Green's functions get no contributions up to this order in λ .

The Fourier transformation of the relativistic wave field field functions in the four dimensional Minkowski space are defined by

$$u_A(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\; \widetilde{u}_A(k)\; \exp\{\,-\,i\,k\cdot x\} \tag{2.24}$$
$$\widetilde{u}_A(k) = \int \mathrm{d}^4 x\; u_A(x)\; \exp\{\,i\,k\cdot x\} \tag{2.25}$$

where the index $A = 1, 2, \ldots, N$ labels, as usual, the components of the real or complex wave field functions.

The Fourier transforms of the $n-$point Green's functions, connected and disconnected, *i.e.* the momentum space Green's functions, are defined by

$$\widetilde{G}_c^{(n)}(k_1, \ldots, k_n)\,(2\pi)^4\,\delta\,(\,k_1 + k_2 + \cdots + k_n\,) \;=$$
$$\int \mathrm{d}x_1 \;\cdots \int \mathrm{d}x_n \; G_c^{(n)}(x_1, \ldots, x_n)\;\; \exp\{\,i\,k_1\cdot x_1 + \cdots + i\,k_n\cdot x_n\,\} \tag{2.26}$$

in such a manner that the total momentum conservation encoded by the $\delta-$distribution does vindicate the space-time translation invariance. If we remember the momentum space scalar Feynman propagator

$$\widetilde{D}_F\,(k) = \widetilde{G}_0^{(2)}(k, -k) = \frac{i\hbar c}{k^2 - (mc/\hbar)^2 + i\varepsilon}$$

it is straightforward to derive the perturbative expansion of the momentum space Green's functions. We find

$$\begin{aligned}
\widetilde{G}_c^{(2)}(k,-k) &= \widetilde{D}_F(k) + \frac{1}{2}(-i\lambda)\,\widetilde{D}_F^2(k)\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{i}{\ell^2-m^2+i\varepsilon}\\
&+ \frac{1}{6}(-i\lambda)^2\,\widetilde{D}_F^2(k)\int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\int\frac{\mathrm{d}^4\ell_2}{(2\pi)^4}\int\frac{\mathrm{d}^4\ell_3}{(2\pi)^4}\\
&\quad (2\pi)^4\,\delta(k-\ell_1-\ell_2-\ell_3)\\
&\times \frac{i}{\ell_1^2-m^2+i\varepsilon}\cdot\frac{i}{\ell_2^2-m^2+i\varepsilon}\cdot\frac{i}{\ell_3^2-m^2+i\varepsilon}\\
&+ \frac{1}{4}(-i\lambda)^2\,\widetilde{D}_F^2(k)\int\frac{\mathrm{d}^4\ell_2}{(2\pi)^4}\,\frac{i}{\ell_2^2-m^2+i\varepsilon}\\
&\times \int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\left(\frac{i}{\ell_1^2-m^2+i\varepsilon}\right)^2\\
&+ \frac{1}{4}(-i\lambda)^2\,\widetilde{D}_F^3(k)\left(\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{i}{\ell^2-m^2+i\varepsilon}\right)^2\\
&+ O(\lambda^3)
\end{aligned}\tag{2.27}$$

$$\begin{aligned}
\widetilde{G}_c^{(4)}(k_1,k_2,k_3,k_4) &= \prod_{a=1}^{4}\frac{i}{k_a^2-m^2+i\varepsilon}\,\Big\{(-i\lambda)\\
&+ \frac{1}{2}(-i\lambda)^2\sum_{j=1}^{4}\frac{i}{k_j^2-m^2+i\varepsilon}\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{i}{\ell^2-m^2+i\varepsilon}\\
&+ \frac{1}{2}(-i\lambda)^2\int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\,\frac{i}{\ell_1^2-m^2+i\varepsilon}\int\frac{\mathrm{d}^4\ell_2}{(2\pi)^4}\,\frac{i}{\ell_2^2-m^2+i\varepsilon}\\
&\times \sum_{(ij)}(2\pi)^4\,\delta(\ell_1+\ell_2-k_i-k_j) + O(\lambda^3)\Big\}
\end{aligned}\tag{2.28}$$

where the sum (ij) runs over the three pairs $(12),(13),(14)$. Finally

$$\begin{aligned}
\widetilde{G}_c^{(6)}(k_1,\dots,k_6) &= \left[\prod_{a=1}^{6}\frac{i}{k_a^2-m^2+i\varepsilon}\right](-i\lambda)^2\\
&\times \sum_{(ijk)}\frac{i}{(k_i+k_j+k_\kappa)^2-m^2+i\varepsilon} + O(\lambda^3)
\end{aligned}\tag{2.29}$$

where again the sum is over the same triples as in equation (2.23).

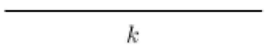
2.2 Self-Interacting Neutral Scalar Field

The above expressions are admittedly rather cumbersome and unwieldy. One urgently needs to device some clever code to generate them to any order in perturbation theory. This is precisely what the genious of Richard Phillips Feynman achieved for us: those rules of correspondence which are universally known as the FEYNMAN RULES

Richard Phillips Feynman
The Theory of Positrons
Phys. Rev. 76, 749 - 759 (1949) [Issue 6 – September 1949]
Space-Time Approach to Quantum Electrodynamics
Phys. Rev. 76, 769 - 789 (1949) [Issue 6 – September 1949]

Keeping in mind the applications to the scattering processes, it is more convenient to express the Feynman rules in the momentum space. Hence, we shall represent the Feynman propagator in momentum space for the neutral spin-less field $\phi(x)$ by a solid line. Moreover, when four lines meet at a vertex, we always understand the momentum flow as running towards the vertex. Then we are left with the Feynman rules in the momentum space to build up the connected $n-$point Green's functions $\widetilde{G}_c^{(n)}(p_1,\ldots,p_n)$ to all orders in perturbation theory for the $\lambda\phi_4^4$ model: namely,

1. For each Feynman propagator $D_F(k) = i\hbar c\,[\,k^2 - (mc/\hbar)^2 + i\varepsilon\,]^{-1}$ draw a solid line with a four-momentum flow
2. For each self-interaction term draw a four-legs vertex and multiply by $-\,i\lambda/\hbar^2c^2$
3. Impose energy-momentum conservation with all momenta incoming at each interaction vertex
$$p_1 + p_2 + p_3 + p_4 \;=\; 0$$
4. Integrate over each internal momentum, *i.e.* each momentum ℓ which is not an argument of the Green's function
$$\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}$$
5. In order to get the contribution to $\widetilde{G}_c^{(n)}(p_1,\ldots,p_n)$, draw all possible arrangements which are topologically nonequivalent, after identification



$$D_F(k) = i\hbar c\,[\,k^2 - (mc/\hbar)^2 + i\varepsilon\,]^{-1}$$

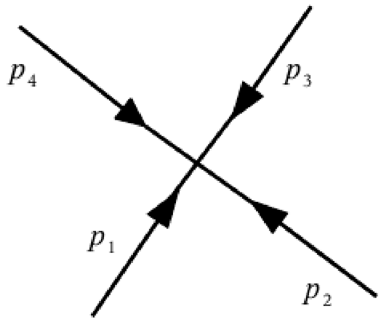


Figure 2.1: $-\,i\lambda/\hbar^2 c^2 \qquad (\,p_1 + p_2 + p_3 + p_4 = 0\,)$

of the so called external legs corresponding to the external momenta $k_1, \ldots, k_n$. The number of ways a given diagram can be drawn is the topological weight w of the diagram. The symmetry factor S of the diagram is equal to its topological weight divided by 4!, *i.e.* $S = w/4!$.

To give an example, consider the so called Tadpole diagram. We need one vertex and three propagators. There are four ways to attach the first propagator from 1 to the vertex, three ways to attach the second propagator from 2 to the vertex. Hence the topological weight is $w = 4 \cdot 3$ and the symmetry factor $S = 4 \cdot 3/4! = \frac{1}{2}$.

Other examples for 2-loops diagrams are illustrated in the figures. Let ℓ the momentum circulating around the closed loop: then the Feynman

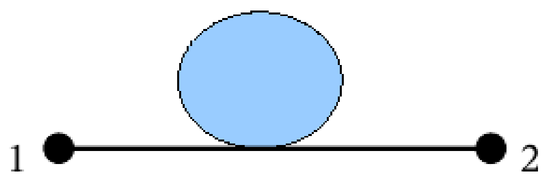


Figure 2.2: the scalar tadpole diagram

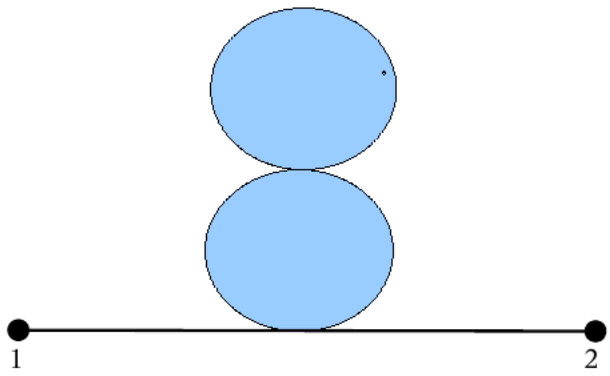


Figure 2.3: Double-Scoop diagram: we need two vertexes; there are four ways to attach the first leg of the first vertex to 1, three ways to link the second leg of the first vertex to 2, four ways to attach the third leg of the first vertex to the first leg of the second vertex and, finally, three ways to tie the fourth leg of the first vertex with anyone of the remaining three legs of the second vertex. Hence the topological weight is $4 \cdot 3 \cdot 4 \cdot 3$ so that the symmetry factor is $1/4$

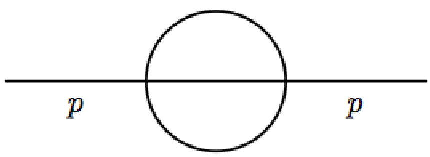


Figure 2.4: Saturn diagram: we need two vertexes; there are four ways to attach the first leg of the first vertex to 1, four ways to link the first leg of the second vertex to 2; this leaves three ways to connect with an internal leg the two vertexes and two further ways to tie the internal lines. Hence the topological weight is $4 \cdot 4 \cdot 3 \cdot 2$ so that the symmetry factor is $1/6$

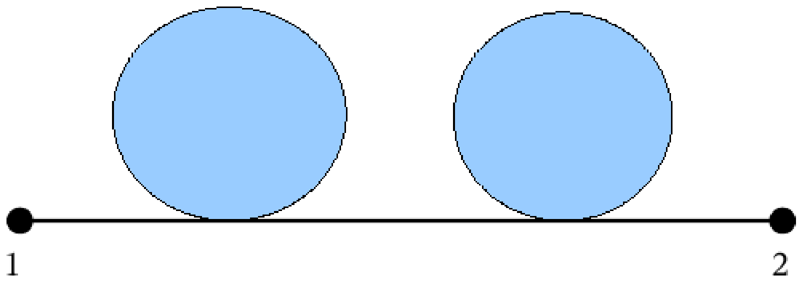


Figure 2.5: Glasses diagram: we need two vertexes; there are four ways to attach the first leg of the first vertex to 1, four ways to link the first leg of the second vertex to 2; this leaves three legs for each vertex free to be tied together. Hence the topological weight is $4 \cdot 3 \cdot 4 \cdot 3$ so that the symmetry factor is 1/4

rules give

$$\frac{1}{2}(-i\lambda)\,\frac{i}{k^2-m^2+i\varepsilon}\quad\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{i}{\ell^2-m^2+i\varepsilon}\cdot\frac{i}{k^2-m^2+i\varepsilon}$$

Thus we see that the Feynman rules reduced essentially the problem of setting up the perturbative expansion of the Green's functions to that faced by a child assembling a *Lego* set. More important, the structure of the propagator and of the vertex, *i.e.* the main tools of the game, can be directly read off of the classical Lagrange density. Consider in fact the functional integral representation of the generating functional

$$\begin{aligned} Z[J] &= \exp\{(i/\hbar)W[J]\} \\ &= \mathcal{N}\int\mathfrak{D}\phi\ \exp\left\{\frac{i}{\hbar}S_0[\phi]-\frac{i}{\hbar}V[\phi]+i\int\mathrm{d}^4x\,\phi(x)J(x)\right\} \end{aligned} \tag{2.30}$$

and let me focus on the first two addenda in the exponent, *viz.*,

$$\frac{i}{\hbar}S_0[\phi]=\frac{i}{2\hbar c}\int\mathrm{d}^4x\left\{g^{\mu\nu}\partial_\mu\phi(x)\,\partial_\nu\phi(x)-\frac{m^2c^2}{\hbar^2}\phi^2(x)\right\}$$

$$-\frac{i}{\hbar}V[\phi]=\frac{-i\lambda}{4!\hbar^2c^2}\int\mathrm{d}^4x\ \phi^4(x)$$

Taking the Fourier transform after a partial integration we come to

$$\frac{i}{\hbar}S_0[\phi]=\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\widetilde{\phi}(-k)\,\frac{i}{\hbar c}\left(k^2-m^2c^2/\hbar^2\right)\widetilde{\phi}(k)$$

$$
-\frac{i}{\hbar}\,V\,[\,\phi\,] = \frac{-\,i\,\lambda}{4!\hbar^2c^2}\ \prod_{\jmath=1}^{4}\int \frac{\mathrm{d}^4k_\jmath}{(\,2\pi\,)^4}\ \widetilde{\phi}(k_\jmath)\,(\,2\pi\,)^4\ \delta\,(\,k_1+k_2+k_3+k_4\,)
$$

whence it appears quite manifest that the momentum space Feynman rules 1.– 3. can be directly read off of the classical Action. More precisely,

- the Feynman propagator is just equal to the opposite of the inverse of the kinetic operator, the inverse being actually defined by the causal prescription, *viz.*

$$
\widetilde{D}_F(k)\ =\ (-1)\ \left[\,\frac{i}{\hbar c}\,\left(k^2-\frac{m^2c^2}{\hbar^2}+i\varepsilon\right)\right]^{-1}
$$

  in the present scalar field case

- the vertex $-i\,\lambda/\hbar^2c^2$ as well as the overall momentum conservation are evidently encoded in the classical interaction potential in momentum space.

2.3 Yukawa Theory

The Feynman rules for spinor fields can be rather easily gathered from the paradigmatic simple model known as the Yukawa theory

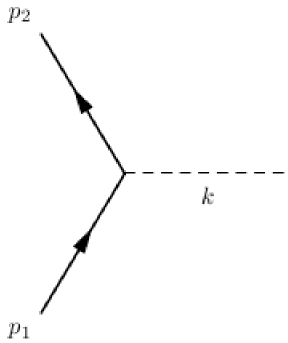
Hideki Yukawa
On the Interaction of Elementary Particles. I
Supplement of the Progress of Theoretical Physics No. 1 (1935) pp. 1-10

The Yukawa model involves a real scalar field interacting with a complex Dirac spinor field, the classical Lagrangian being given by

$$
\mathcal{L}_{\mathrm{Yukawa}} = \tfrac{1}{2}\,g^{\,\mu\nu}\,\partial_\mu\,\phi\,\partial_\nu\,\phi - \frac{m^2c^2}{2\,\hbar^2}\,\phi^{\,2} + \tfrac{1}{2}\,\bar\psi\,\gamma^{\,\mu}\,i\,\overleftrightarrow{\partial}_{\!\mu}\psi - \left(\frac{Mc}{\hbar}+\frac{y\,\phi}{\sqrt{\hbar c}}\right)\,\bar\psi\,\psi
$$

where y is the dimensionless Yukawa coupling parameter. Notice that the spinor field in the physical C.G.S. system of units has engineering dimensions given by $[\,\psi\,] = \mathrm{eV}^{\frac{1}{2}}\ \mathrm{cm}^{-1}$. Turning to the momentum space we find

$$
\begin{aligned}
\frac{i}{\hbar}\,S_{\mathrm{Yukawa}}\,[\,\phi,\psi,\bar\psi\,]\ &=\ \frac{i}{\hbar c}\int \mathrm{d}^4x\ \mathcal{L}_{\mathrm{Yukawa}}\,[\,\phi,\psi,\bar\psi\,] \\
=\ \frac{i}{\hbar c}\int \frac{\mathrm{d}^4k}{(2\pi\,)^4}\ \tfrac{1}{2}\,\widetilde{\phi}(-k)\,\left(k^2-\frac{m^2c^2}{\hbar^2}\right)\,\widetilde{\phi}(k)
\end{aligned}
$$



$$-\,iy \qquad\qquad (\,p_1 + k - p_2 = 0\,)$$

$$
\begin{aligned}
&+\ \frac{i}{\hbar c}\int\frac{\mathrm{d}^4 p}{(\,2\pi\,)^4}\ \widetilde{\psi}^{\,\dagger}(p)\,\gamma^0\left(\not p - \frac{Mc}{\hbar}\right)\widetilde{\psi}\,(p) \\
&-\ \frac{iy}{\sqrt{(\hbar c)^3}}\int\frac{\mathrm{d}^4 p_1}{(\,2\pi\,)^4}\int\frac{\mathrm{d}^4 p_2}{(\,2\pi\,)^4}\,\widetilde{\phi}(p_2 - p_1)\widetilde{\psi}^{\,\dagger}(p_2)\gamma^0\widetilde{\psi}(p_1)
\end{aligned}
$$

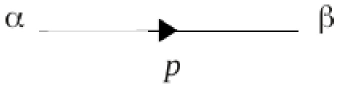
in such a manner that the scalar and spinor propagators correspond, as explained before, to the opposite of the inverse of the Klein-Gordon and Dirac kinetic operators respectively. On the other side, the vertex is clearly given by $-iy/\sqrt{(\hbar c)^3}$, the energy momentum conservation being here $k + p - q \;=\; 0\,$. Several comments are now in order.

1. It is worthwhile to notice that the direction of the energy momentum on a fermion line is always significant. On a fermion propagator, the 4-momentum must be assigned in the direction of the charge flow. Here, the momentum of a particle is always taken towards the vertex, the particles being assumed to carry negative elementary charge $-\,e$ with $e\,>\,0\,$. Hence, the direction of the momentum is always understood by definition to follow the negative charge flow. Thus, the momentum of the antiparticle will go out of the vertex, which corresponds to the term $-\,q$ in the argument of the energy momentum $\delta-$distribution.

2. The diagrams of the Yukawa theory never exhibit topological weights nor symmetry factors, since the three fields $(\,\phi\,\bar\psi\,\psi\,)$ in the interaction Lagrangian can not be interchanged one another.

3. Finally, the Graßmann nature of the spinor field is reflected in one crucial change in the Feynman rules : whenever a closed fermion line, which is usually named a fermion cycle or more commonly a fermion loop, appears in a diagram, then one must multiply the diagram by a factor (-1) for each fermion loop of the diagram. This latter rule can be illustrated by the following enlightening example.

$$D_F(k) = i\,[\,k^2 - m^2 + i\varepsilon\,]^{-1}$$



$$S^F_{\alpha\beta}(p) = i(\not p + M)_{\alpha\beta}\,(p^2 - M^2 + i\varepsilon)^{-1}$$

2.3.1 Yukawa Determinant

Consider the generating functional for the Yukawa field theory

$$\begin{aligned} Z[\,\zeta\,,\,\bar\zeta\,,\,J\,] \;&\stackrel{\rm def}{=}\; \mathcal{N}\int\mathfrak{D}\phi\int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi\;\exp\left\{\,i\,S_{\rm Yukawa}[\,\bar\psi\,,\,\psi\,,\,\phi\,]\,\right\} \\ &\times\;\exp\left\{i\!\int{\rm d}x\,\left[\,\bar\zeta(x)\,\psi(x) + \bar\psi(x)\,\zeta(x) + J(x)\,\phi(x)\,\right]\right\} \end{aligned} \tag{2.31}$$

By making use of the same trick I have employed before in the case of the perturbative definition of the generating functional for the self-interacting real scalar field theory I can write

$$\begin{aligned} Z[\,J\,,\,\zeta\,,\,\bar\zeta\,] \;&=\; \exp\{-\,i\,V\,[\,\delta/i\,\delta J\,,\,\delta/i\,\delta\bar\zeta\,,\,\delta/i\,\delta\zeta\,]\,\}\;Z_0[\,\zeta\,,\,\bar\zeta\,,\,J\,] \\ V \;&=\; y\int{\rm d}x\;\left(\delta/i\,\delta J(x)\right)\left(\delta/i\,\delta\zeta_\alpha(x)\right)\left(\delta/i\,\delta\bar\zeta_\alpha(x)\right) \end{aligned} \tag{2.32}$$

while

$$Z_0[\,\zeta\,,\,\bar\zeta\,,\,J\,] = Z_0^{\rm F}[\,\zeta\,,\,\bar\zeta\,]\cdot Z_0^{\rm B}[\,J\,] = \exp\{\,i W_0^{\rm B}[\,J\,] + i W_0^{\rm F}[\,\zeta\,,\,\bar\zeta\,]\,\}$$

$$Z_0^{\rm F}[\,\zeta\,,\,\bar\zeta\,] = \exp\left\{\,-\int{\rm d}x\int{\rm d}z\;\bar\zeta(x)\,S_F(x-z)\,\zeta(z)\right\}$$

$$Z_0^{\rm B}[\,J\,] = \exp\left\{\,-\tfrac{1}{2}\int{\rm d}x\int{\rm d}z\;J(x)\,D_F(x-z)\,J(z)\right\}$$

It is important to remark that in the definition (2.32), where I have explicitly written the repeated summed spinor indexes for the sake of clarity, the order of the anti-commuting Graßmann-like functional derivatives is crucial.

Now, to our task, it turns out to be convenient to rewrite the generating functional in the following equivalent form: namely,

$$\begin{aligned} Z[\,J\,,\,\zeta\,,\,\bar\zeta\,] \;&=\; \mathcal{N}'\int\mathfrak{D}\phi\;\exp\left\{\,i\,S_0[\,\phi\,] + i\int{\rm d}z\;J(z)\,\phi(z)\right\} \\ \times\;\exp&\left\{iy\!\int{\rm d}x\,\phi(x)\left(\,\delta^{(2)}/\delta\zeta_x\,\delta\bar\zeta_x\right)\right\}\;Z_0^{\rm F}[\,\zeta\,,\,\bar\zeta\,] \end{aligned}$$

where $S_0[\phi]$ denotes the classical Action for the real Klein-Gordon field. As a consequence, if we set

$$\exp\{-iV_\phi[\delta/i\delta\bar\zeta\,,\,\delta/i\delta\zeta]\}\ \equiv$$
$$\exp\left\{-iy\int \mathrm{d}x\ \phi(x)\left(\delta^{(2)}/i\,\delta\zeta_x\,i\,\delta\bar\zeta_x\right)\right\}$$
$$Z_\phi^{\mathrm{F}}[\zeta\,,\bar\zeta]\ =\ \exp\{-iV_\phi[\delta/i\delta\bar\zeta\,,\,\delta/i\delta\zeta]\}\ Z_0^{\mathrm{F}}[\zeta\,,\bar\zeta] \tag{2.33}$$

we obtain the functional integral representation

$$Z_\phi^{\mathrm{F}}[\zeta\,,\bar\zeta]\ =\ \mathcal{N}_\phi\int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi$$
$$\exp\left\{i\int \mathrm{d}x\ \bar\psi(x)\left[\tfrac12\gamma^\mu\,i\overleftrightarrow{\partial}_\mu - M - y\,\phi(x)\right]\psi(x)\right\}$$
$$\times\ \exp\left\{i\int \mathrm{d}x\ \left[\bar\zeta(x)\,\psi(x)+\bar\psi(x)\,\zeta(x)\right]\right\} \tag{2.34}$$

with

$$Z_\phi^{\mathrm{F}}[0\,,0]\ =\ \mathcal{N}_\phi\int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi$$
$$\exp\left\{i\int \mathrm{d}x\ \bar\psi(x)\left[\tfrac12\gamma^\mu\,i\overleftrightarrow{\partial}_\mu - M - y\,\phi(x)\right]\psi(x)\right\}$$
$$\stackrel{\mathrm{def}}{=}\ \mathcal{N}_\phi\det\|\,i\partial\!\!\!/ - M - y\,\phi\,\| \tag{2.35}$$

the latter definition being understood, as usual, up to the Wick rotation to the Euclidean space. Notice that the constant $\mathcal{N}_\phi$ is conveniently fixed by the requirement that in the limit $\phi\to 0$ we recover $Z_0^{\mathrm{F}}[0\,,0]=1$.

As a consequence, we eventually come to the symbolic equality

$$\begin{aligned}\frac{\det\|\,i\partial\!\!\!/ - M - y\,\phi\,\|}{\det\|\,i\partial\!\!\!/ - M\,\|} &= \det\|\,\mathbb{I}-y\,(i\partial\!\!\!/ - M)^{-1}\phi\,\| \qquad (2.36)\\ &= \exp\{\,\mathrm{Tr}\ \ln\|\,\mathbb{I}-y\,(i\partial\!\!\!/ - M)^{-1}\phi\,\|\,\}\\ &= \exp\left\{(-1)\sum_{n=1}^{\infty}\frac1n\ \mathrm{Tr}\,[\,(i\partial\!\!\!/ - M)^{-1}\,y\,\phi\,]^n\right\}\end{aligned}$$

in which we understand

$$\langle x_1|\,(i\partial\!\!\!/ - M)^{-1}|x_2\rangle\ =\ \frac1i\,S_F(x_1-x_2)$$

while the symbol [Tr] indicates the sum over spinor indexes and integration over space-time coordinates. After setting

$$X_\phi^n = (-1)\,\mathrm{Tr}\,[\,(i\partial\!\!\!/ - M)^{-1}\,y\,\phi\,]^n$$

in such a manner that we can write

$$\ln\det \| \, i\,\partial\!\!\!/ - M - y\,\phi\, \| = \ln\det \| \, i\,\partial\!\!\!/ - M \, \| + \sum_{n=1}^{\infty} n^{-1}\, X_\phi^n \tag{2.37}$$

explicit evaluation for $n = 1$ yields

$$\begin{aligned} X_\phi^1 &= (-1)\,\mathrm{Tr}\,[\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,y\,\phi\,] \\ &= -y \int \mathrm{d}x_1 \, \langle\, x_1 \,|\, \mathrm{tr}\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,\phi\,|\, x_1\,\rangle \\ &= -y \int \mathrm{d}x_1 \int \mathrm{d}x_2 \, \langle\, x_1 \,|\, \mathrm{tr}\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,|\, x_2\rangle\,\langle\, x_2\,|\,\phi\,|\,x_1\,\rangle \\ &= iy \int \mathrm{d}x_1 \int \mathrm{d}x_2 \; \phi(x_1)\,\delta\,(x_1 - x_2)\; \mathrm{tr}\, S_F\,(\,x_1 - x_2\,) \\ &= iy \int \mathrm{d}x_1 \; \phi(x_1)\; \mathrm{tr}\, S_F\,(\,x_1 - x_1\,) \\ &= iy \int \mathrm{d}x_1 \; \phi(x_1)\; \mathrm{tr}\, S_F\,(\,0\,) \; \stackrel{\mathrm{def}}{=} \; iy\;\mathrm{Tr}\,(\,\phi\, S_F\,) \end{aligned} \tag{2.38}$$

in which the symbol [tr] denotes the sum over spinor indexes. The next term can be handled in a quite similar way by making repeatedly use of the completeness relation

$$\int \mathrm{d}x \; |\,x\,\rangle\,\langle\,x\,| \; = \; \mathbb{I}$$

Then we obtain

$$\begin{aligned} X_\phi^2 &= (-\,iy\,)^2 \int \mathrm{d}x_1 \int \mathrm{d}x_2 \; \phi\,(x_1)\,\phi\,(x_2) \\ &\times \;\; (-1)\,\mathrm{tr}\; S_F\,(\,x_2 - x_1\,)\, S_F\,(\,x_1 - x_2\,) \end{aligned} \tag{2.39}$$

It is convenient to introduce the center of mass and relative coordinates

$$\bar{x} = \frac{1}{2}\,(\,x_1 + x_2\,) \qquad\qquad x = x_1 - x_2 \qquad\qquad \frac{\partial\,(\,x_1, x_2\,)}{\partial\,(\,X, x\,)} = 1$$

so that

$$\begin{aligned} X_\phi^2 &= y^2 \int \mathrm{d}\bar{x} \int \mathrm{d}x \; \phi\,(\,\bar{x} + x/2\,)\,\phi\,(\,\bar{x} - x/2\,)\; \mathrm{tr}\; S_F\,(-x)\, S_F\,(\,x\,) \\ &= y^2 \int \mathrm{d}\bar{x} \int \mathrm{d}x \int \frac{\mathrm{d}\,\ell}{(2\pi)^4}\; \widetilde{\phi}(\ell) \int \frac{\mathrm{d}\,k}{(2\pi)^4}\; \widetilde{\phi}(k) \\ &\times \;\; \exp\{\,-\,i\bar{x}\cdot(\ell + k) - \,ix\cdot(\ell - k)/2\} \int \frac{\mathrm{d}\,p}{(2\pi)^4} \int \frac{\mathrm{d}\,q}{(2\pi)^4} \end{aligned}$$

$$
\begin{aligned}
\times & \quad \exp\{ i\,(p-q)\cdot x\}\ \mathrm{tr}\left[\frac{i}{\not p - M + i\varepsilon}\cdot\frac{i}{\not q - M + i\varepsilon}\right] \\
= & \quad -\,y^2\int\frac{\mathrm{d}k}{(2\pi)^4}\ \widetilde{\phi}(k)\,\widetilde{\phi}(-k)\int\frac{\mathrm{d}p}{(2\pi)^4}\int\frac{\mathrm{d}q}{(2\pi)^4} \\
\times & \quad \frac{\mathrm{tr}\,[\,(\not p + M\,)\,(\not q + M\,)\,]}{(\,p^2 - M^2 + i\varepsilon\,)\,(\,q^2 - M^2 + i\varepsilon\,)}\ (2\pi)^4\,\delta\,(\,k+p-q\,) \\
= & \quad (-\,iy\,)^2\int\frac{\mathrm{d}k}{(2\pi)^4}\ \widetilde{\phi}(k)\,\widetilde{\phi}(-k) \\
\times & \quad (-1)\int\frac{\mathrm{d}p}{(2\pi)^4}\ \mathrm{tr}\left[\frac{i}{\not p - M + i\varepsilon}\cdot\frac{i}{\not p + \not k - M + i\varepsilon}\right]
\end{aligned}
$$

the very last line corresponding to a fermion loop with two propagators.

- Hence, whenever a fermion loop appears, it always involves a trace operation over the spinor indexes as well as a multiplication by a factor (-1), the ultimate reason of which being due to the anti-commuting Graßmann-like nature of the fermion fields. This is the Feynman rule for fermion loops.

A little thought will convince the reader [1] that the iteration of the above machinery leads to the result

$$
\begin{aligned}
X_\phi^n &= (-\,iy\,)^n\int\frac{\mathrm{d}k_1}{(2\pi)^4}\ \widetilde{\phi}(k_1)\ \cdots\int\frac{\mathrm{d}k_{n-1}}{(2\pi)^4}\ \widetilde{\phi}(k_{n-1}\,)\,\widetilde{\phi}(k_1+\cdots+k_{n-1}\,) \\
&\times(-1)\int\frac{\mathrm{d}p}{(2\pi)^4}\ \mathrm{tr}\left[\,\widetilde{S}_F\,(p)\,S_F\,(p+k_1\,)\ \cdots\ S_F\,(p+k_1+\cdots+k_{n-1}\,)\right] \\
&= (-1)\ (-\,iy\,)^n\ \mathrm{Tr}\,(\,\phi\,S_F\,)^n \qquad (\,k_n = k_1+\cdots+k_{n-1}\,) \qquad (2.40)
\end{aligned}
$$

which corresponds to a fermion loop with $n-$external legs associated to the scalar field vertexes, as depicted in the figure. As a consequence, we can see by direct inspection that the symbolic equality (2.36) can be understood in a perturbative sense as a power series in the Yukawa coupling, the $n-$th coefficient of which does involve the 1-loop fermion boxes with $n-$external scalar legs with momenta $k_1, k_2, \ldots, k_n:$ namely,

$$
\frac{\det\ \|\ i\not\partial - M - y\,\phi\ \|}{\det\ \|\ i\not\partial - M\ \|}\ =\ \exp\left\{(-1)\sum_{n=1}^{\infty}\ \frac{1}{n}\ (-\,iy\,)^n\,\mathrm{Tr}\,(\,\phi\,S_F\,)^n\right\}
$$

[1] The trick is to introduce a change of variables to a new system of coordinates with the center of mass and $(n-1)$ relative coordinates with a unit Jacobian.

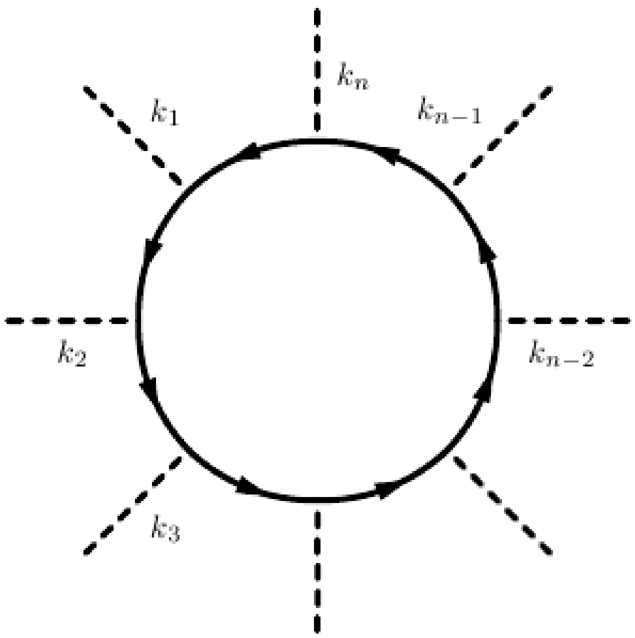


Figure 2.6: The $n-$point Yukawa 1-loop box yielding the factor (-1)

or even

$$\det \parallel i\,\partial\!\!\!/ - M - y\,\phi \parallel = \det \parallel i\,\partial\!\!\!/ - M \parallel \exp\{\,\mathrm{Tr}\,\ln(\mathbb{I} + iy\,\phi\,S_F\,)\,\}$$

as indeed expected by taking into account the suggestive symbolic relation $(\,i\,\partial\!\!\!/ - M\,)^{-1} = -\,iS_F$ and the cyclic property of the [Tr] operation $\mathrm{Tr}\,(\phi S_F) = \mathrm{Tr}\,(S_F\phi)$. Notice that the first four 1-loop fermion cycles of the perturbation expansion, that is $n = 1, 2, 3, 4$, do appear to be ultraviolet divergent by naïve power counting: namely,

$$iy\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \mathrm{tr}\,\widetilde{S}_F(p)$$

$$y^2\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \mathrm{tr}\,\Big[\,\widetilde{S}_F(p)\,\widetilde{S}_F(p+k)\,\Big]$$

$$-\,iy^3\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \mathrm{tr}\,\Big[\,\widetilde{S}_F(p)\,\widetilde{S}_F(p+k_1)\,\widetilde{S}_F(p+k_1+k_2)\Big]$$

$$-\,y^4\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \mathrm{tr}\,\Big[\widetilde{S}_F(p)\,\widetilde{S}_F(p+k_1)\,\widetilde{S}_F(p+k_1+k_2)\,\widetilde{S}_F(p+k_1+k_2+k_3)\Big]$$

with $\widetilde{S}_F(p) = i(\not p - M + i\varepsilon)^{-1}$. Hence, they call for some regularization method in order to be properly defined. I shall deal at length with this in the sequel.

`Effective Action.` I conclude this paragraph by introducing the very customary and useful concepts and definitions of *effective Action* and *effective Potential* related to the Yukawa determinant. These latter encode the effects of the coupling of quantized matter, described by fermionic fields, on some external bosonic fields – typically scalar or vector. We can start from (1.61) that gives the effective Action for the (free) Dirac operator

$$\det \| \, (i\not\partial - M)\,/\mu \, \| \equiv \exp\left\{ i\,\Gamma_{\rm eff}^{(0)} \right\} \tag{2.41}$$

$$\Gamma_{\rm eff}^{(0)} = \frac{VM^4}{8\pi^2}\left(\ln \frac{M}{\mu} - \frac{3}{4} \right) \tag{2.42}$$

Thus we can suitably set

$$\det \| \, (i\not\partial - M - y\,\phi)\,/\mu \, \| \equiv \exp\{ i\,\Gamma_{\rm eff}[\phi] \} \tag{2.43}$$

$$\Gamma_{\rm eff}[\phi] = \Gamma_{\rm eff}^{(0)} - i\,{\rm Tr}\,\ln(\mathbb{I} + iy\,\phi\,S_F) = \Gamma_{\rm eff}^{(0)} + \sum_{n=1}^{\infty} \Gamma_{\rm eff}^{(n)}[\phi] \tag{2.44}$$

$$\Gamma_{\rm eff}^{(n)}[\phi] = -\frac{i}{n}\,X_{\phi}^{\,n} = \frac{i}{n}\,(-iy)^n\,{\rm Tr}\,(\phi\,S_F)^n \qquad (n \in \mathbb{N}) \tag{2.45}$$

In the case of a constant external field configuration

$$\phi(x) = \text{constant} \equiv \varphi$$

we can write the constant effective Action, which turns out to be an ordinary function,

$$\begin{aligned} \Gamma_{\rm eff}(\varphi) &= -i\,\ln\det \| \,(i\not\partial - M - y\,\varphi)/\mu \,\| \\ &= \Gamma_{\rm eff}^{(0)} - i\,{\rm Tr}\,\ln[\,\mathbb{I} + iy\,\varphi\,S_F\,] \\ &= (M + y\,\varphi)^4\,\frac{V}{8\pi^2}\left[\ln(M + y\,\varphi) - \ln\mu - \tfrac{3}{4}\right] \end{aligned} \tag{2.46}$$

and the quantity

$$\mathcal{V}_{\rm eff}(\varphi) = (M + y\,\varphi)^4\,\frac{1}{8\pi^2}\left(\frac{3}{4} - \ln\frac{M + y\,\varphi}{\mu} \right) \tag{2.47}$$

$$\Gamma_{\rm eff}(\varphi) = -\int {\rm d}^4x\,\mathcal{V}_{\rm eff}(\varphi) = -V_4\,\mathcal{V}_{\rm eff}(\varphi) = -(2\pi)^4\,\delta(0)\,\mathcal{V}_{\rm eff}(\varphi) \tag{2.48}$$

that is called the *effective Potential* of the Yukawa determinant. Of course we find

$$\Gamma_{\rm eff}(0) = \Gamma_{\rm eff}^{(0)}$$

while we have

$$\begin{aligned} \Gamma_{\rm eff}^{(1)}(\varphi) &= -i\,X_{\varphi}^{(1)} = y\varphi\int{\rm d}x\,{\rm tr}\,S_F(0) = y\varphi V_4 \int \frac{{\rm d}^4p}{(2\pi)^4}\,{\rm tr}\,\widetilde{S}_F(p) \\ &= iy\varphi V_4\,M \int \frac{{\rm d}^4p}{(2\pi)^4}\,{\rm tr}\,\mathbb{I}\,\left(p^2 - M^2 + i\varepsilon\right)^{-1} \equiv \varphi\,\frac{\rm d}{{\rm d}\varphi}\,\Gamma_{\rm eff}(0) \end{aligned}$$

Hence in order to match with the McLaurin expansion of the $\zeta-$function regularization result (2.46) we must assign a finite value to a formally divergent integral: namely,

$$\frac{M^2}{2\pi^2}\left(\ln\frac{M}{\mu}-\frac{1}{2}\right)\equiv i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{tr}\,\mathbb{I}\,\left(p^2-M^2+i\varepsilon\right)^{-1}\Big\rfloor_{\text{finite part}}$$

Next we get

$$\begin{aligned}\Gamma_{\text{eff}}^{(2)}(\varphi) &= -\frac{i}{2}\,X_\varphi^{\,2}=\frac{i}{2}\,(-iy\,\varphi)^2\,\mathrm{Tr}\,S_F^2\\ &= \frac{i}{2}\,(y\varphi)^2\,V_4\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{tr}\,\widetilde{S}_F^2(p)\end{aligned}$$

and, once again, if we want to reproduce the output of the Zeta function regularization

$$\Gamma_{\text{eff}}^{(2)}(\varphi)=\frac{V_4}{4\pi^2}\,(y\,\varphi M)^2\left(3\ln\frac{M}{\mu}-\frac{1}{2}\right)=\tfrac{1}{2}\,\varphi^2\,\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\,\Gamma_{\text{eff}}(0)$$

we have to set

$$\frac{M^2}{2\pi^2}\left(3\ln\frac{M}{\mu}-\frac{1}{2}\right)\equiv i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{tr}\,\widetilde{S}_F(p)\,\widetilde{S}_F(p)\Big\rfloor_{\text{finite part}} \tag{2.49}$$

We can thereby proceed in the same way to give a finite value to the two further formally divergent quantities: namely,

$$\begin{aligned}\Gamma_{\text{eff}}^{(3)}(\varphi) &= -\frac{i}{3}\,X_\varphi^{\,3}=\frac{i}{3}\,(-iy\varphi)^3\,\mathrm{Tr}\,S_F^3\\ &= -\frac{1}{3}\,(y\varphi)^3\,V_4\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{tr}\,\widetilde{S}_F^3(p)\Big\rfloor_{\text{finite part}}\\ &= \frac{V_4}{2\pi^2}\,(y\varphi)^3\,M\left(\ln\frac{M}{\mu}+\frac{1}{3}\right)=\tfrac{1}{6}\,\varphi^3\,\frac{\mathrm{d}^3}{\mathrm{d}\varphi^3}\,\Gamma_{\text{eff}}(0)\\ \Gamma_{\text{eff}}^{(4)}(\varphi) &= -\frac{i}{4}\,X_\varphi^{\,4}=\frac{i}{4}\,(-iy\varphi)^4\,\mathrm{Tr}\,S_F^4\\ &= \frac{i}{4}\,(y\varphi)^4\,V_4\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{tr}\,\widetilde{S}_F^4(p)\Big\rfloor_{\text{finite part}}\\ &= \frac{V_4}{24\pi^2}\,(y\varphi)^4\left(3\ln\frac{M}{\mu}-2\right)=\tfrac{1}{24}\,\varphi^4\,\frac{\mathrm{d}^4}{\mathrm{d}\varphi^4}\,\Gamma_{\text{eff}}(0)\end{aligned}$$

Notice that, since the trace of an odd number of gamma matrices is null, we get

$$\begin{aligned}\mathrm{tr}\,\widetilde{S}_F^3(p) &= \frac{\mathrm{tr}\,(\not p+M)^3}{(p^2-M^2+i\varepsilon)^3}=\frac{M(3p^2+M^2)}{(p^2-M^2+i\varepsilon)^3}\,\mathrm{tr}\,\mathbb{I}\\ \mathrm{tr}\,\widetilde{S}_F^4(p) &= \frac{\mathrm{tr}\,(\not p+M)^4}{(p^2-M^2+i\varepsilon)^4}=\frac{(p^2)^2+6p^2M^2+M^4}{(p^2-M^2+i\varepsilon)^4}\,\mathrm{tr}\,\mathbb{I}\end{aligned}$$

Moreover one can readily check that all the coefficients $\Gamma_{\text{eff}}^{(n)}(\varphi)$ with $n\geq 5$ are finite and do not depend upon the arbitrary mass scale μ of the Zeta function regularization.

2.4 Quantum ElectroDynamics (QED)

Let me finally come to a fully realistic and fundamental field theoretic model: the Spinor Quantum ElectroDynamics, *i.e.* the Lorentz covariant theory of a Dirac spinor quantum matter interacting with the electromagnetic radiation quanta. The electromagnetic field is described by the vector potential $A^\mu = (\varphi, \mathbf{A})$ with

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\varphi - \frac{1}{c}\,\dot{\mathbf{A}}$$

To this aim, let us first recall the canonical engineering dimensions of some quantities in the Heaviside-Lorentz C. G. S. electromagnetic system of units. We have

$$[\,A^\mu\,] = \text{Gauss cm} = \text{eV}^{\frac12}\ \text{cm}^{-\frac12}$$

$$[\,\mathbf{E}\,] = [\,\mathbf{B}\,] = [\,F_{\mu\nu}\,] = \text{Gauss} = \text{eV}^{\frac12}\ \text{cm}^{-\frac32}$$

because

$$W = c\,P_0 = \int \mathrm{d}\mathbf{x}\,\tfrac12\,[\,\mathbf{E}^2(t,\mathbf{x}) + \mathbf{B}^2(t,\mathbf{x})\,] \qquad P_0 = \mid \mathbf{P} \mid$$

is the energy of the radiation field[2]. Notice that A^μ shares the very same dimensions with a scalar field, *i.e.* $[\,A^\mu\,] = \sqrt{\text{eV/cm}}$. The positive proton charge $e > 0$ has dimensions $[\,e\,] = \text{eV}^{\frac12}\ \text{cm}^{\frac12} = \text{Gauss cm}^2$, while the unit of quantum magnetic flux reads

$$\Phi_0 \;\equiv\; \frac{hc}{e} \;\simeq\; 4.136 \times 10^{-7}\ \text{Gauss cm}^2 \qquad [\,\Phi_0\,] = \text{eV}^{\frac12}\ \text{cm}^{\frac12}$$

Finally we recall the fine-structure constant

$$\alpha = \frac{e^2}{4\pi\hbar c} = 7.297\,352\,568(24) \times 10^{-3} \;\simeq\; \frac{1}{137}$$

as well as the electron current density

$$\jmath^{\,\mu}(x) = -\,\frac{e}{\hbar}\,\overline{\psi}(x)\,\gamma^\mu\,\psi(x)$$

$$[\,\jmath^{\,\mu}\,] = \text{esu cm}^{-2}\ \text{s}^{-1} = \text{eV}^{\frac12}\ \text{cm}^{-\frac32}\ \text{s}^{-1} = \text{Gauss/s}$$

where 1 esu = Gauss cm^2. Then the corresponding Lagrange density is provided by

$$\begin{aligned} \mathcal{L} \;&=\; -\,\frac14\,F^{\,\mu\nu}\,F_{\mu\nu} + \bar{\psi}\left(\tfrac12\,i\,\overset{\leftrightarrow}{\not\partial} - \frac{Mc}{\hbar} + \frac{e}{\hbar c}\,\not A\right)\psi \qquad (2.50)\\ &\doteq\; -\,\frac14\,F^{\,\mu\nu}\,F_{\mu\nu} + \bar{\psi}\left(\gamma^{\,\nu}\,i\partial_\nu - \frac{Mc}{\hbar}\right)\psi - \frac1c\,\jmath^{\,\mu}\,A_\mu \end{aligned}$$

[2]The key dimensional equality in the Heaviside-Lorentz C. G. S. electromagnetic system of units are $[\,W\,] = \text{Gauss}^2\ \text{cm}^3 = [\,e^2\,]/\text{cm} = \text{eV} = [\,e\,]$ Gauss cm .

and turns out to be invariant under the so called local phase transformations and gauge transformations, *i.e.* the space-time point dependent symmetry

$$\begin{cases} A_\mu(x) \quad \mapsto \quad A'_\mu(x) = A_\mu(x) + \partial_\mu f(x) \\ \psi(x) \quad \mapsto \quad \psi'(x) = \exp\{(ie/\hbar c) f(x)\}\,\psi(x) \\ \bar{\psi}(x) \quad \mapsto \quad \bar{\psi}'(x) = \bar{\psi}(x)\, \exp\{(-\,ie/\hbar c) f(x)\} \end{cases} \tag{2.51}$$

in which $f(x)$ is any arbitrary real function with the dimensions of a magnetic flux. Notice that in the limit of a constant value

$$f(x) \longrightarrow f = \frac{\Phi_0}{2\pi}\,\theta \qquad 0 \le \theta < 2\pi$$

the gauge transformation for the spinor field reduces to the ordinary U(1) phase transformations

$$\psi(x) \quad \longmapsto \quad \psi'(x) = \mathrm{e}^{i\theta}\,\psi(x)$$

leading to the Nöther tetra-current

$$\frac{\delta\,\mathcal{L}}{\delta\,\partial_\mu\,\psi}\,\Delta\,\psi = -\,\overline{\psi}\,\gamma^\mu\,\psi\,\delta\theta \equiv J^\mu\,\delta\theta \quad \Longrightarrow \quad \jmath^\nu = \frac{e}{\hbar}\,J^\nu$$

The Lagrangian (2.51) gives rise to the Euler-Lagrange coupled and gauge invariant field equations

$$\partial_\mu\,F^{\mu\nu}(x) = \; -\,\frac{e}{\hbar c}\,\overline{\psi}(x)\,\gamma^\nu\,\psi(x) = \frac{1}{c}\,\jmath^\nu(x) \tag{2.52}$$

$$\left(i\,\partial\!\!\!/ - \frac{Mc}{\hbar}\right)\psi(x) = \; -\,\frac{e}{\hbar c}\,A\!\!\!/(x)\,\psi(x) \tag{2.53}$$

It is worthwhile to remark that the QED spinor Lagrangian can also be written, up to a boundary term, in the form

$$\mathcal{L}_{\,\mathrm{QED}} \doteq \bar{\psi}\left(\gamma^\mu\,i\,D_\mu - \frac{Mc}{\hbar}\right)\psi - \frac{1}{4}\,F^{\rho\nu}\,F_{\rho\nu} \tag{2.54}$$

where the gauge covariant derivative

$$D_\mu \equiv \partial_\mu + \frac{iq}{\hbar c}\,A_\mu \tag{2.55}$$

has been introduced for a charged particle of charge q, that corresponds to the well known minimal substitution

$$p_\mu \quad \longrightarrow \quad p_\mu - \frac{q}{c}\,A_\mu \tag{2.56}$$

in order to introduce the electromagnetic interaction in the classical and quantum physics of point-like particles with a charge q.

Now, just owing to the gauge symmetry, the kinetic term for the vector potential

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \doteq \frac{1}{2}A^{\mu}(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^{\nu}$$

can not be inverted and thereby a Feynman propagator can not be defined. Hence, in the aim to develop Lorentz covariant perturbation theory, the so called general covariant gauge fixing Lagrangian must be added, *viz.*

$$\mathcal{L}_{\text{g.f.}} = A^{\mu}(x)\,\partial_\mu B(x) \;+\; \frac{\xi}{2}\,B^2(x) \tag{2.57}$$

where $B(x)$ is an auxiliary non-physical scalar field of canonical engineering dimension $[\,B\,] = \text{erg}^{\frac{1}{2}}\ \text{cm}^{-\frac{3}{2}}$, while the dimensionless parameter $\xi \in \mathbb{R}$ is named the gauge fixing parameter, the Abelian field strength being as usual $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, in such a manner that the action results to be Poincaré invariant. If we definitely make the simplest choice $\xi = 1$, that is called the Feynman gauge, after turning to the momentum space we find

$$\begin{aligned}
\frac{i}{\hbar}S[\,A^{\mu},\psi,\bar\psi\,] \;&=\; -\,\frac{i}{2\hbar c}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\widetilde{A}^{\mu}(k)\,k^2\,g_{\mu\nu}\,\widetilde{A}^{\nu}(-k)\\
&+\;\frac{i}{\hbar c}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\widetilde{\psi}^{\,\dagger}(p)\,\gamma^0\left(\not p - \frac{Mc}{\hbar}\right)\widetilde{\psi}(p)\\
&+\;\frac{ie}{(\hbar c)^2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\widetilde{A}_{\mu}(q-p)\,\widetilde{\psi}^{\,\dagger}(q)\,\gamma^0\,\gamma^{\mu}\,\widetilde{\psi}(p)
\end{aligned}$$

in such a manner that, in close resemblance with the Yukawa theory, we immediately come to the Feynman rules in the Feynman gauge: namely,

spinor propagator : $$\left(\frac{i\hbar c}{\not p + i\varepsilon - Mc/\hbar}\right)_{\alpha\beta} \tag{2.58}$$

photon propagator : $$-\,i\hbar c\,\frac{g^{\mu\nu}}{k^2 + i\varepsilon} \tag{2.59}$$

vertex : $$\frac{ie}{(\hbar c)^2}\,\gamma^{\mu}_{\alpha\beta} \qquad (\,k + p - q = 0\,) \tag{2.60}$$

while the energy momentum conservation is again $(2\pi)^4\,\delta\,(\,k + p - q\,)$ and a factor (-1) must be included for each fermion loop.

`Gauge Fixing Lagrangian`. It is worthwhile to notice that an alternative and very useful method to obtain a gauge fixing Lagrange density $\mathcal{L}_{\text{g.f.}}$ is as follows. Let us begin from

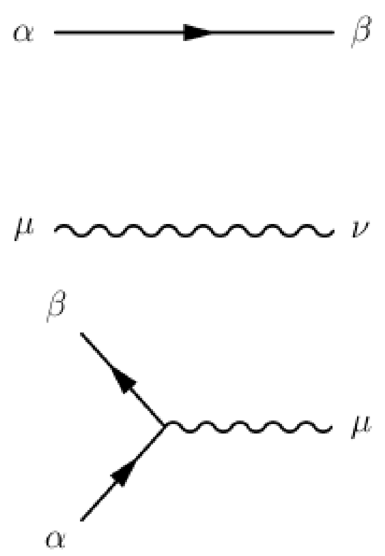


Figure 2.7: Feynman rules for the spinor propagator $S_{\alpha\beta}(p)$, the photon propagator $D_{\mu\nu}(k)$ and fermion-fermion-photon vertex $\gamma^{\mu}_{\alpha\beta}$

the expression (2.57) and substitute the equations of motion for the auxiliary non-physical field $\partial \cdot A = \xi B$: then we get, up to an irrelevant boundary term,

$$\mathcal{L}_{\,\mathrm{g.f.}} \doteq \frac{1}{2\xi}\; A^{\,\mu}\,(x)\,\partial_{\mu}\,\partial_{\nu} A^{\nu}\,(x) \tag{2.61}$$

This entails that, if we sum the latter gauge fixing Lagrangian with the gauge invariant Lagrange density of the radiation field, then we eventually get

$$\begin{aligned} \mathcal{L}_{\,\xi}\,[\,A^{\mu}\,] \;&\doteq\; \tfrac{1}{2}\,A^{\,\nu}\,(\,\delta^{\mu}_{\nu}\,\Box - \partial^{\,\mu}\partial_{\nu}\,)\,A_{\,\mu} + \tfrac{1}{2}\,\xi^{\,-1}\,A^{\,\mu}\,\partial_{\mu}\,\partial_{\nu}\,A^{\,\nu} \\ &\doteq\; \tfrac{1}{2}\,A^{\,\nu}\,\left[\,\delta^{\mu}_{\nu}\,\Box - \left(\,1 - \xi^{\,-1}\,\right)\partial^{\,\mu}\,\partial_{\nu}\,\right]A_{\,\mu} \end{aligned}$$

It follows that the momentum space kinetic operator is provided by

$$\mathcal{K}^{\,\mu}_{\nu} = -\,\delta^{\,\mu}_{\nu}\,k^{\,2} + \left(\,1 - \xi^{\,-1}\,\right)k^{\,\mu}\,k_{\nu}$$

In order to find its algebraic inverse for $k^{\,2} \neq 0$ let us set

$$\mathcal{K}^{\,-1}_{\lambda\mu} = \frac{A}{k^{\,2}}\left(\,g_{\,\lambda\mu} - \frac{a}{k^{\,2}}\,k_{\lambda}\,k_{\mu}\right)$$

so that

$$\begin{aligned} \mathcal{K}^{\,-1}_{\lambda\mu}\,\mathcal{K}^{\,\mu}_{\nu} \;&=\; \frac{A}{k^{\,2}}\left(g_{\,\lambda\mu} - \frac{a}{k^{\,2}}\,k_{\lambda}\,k_{\mu}\right)\left[\,-\,\delta^{\,\mu}_{\nu}\,k^{\,2} + \left(\,1 - \xi^{\,-1}\,\right)k^{\,\mu}\,k_{\nu}\,\right] \\ &=\; -\,A\,g_{\lambda\nu} + \frac{k_{\,\lambda}\,k_{\,\nu}}{k^{\,2}}\;A\left[\,1 - \frac{1}{\xi} + \frac{a}{\xi}\,\right] = g_{\lambda\nu} \end{aligned}$$

and consequently

$$A = -1 \qquad a = 1 - \xi$$

$$\mathcal{K}^{-1}_{\lambda\mu} = \frac{1}{k^2}\left(-g_{\lambda\mu} + \frac{1-\xi}{k^2}\, k_\lambda\, k_\mu \right)$$

This drives to the Wick propagator or causal Green's function

$$\widetilde{D}^{\,c}_{\lambda\mu}(k\,;\xi) = \frac{i\hbar c}{k^2 + i\varepsilon}\left(-g_{\lambda\mu} + \frac{1-\xi}{k^2 + i\varepsilon}\, k_\lambda\, k_\mu \right)$$

where the causal Feynman prescription has been endorsed allowing for the Wick rotation and thereby

$$D^{\,c}_{\lambda\mu}(x-y\,;\xi) = \frac{i\hbar c}{(2\pi)^4}\int \mathrm{d}^4 k\ \frac{\mathrm{e}^{-ik\cdot(x-y)}}{k^2 + i\varepsilon}\left(-g_{\lambda\mu} + \frac{1-\xi}{k^2 + i\varepsilon}\, k_\lambda\, k_\mu \right)$$

which satisfies

$$\left[\, \delta^{\,\mu}_{\nu}\,\Box - \left(1 - \xi^{-1} \right) \partial^{\,\mu}\,\partial_\nu \,\right] D^{\,c}_{\lambda\mu}(x-y\,;\xi) = i\hbar c\, g_{\nu\lambda}\, \delta^{(4)}(x-y)$$

For $\xi = 1$ one gets the photon propagator in the Feynman gauge, which turns out to be the simplest and most convenient choice, while for $\xi = 0$ one gets the so called Landau gauge photon propagator, which fulfill the Lorenz condition $\partial_\mu A^\mu(x) = 0$ as an operator equality at the quantum level.

As a final important remark, I recall the Euler-Lagrange field equations, that hold true in the Feynman gauge, for the operator valued tempered distributions $A^{\mu}(x)$, $B(x)$ and $\psi(x)$: namely,

$$c\,\Box\, A^{\mu}(x) = \jmath^{\,\mu}(x) \tag{2.62}$$

$$i\,\partial\!\!\!/\,\psi(x) + \frac{e}{\hbar c}\, A\!\!\!/\,(x)\,\psi(x) = \frac{Mc}{\hbar}\,\psi(x) \tag{2.63}$$

$$\partial_\mu\, A^{\mu}(x) = B(x) \tag{2.64}$$

where the local electric current quantum operator is defined as

$$\jmath^{\,\mu}(x) = -\frac{e}{\hbar}\,\lim_{\epsilon\to 0}\,\tfrac{1}{2}\left(\bar\psi(x+\epsilon)\,\gamma^{\mu}\,\psi(x) - \bar\psi(x)\,\gamma^{\mu}\,\psi(x+\epsilon) \right)$$

in such a manner to avoid the ill-defined product of tempered distribution at the same space-time point, the relative minus sign being due to the canonical equal time anti-commutation relations

$$\{\,\psi(t,\mathbf{x})\,,\,\psi^{\dagger}(t,\mathbf{y})\,\} \;=\; \hbar c\,\delta\,(\mathbf{x}-\mathbf{y})$$

all the other anti-commutators vanishing. From the 4-divergence of equation (2.62) and taking equation (2.64) into account we find

$$c\,\Box\, B(x) = \partial_\mu\,\jmath^{\,\mu}(x)$$

On the other side, from the invariance of the Lagrangian with respect to the $U(1)$ global, *i.e.* space-time point independent, phase transformations

$$\begin{cases} \psi(x) & \mapsto \quad \psi'(x) = \exp\{i\theta\}\,\psi(x) \\ \bar{\psi}(x) & \mapsto \quad \bar{\psi}'(x) = \bar{\psi}(x)\,\exp\{-i\theta\} \end{cases}$$

it follows that the electric current density four vector satisfies the continuity equation and the total electric charge is conserved, which is nothing but the Nöther's theorem. Hence

$$\partial_\mu \jmath^\mu(x) \equiv 0 \qquad \Leftrightarrow \qquad \Box B(x) = 0 \tag{2.65}$$

which means that the auxiliary scalar $B(x)$ is still a free field obeying the d'Alembert wave equation **even in the presence of the interaction.** It follows that it is still possible to select the Hilbert space of the physical states, with a positive semi-definite norm, from the subsidiary condition

$$B^{(-)}(x)\,|\,\mathrm{phys}\,\rangle = 0$$

just like in the Gupta-Bleuler or Nakanishi-Lautrup formalism for the free radiation quantum field.

Scalar Electrodynamics. Consider a complex scalar field of mass m and electric charge q interacting with the electromagnetic field, the dynamics being governed by the classical Lagrangian in the Feynman gauge

$$\mathcal{L} = g^{\mu\nu} D_\mu\phi\,(D_\nu\phi)^* - m^2\phi\phi^* - \lambda\,(\phi\phi^*)^2 - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\partial_\mu A^\mu\,\partial_\nu A^\nu$$

$$D_\mu\phi \equiv (\partial_\mu + iqA_\mu)\,\phi \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

with λ and q positive coupling constants. The Lagrangian can be rewritten in the more explicit form

$$\begin{aligned} \mathcal{L} &= \partial^\mu\phi\,\partial_\mu\phi^* - m^2\phi^*\phi - \lambda\,(\phi^*\phi)^2 - \tfrac{1}{2}\partial^\mu A^\nu\,\partial_\mu A_\nu \\ &- iq\,A^\mu\,(\phi^*\partial_\mu\phi - \phi\,\partial_\mu\phi^*) + q^2 A^\mu\phi\,A_\mu\phi^* \end{aligned}$$

Then we evidently have

$$\delta\mathcal{L}/\delta\partial^\mu\phi = (D_\mu\phi)^* = \partial_\mu\phi^* - iq\,A_\mu\phi^*$$
$$\delta\mathcal{L}/\delta\partial^\mu\phi^* = D_\mu\phi = \partial_\mu\phi + iq\,A_\mu\phi$$
$$\delta\mathcal{L}/\delta\partial_\mu A_\nu = -\,\partial^\mu A^\nu$$

$$\delta\mathcal{L}/\delta\phi = iq\,A\cdot(D\phi)^* - m^2\phi^* - 2\lambda\,\phi^*(\phi\phi^*)$$
$$\delta\mathcal{L}/\delta\phi^* = -\,iq\,A\cdot D\phi - m^2\phi - 2\lambda\,\phi\,(\phi\phi^*)$$

$$\begin{aligned} \delta\mathcal{L}/\delta\partial^\mu A^\nu &= -\,\partial_\mu A_\nu \\ \delta\mathcal{L}/\delta A^\nu &= -\,q\,\phi^*\,i\overleftrightarrow{\partial}_\nu\phi + 2q^2 A_\nu\phi\phi^* \\ &= -\,iq\,(\phi^* D_\nu\phi - \phi\,D_\nu^*\phi^*) \equiv -\,iq\,\phi^*\overleftrightarrow{D}_\nu\,\phi \end{aligned}$$

so that the Euler-Lagrange field equations read

$$
\begin{aligned}
& D_\mu^* \,(D^\mu \phi)^* + m^2 \phi^* + 2\lambda\, \phi^* \phi \phi^* = 0 \\
& D^\mu D_\mu \phi + m^2 \phi + 2\lambda\, \phi\, \phi^* \phi = 0 \\
& \Box A_\mu = iq\, \phi^* \overleftrightarrow{D}_\mu \ \phi \equiv J_\mu
\end{aligned}
$$

The Lagrangian is invariant under the full non-homogeneous Lorentz group IO(1,3) and the internal U(1) phase transformations

$$
\phi(x) \ \mapsto \ \phi'(x) = \phi(x)\, \mathrm{e}^{-\, iq\, \theta} \qquad (\, 0 \le \theta < 2\pi \,)
$$

in such a manner that the electric current does satisfy the continuity equation

$$
\partial_\mu J^\mu = 0 = \partial^\mu \left(\phi^* \, i \overleftrightarrow{D}_\mu \ \phi \right)
$$

which guarantees charge conservation in time

$$
Q \equiv q \int \mathrm{d}\mathbf{r}\ \phi^*(t, \mathbf{r})\, i \overleftrightarrow{D}_t\, \phi(t, \mathbf{r}) \qquad \dot{Q} = 0
$$

Notice that, by taking the four divergence of the equations of motion

$$
\partial^\nu (\Box A_\nu - J_\nu) = 0 \quad \Rightarrow \quad \Box\, \partial \cdot A = 0
$$

which means that $\partial \cdot A$ is always a free field even in the presence of interaction with the charged scalar matter. It is also important to realize that the covariant derivative $D_\mu \phi$ of the charged scalar field transforms homogeneously with respect to the local phase transformations, *i.e.* the gauge transformations: namely,

$$
\begin{aligned}
\phi(x) \ &\mapsto \ \phi'(x) = \phi(x)\, \exp\{-\, iq\, \theta(x)\} \\
A_\mu(x) \ &\mapsto \ A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x)
\end{aligned}
$$

In fact we obtain

$$
\begin{aligned}
D'_\mu \phi'(x) \ &= \ \partial_\mu \left[\, \phi(x)\, \exp\{-\, iq\, \theta(x)\} \,\right] \\
&+ \ iq \left[\, A_\mu(x) + \partial_\mu \theta(x) \,\right] \phi(x)\, \exp\{-\, iq\, \theta(x)\} \\
&= \ \exp\{-\, iq\, \theta(x)\}\, D_\mu \phi(x)
\end{aligned}
$$

which entails that the electric current is also gauge invariant, *viz.*,

$$
J'_\mu(x) = \left(iq\, \phi^*(x) \overleftrightarrow{D}_\mu \phi(x) \right)' = J_\mu(x)
$$

Let us calculate the canonical energy-momentum tensor, as provided by Nöther theorem

$$
\begin{aligned}
T_{\mu\nu} \ &\equiv \ (\, \delta \mathcal{L} / \delta \partial^\mu \phi \,)\, \partial_\nu \phi + (\, \delta \mathcal{L} / \delta \partial^\mu \phi^* \,)\, \partial_\nu \phi^* \\
&+ \ (\, \delta \mathcal{L} / \delta \partial^\mu A_\lambda \,)\, \partial_\nu A_\lambda - \mathcal{L}\, g_{\mu\nu} \\
&= \ (D_\mu \phi)^*\ \partial_\nu \phi + (\, D_\mu \phi)\ \partial_\nu \phi^* - \partial_\mu A_\lambda\, \partial_\nu A^\lambda - \mathcal{L}\, g_{\mu\nu}
\end{aligned}
$$

Then we obtain the energy of the classical system

$$
\begin{aligned}
P_0 &= \int \mathrm{d}\mathbf{r}\, \Big\{ \dot{\phi}(t,\mathbf{r})\, D_0^* \phi^*(t,\mathbf{r}) + \dot{\phi}^*(t,\mathbf{r})\, D_0\, \phi(t,\mathbf{r}) \\
&- \dot{\varphi}^2(t,\mathbf{r}) + \dot{\mathbf{A}}^2(t,\mathbf{r}) - \mathcal{L}(t,\mathbf{r}) \Big\}
\end{aligned}
$$

where we have set

$$A^\mu(t,\mathbf{r}) \equiv \Big(\varphi(t,\mathbf{r})\,,\, \mathbf{A}(t,\mathbf{r}) \Big)$$

and taking into account that we have the canonical field momenta

$$
\begin{aligned}
\delta\mathcal{L}/\delta\dot{\phi}(t,\mathbf{r}) &\equiv \Pi^*(t,\mathbf{r}) = D_0^* \phi^*(t,\mathbf{r}) \\
\delta\mathcal{L}/\delta\dot{\phi}^*(t,\mathbf{r}) &\equiv \Pi(t,\mathbf{r}) = D_0\, \phi(t,\mathbf{r}) \\
\delta\mathcal{L}/\delta\dot{\varphi}(t,\mathbf{r}) &\equiv \Pi_\varphi(t,\mathbf{r}) = -\dot{\varphi}(t,\mathbf{r}) \\
\delta\mathcal{L}/\delta\dot{\mathbf{A}}(t,\mathbf{r}) &\equiv \mathbf{\Pi}(t,\mathbf{r}) = \dot{\mathbf{A}}(t,\mathbf{r})
\end{aligned}
$$

so that

$$\dot{\phi}(t,\mathbf{r}) = \Pi(t,\mathbf{r}) - iq\,\varphi(t,\mathbf{r})\,\phi(t,\mathbf{r})$$

we find the classical Hamiltonian

$$
\begin{aligned}
H &= \int \mathrm{d}\mathbf{r}\, \Big\{ \Big[\Pi(t,\mathbf{r}) - iq\,\varphi(t,\mathbf{r})\,\phi(t,\mathbf{r}) \Big] \Pi^*(t,\mathbf{r}) - \mathcal{L}(t,\mathbf{r}) \\
&+ \Big[\Pi^*(t,\mathbf{r}) + iq\,\varphi(t,\mathbf{r})\,\phi^*(t,\mathbf{r}) \Big] \Pi(t,\mathbf{r}) - \dot{A}_\mu(t,\mathbf{r}) \dot{A}^\mu(t,\mathbf{r}) \Big\}
\end{aligned}
$$

Since we can write

$$V(\phi\phi^*) \equiv m^2\, \phi\phi^* + \lambda\, (\phi\phi^*)^2$$

$$
\begin{aligned}
\mathcal{L}(t,\mathbf{r}) &= \Pi(t,\mathbf{r})\,\Pi^*(t,\mathbf{r}) - D_k\,\phi(t,\mathbf{r})\, D_k^*\,\phi^*(t,\mathbf{r}) - V(\phi\phi^*) \\
&- \tfrac{1}{2}\,\dot{A}_\mu(t,\mathbf{r})\dot{A}^\mu(t,\mathbf{r}) + \tfrac{1}{2}\,\nabla A_\mu(t,\mathbf{r}) \cdot \nabla A^\mu(t,\mathbf{r}) \\
&= \Pi(t,\mathbf{r})\,\Pi^*(t,\mathbf{r}) - D_k\,\phi(t,\mathbf{r})\, D_k^*\,\phi^*(t,\mathbf{r}) - V(\phi\phi^*) \\
&- \tfrac{1}{2}\,\Pi_\varphi^2(t,\mathbf{r}) + \tfrac{1}{2}\,\mathbf{\Pi}^2(t,\mathbf{r}) + \tfrac{1}{2}\,\nabla\varphi(t,\mathbf{r}) \cdot \nabla\varphi(t,\mathbf{r}) \\
&- \tfrac{1}{2}\,\nabla A_k(t,\mathbf{r}) \cdot \nabla A_k(t,\mathbf{r})
\end{aligned}
$$

we eventually obtain the classical Hamiltonian as a functional of the fields and their conjugate momenta, *viz.,*

$$
\begin{aligned}
H &= \int \mathrm{d}\mathbf{r}\, \Big\{ \Pi(t,\mathbf{r})\,\Pi^*(t,\mathbf{r}) + D_k\,\phi(t,\mathbf{r})\, D_k^*\,\phi^*(t,\mathbf{r}) + V(\phi\phi^*) \\
&- iq\,\varphi(t,\mathbf{r})\,\phi(t,\mathbf{r})\,\Pi^*(t,\mathbf{r}) + iq\,\varphi(t,\mathbf{r})\,\phi^*(t,\mathbf{r})\,\Pi(t,\mathbf{r}) \\
&- \tfrac{1}{2}\,\Pi_\varphi^2(t,\mathbf{r}) - \tfrac{1}{2}\,\nabla\varphi(t,\mathbf{r}) \cdot \nabla\varphi(t,\mathbf{r}) \\
&+ \tfrac{1}{2}\,\mathbf{\Pi}^2(t,\mathbf{r}) + \tfrac{1}{2}\,\nabla A_k(t,\mathbf{r}) \cdot \nabla A_k(t,\mathbf{r}) \Big\}
\end{aligned}
$$

which in not positive semi-definite owing to the presence of the electric potential $\varphi(t,\mathbf{r})$, as well as the total momentum

$$
\begin{aligned}
\mathbf{P} &= -\int \mathrm{d}\mathbf{r}\, \Big\{ \Pi_\varphi(t,\mathbf{r}) \boldsymbol{\nabla}\varphi(t,\mathbf{r}) + \Pi^k(t,\mathbf{r})\, \boldsymbol{\nabla} A^k(t,\mathbf{r}) \\
&+ \Pi^*(t,\mathbf{r}) \boldsymbol{\nabla}\phi(t,\mathbf{r}) + \Pi(t,\mathbf{r}) \boldsymbol{\nabla}\phi^*(t,\mathbf{r}) \Big\}
\end{aligned}
$$

The Feynman rules can be obtained in a straightforward manner from the classical Action multiplied by the imaginary unit. Actually we get the scalar propagator in momentum space – with an arrow because the charged scalar particles and antiparticles have opposite charges

$$\widetilde{D}(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

the usual photon propagator in the Feynman gauge

$$\widetilde{D}^{\,\mu\nu}(k) = \frac{-\,ig^{\,\mu\nu}}{k^2 + i\varepsilon}$$

Furthermore, after setting as usual

$$\phi\,(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\widetilde{\phi}\,(\,p\,)\,\mathrm{e}^{-\,ip\cdot x} \qquad\qquad A_{\mu}\,(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\widetilde{A}_{\mu}\,(k)\,\mathrm{e}^{-\,ik\cdot x}$$

we obtain the particle antiparticle photon 3–point vertex

$$-\,iq\,(\,p + p\,'\,)^{\mu} \qquad\qquad (\,p + k - p\,' = 0\,)$$

the four point scalar vertex with two particles and two antiparticles

$$-\,4i\lambda \qquad\qquad (\,p + q - p\,' - q\,' = 0\,)$$

and the particle antiparticle two photons four point vertex

$$2iq^{2}\,g^{\,\mu\nu} \qquad\qquad (\,p + k - p\,' + k\,' = 0\,)$$

As already noticed, it turns out that the d'Alembert equation $\Box\,\partial_{\mu}A^{\mu}(x) = 0$ holds true even in the interacting case, so that we can select the physical subspace $\mathfrak{H}_{\mathrm{phys}}$ of the indefinite metric Fock space $\mathfrak{F}$ by the subsidiary condition

$$\partial^{\,\mu}A_{\mu}^{(-)}(x)|\,\mathrm{phys}\,\rangle = 0$$

which guarantees a positive semi-definite metric of the physical Hilbert space $\mathfrak{H}_{\mathrm{phys}} \subset \mathfrak{F}\,$.

2.5 Non-Abelian Gauge Theories

The setting up of quantum electrodynamics can be generalized to the so called non-Abelian gauge field theories. To this purpose, let us consider a classical Action invariant under an internal symmetry Lie group G of local, *i.e.* space-time point dependent, transformations. We assume the non-Abelian gauge group G to be simple and compact: without loss of generality we shall choose $SU(N)$ with $N \geq 2\,$.

2.5.1 Covariant Derivative and Related Properties

To start up, let me denote by $\psi(x)$ a column multiplet of N Dirac spinor fields which transforms according to an irreducible fundamental representation of

$SU(N)$, which is identified by a set of generators or infinitesimal operators $\boldsymbol{\tau}_F^a$ ($a = 1, \dots, N^2 - 1$) in such a manner that

$$\psi(x) \equiv \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{pmatrix} \tag{2.66}$$

$$\psi(x) \mapsto \psi_\omega(x) = \exp\{ ig\, \omega^a(x) \boldsymbol{\tau}_F^a \}\, \psi(x) \equiv U_\omega(x)\, \psi(x) \tag{2.67}$$

with

$$U_\omega^{-1}(x) = \exp\{-\, ig\, \omega^a(x) \boldsymbol{\tau}_F^a\} = U_\omega^\dagger(x) = U_{-\omega}(x)$$

Since the fundamental representations of $SU(N)$ have dimension N it follows that the number of independent real functions in the spinor multiplet (2.66) is $8N$. Thus, the Hermitean $N \times N$ matrices $\boldsymbol{\tau}_F^a$ ($a = 1, \dots, N^2 - 1$) form an $N-$dimensional representation of the $su(N)$ Lie algebra. The $SU(N)$ structure constants f^{abc} are completely anti-symmetric real quantities defined by the commutation relations or inner Lie products

$$[\, \boldsymbol{\tau}_F^a \,,\, \boldsymbol{\tau}_F^b \,] = i f^{abc}\, \boldsymbol{\tau}_F^c \qquad (a, b, c = 1, \dots, N^2 - 1) \tag{2.68}$$

For example, the structure constants of the unitary group $SU(2)$, which is the universal covering of the proper rotation group $SO(3)$, are provided by $f^{abc} = -\, \varepsilon_{abc}$, while the Hermitean generators $\boldsymbol{\tau}_F^a$ are nothing but half the Pauli matrices $\frac{1}{2}\sigma_a$.

Conjugated representations. For any finite dimensional representation $\{T(g)\,|\,g \in G\}$ of a Lie group G of finite dimension n

$$T(g_1)\, T(g_2) = T(g_1 g_2) \qquad (\, \forall\, g_1, g_2 \in G\,)$$

it follows that there always exist its complex conjugate representation $\{\overline{T}(g)\,|\,g \in G\}$ of the same dimension and with matrix elements

$$\overline{T}_{\imath\jmath}(g) \equiv T^*_{\imath\jmath}(g) \qquad (\, \forall\, g \in G\,)$$

because

$$\overline{T}_{\imath\jmath}(g_1 g_2) = T^*_{\imath\jmath}(g_1 g_2) = T^*_{\imath\kappa}(g_1)\, T^*_{\kappa\jmath}(g_2) = \overline{T}_{\imath\kappa}(g_1)\, \overline{T}_{\kappa\jmath}(g_2) \tag{2.69}$$

Now, if we denote by I_a ($a = 1, \dots, n$) and $\bar{I}_a$ ($a = 1, \dots, n$) the corresponding generators, it turns out that the finite dimensional representations of a Lie group fall into three categories: real, pseudo-real and complex.

- Real representations: a representation is said to be real iff there exists a non-singular symmetric matrix S which realizes the transformations

$$\bar{I}_a = -\,S I_a S^{-1} \qquad S = S^\top \qquad (\,\forall\, a = 1,\ldots,n\,)$$

This is for example the case of the adjoint representation $SO(3)$ of the rotation group.

- Pseudo-real representations: a representation is said to be pseudo-real iff there exists a non-singular anti-symmetric matrix S which realizes the transformations

$$\bar{I}_a = -\,S I_a S^{-1} \qquad S = -\,S^\top \qquad (\,\forall\, a = 1,\ldots,n\,)$$

This is for example the case of the fundamental representation $SU(2)$ of the rotation group.

- Complex representations: a representation is said to be complex iff its set of the generators $\{I_a \,|\, a = 1,\ldots,n\}$ and that one of its conjugated $\{\bar{I}_a \,|\, a = 1,\ldots,n\}$ are never related by a similarity transformation. As an example one can take the two non-equivalent three dimensional fundamental representations **3** and $\overline{\mathbf{3}}$ of the unitary group $SU(3)$.

In order to build a gauge invariant Action, we need the covariant derivative of the spinor multiplet $\psi(x)$, which generalizes the concept of minimal coupling from the electromagnetic Abelian case to the non-Abelian framework. This can be readily achieved by setting up a matrix-valued derivative operator D_μ such that $D_\mu\,\psi(x)$ transforms like $\psi(x)$ itself. To this aim we define

$$D_\mu\,\psi\,(x) \equiv \left(\,\partial_\mu - ig\,A_\mu^a(x)\,\boldsymbol{\tau}_F^{\,a}\,\right)\psi\,(x) \tag{2.70}$$

where we have introduced the non-Abelian real vector potentials

$$A_\mu^a(x) \qquad (\,a = 1,\ldots,N^2-1\,)$$

the number of which is equal to the dimension of the local symmetry group $SU(N)$. As a matter of fact, if we impose the homogeneous transformation law

$$\psi\,(x)\,\mapsto\,U_\omega(x)\,\psi\,(x) \quad\Leftrightarrow\quad D_\mu\,\psi\,(x)\,\mapsto\,U_\omega(x)\,D_\mu\,\psi\,(x) \tag{2.71}$$

we necessarily obtain the non-homogeneous transformation law for the gauge vector potentials: namely,

$$A_\mu(x)\,\mapsto\,A_\mu^\omega(x) = U_\omega(x)\,A_\mu(x)\,U_\omega^\dagger(x) - \frac{i}{g}\,[\,\partial_\mu U_\omega(x)\,]\,U_\omega^\dagger(x) \tag{2.72}$$

in which I have introduced as customary the matrix valued vector potential in the fundamental irreducible representations of the $su(N)$ Lie algebra, *viz.*

$$A_\mu(x) = A_\mu^a(x)\,\boldsymbol{\tau}_F^{\,a} \tag{2.73}$$

Notice that the homogeneous transformation law (2.71) actually entails

$$D'_\mu\,\psi'(x) = D'_\mu\,U_\omega\,\psi(x) = U_\omega(x)\,D_\mu\,\psi(x) \tag{2.74}$$

and consequently

$$U^\dagger_\omega(x)\,D'_\mu\,U_\omega(x) = D_\mu \qquad\qquad D'_\mu \;=\; U_\omega(x)\,D_\mu\,U^\dagger_\omega(x) \tag{2.75}$$

In order the gauge potentials acquire the role of dynamical fields, we need an Action that describes their dynamics and is invariant under local, *i.e.* space-time point dependent, $SU(N)$ transformations. This task is most easily achieved in terms of the so called non-Abelian field strength anti-symmetric matrix valued tensor $F_{\mu\nu}(x) = -F_{\nu\mu}(x)$ defined by

$$\begin{aligned}
F_{\mu\nu}(x) &= F^a_{\mu\nu}(x)\,\boldsymbol{\tau}^a_F \equiv \frac{i}{g}\,[\,D_\mu\,,\,D_\nu\,] \\
&= \frac{i}{g}\,\big[\,\partial_\mu - ig\,A^a_\mu(x)\,\boldsymbol{\tau}^a_F\,,\partial_\nu - ig\,A^b_\nu(x)\,\boldsymbol{\tau}^b_F\,\big] \\
&= [\,\partial_\mu\,A^a_\nu(x) - \partial_\nu\,A^a_\mu(x)\,]\,\boldsymbol{\tau}^a_F - ig\,A^a_\mu(x)\,A^b_\nu(x)\,\big[\,\boldsymbol{\tau}^a_F\,,\,\boldsymbol{\tau}^b_F\,\big] \\
&= \boldsymbol{\tau}^a_F\Big(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + gf^{abc}A^b_\mu(x)\,A^c_\nu(x)\Big) \\
&= \partial_\mu\,A_\nu(x) - \partial_\nu\,A_\mu(x) - ig\,[\,A_\mu(x)\,,\,A_\nu(x)\,]
\end{aligned} \tag{2.76}$$

and which evidently transforms in a homogeneous way

$$F'_{\mu\nu}(x) = \frac{i}{g}\,\big[\,D'_\mu\,,\,D'_\nu\,\big] = U_\omega(x)\,F_{\mu\nu}(x)\,U^\dagger_\omega(x) \tag{2.77}$$

This means that the field strengths tensor transforms according to the adjoint representation of the gauge group. In fact we find

$$\begin{aligned}
F'_{\mu\nu}(x) &= \exp\{\,ig\,\omega^a(x)\boldsymbol{\tau}^a_F\}\,F_{\mu\nu}(x)\,\exp\{-\,ig\,\omega^a(x)\boldsymbol{\tau}^a_F\} \\
&= F_{\mu\nu}(x) + ig\,\omega^a(x)\,\,[\,\boldsymbol{\tau}^a_F\,,\,F_{\mu\nu}(x)\,] + O(\omega^2) \\
&= \Big(F^c_{\mu\nu}(x) - gf^{abc}\,\omega^a(x)\,F^b_{\mu\nu}(x)\;+\;O(\omega^2)\Big)\boldsymbol{\tau}^c_F
\end{aligned}$$

that yields, after reshuffling of the group indexes,

$$\delta\,F^a_{\mu\nu}(x) = \;g\,\|\;I_a\;\|_{bc}\;\;F^c_{\mu\nu}(x)\,\omega^b(x) \tag{2.78}$$

where I used the well known property of the anti-Hermitean infinitesimal operators in the N^2-1 dimensional adjoint representation, *viz.*

$$\|\;I_a\;\|_{bc} \equiv \;f^{acb} \qquad\qquad (\,a,b,c = 1,2,\ldots,N^2-1\,)$$

with $I_a + I_a^\dagger = 0$ – see the first chapter of the notes of the first semester course. Hence, if we introduce the anti-symmetric tensor field strengths multiplet

$$\{F^{\,a}_{\mu\nu}(x)\,|\,a = 1, 2, \ldots, N^2 - 1\}$$

we can write its finite transformation law by raising the infinitesimal one (2.78) to the exponential form, *viz.*,

$$\begin{aligned}
\big(F^{\,a}_{\mu\nu}(x)\big)' &= \|\exp\{-gI_c\,\omega^{\,c}(x)\}\,\|_{ab}\;F^{\,b}_{\mu\nu}(x) \\
&= \Big(\delta^{ab} + gf^{cab}\,\omega^{\,c}(x) + \cdots\Big)F^{\,b}_{\mu\nu}(x) \qquad (2.79) \\
&= F^{\,a}_{\mu\nu}(x) + gf^{abc}\,F^{\,b}_{\mu\nu}(x)\,\omega^{\,c}(x) + O(\omega^2) \qquad (2.80)
\end{aligned}$$

which is fully equivalent to equation (2.77). Notice that one can also define the adjoint covariant derivative, *i.e.* the covariant derivative in the adjoint representation: namely,

$$\nabla_\mu = \partial_\mu + gA^c_\mu\,I_c \qquad\qquad \nabla^{ab}_\mu \;\equiv\; \partial_\mu\,\delta^{ab} - gf^{abc}\,A^c_\mu(x)$$

Then we obtain

$$\begin{aligned}
\frac{1}{g}\,[\,\nabla_\mu\,,\,\nabla_\nu\,] &= \partial_\mu\,A^c_\nu(x)\,I_c - \partial_\nu\,A^c_\mu(x)\,I_c + g\,A^a_\mu(x)\,A^b_\nu(x)\,[\,I_a\,,\,I_b\,] \\
&= \Big(\partial_\mu\,A^c_\nu(x) - \partial_\nu\,A^c_\mu(x) + gf^{abc}\,A^a_\mu(x)\,A^b_\nu(x)\Big)I_c \\
&= \Big(\partial_\mu\,A^a_\nu(x) - \nabla^{ab}_\nu\,A^b_\mu(x)\Big)I_a = F^{\,a}_{\mu\nu}(x)\,I_a \qquad (2.81)
\end{aligned}$$

in close analogy with the corresponding formula (2.76) for the fundamental representation. If we rewrite the above commutator (2.81) by exhibiting the group indexes we evidently obtain

$$\big[\,\nabla^{ab}_\mu\,,\,\nabla^{bc}_\nu\,\big] \;=\; g\,F^{\,d}_{\mu\nu}(x)\,\|\,I_d\,\|_{ac} = -\,g\,f^{\,acd}\,F^{\,d}_{\mu\nu}(x) \qquad (2.82)$$

It is also very useful to collect the infinitesimal form of the non-Abelian gauge transformations that reads, up to the first order in the small parameter functions $\delta\omega^{\,a}(x)$

$$\left\{\begin{array}{c}
\delta\psi\,(x) = ig\,\delta\,\omega^{\,a}(x)\,\boldsymbol{\tau}^{\,a}_F\,\psi\,(x) \\
\delta A_\mu(x) = \nabla_\mu\,\delta\omega\,(x) = \boldsymbol{\tau}^{\,a}_F\,\delta A^a_\mu(x) = \boldsymbol{\tau}^{\,a}_F\,\nabla^{ab}_\mu\delta\omega^{\,b}(x) \\
\delta F_{\mu\nu}(x) = gf^{abc}\,F^{\,b}_{\mu\nu}(x)\,\delta\omega^c(x)\,\boldsymbol{\tau}^{\,a}_F
\end{array}\right.$$

$$\left\{\begin{array}{c}
\delta A^a_\mu(x) = \partial_\mu\,\delta\,\omega^{\,a}(x) - gf^{abc}\,\delta\,\omega^b(x)\,A^c_\mu(x) \\
\delta F^{\,a}_{\mu\nu}(x) = -\,gf^{abc}\,\delta\omega^b(x)\,F^{\,c}_{\mu\nu}(x)
\end{array}\right.$$

2.5.2 Classical Dynamics of Non-Abelian Fields

According to the general requirements to set up a local and gauge invariant classical Lagrange density, which describes the dynamics of the non-Abelian gauge fields interacting with the matter spinor fields, we are necessarily driven to the Yang-Mills construction

$$\mathcal{L}_{\rm YM} \equiv -\tfrac{1}{2}\,{\rm tr}\,[\,F_{\mu\nu}(x)F^{\mu\nu}(x)\,] + \bar\psi(x)\,(i\not{D} - M)\,\psi(x) \tag{2.83}$$

where

$$\not{D} \equiv \gamma^{\mu}D_{\mu} = \not\partial - ig\,\gamma^{\mu}A_{\mu}(x) = \not\partial - ig\,\gamma^{\mu}A_{\mu}^{a}(x)\,\boldsymbol{\tau}_F^{\,a}$$
$${\rm tr}\left(\boldsymbol{\tau}_F^{\,a}\,\boldsymbol{\tau}_F^{\,b}\right) = \tfrac{1}{2}\,\delta^{\,ab}$$

Chen-Ning Franklin Yang & Robert Laurence Mills
Conservation of Isotopic Spin and Isotopic Gauge Invariance
The Physical Review **96**, 191 (1954)

The Euler-Lagrange field equations can be easily obtained and read

$$\begin{cases} g^{\mu\lambda}\,\nabla_{\mu}^{ab}\,F_{\lambda\nu}^{b}(x) = \jmath_{\nu}^{a}(x) \\ (i\not{D} - M)\,\psi(x) = 0 \end{cases} \tag{2.84}$$

where the so called non-Abelian fermionic vector current reads

$$\jmath_{\nu}^{a}(x) \equiv (-\,g)\,\bar\psi(x)\,\gamma^{\rho}\,\boldsymbol{\tau}_F^{\,a}\,\psi(x)\,g_{\nu\rho}$$

`Proof`: in fact we find

$$\begin{aligned} [\delta\mathcal{L}_{\rm YM}/\delta\partial_\mu A_\nu^a(x)] &= [\delta/\delta\partial_\mu\,A_\nu^a(x)]\left\{-\tfrac{1}{4}\,F^{\lambda\kappa d}\left(\partial_\lambda A_\kappa^d - \partial_\kappa A_\lambda^d + gf^{dbc}A_\lambda^b A_\kappa^c\right)\right\} \\ &= -\,F^{\mu\nu a}(x) \\ [\delta\mathcal{L}_{\rm YM}/\delta A_\nu^a(x)] &= -\,F^{\mu\nu d}(x)\,gf^{dba}A_\mu^b(x) + g\bar\psi(x)\gamma^\nu\boldsymbol{\tau}_F^a\psi(x) \\ &= gf^{abd}A_\mu^b(x)\,F^{\mu\nu d}(x) + g\bar\psi(x)\gamma^\nu\boldsymbol{\tau}_F^a\psi(x) \end{aligned}$$

so that consequently

$$\left[\,\partial_\mu\delta^{ad} + gf^{abd}A_\mu^b(x)\,\right]\,F^{\mu\nu d}(x) + g\bar\psi(x)\gamma^\nu\boldsymbol{\tau}_F^a\psi(x) = 0$$

and thereby

$$\nabla_\mu^{ab}F^{\mu\nu b}(x) = (-g)\bar\psi(x)\gamma^\nu\boldsymbol{\tau}_F^a\psi(x)$$

which is the claim to be proved □

Then it can be readily proved by symmetry reasons that the non-Abelian fermionic vector current

$$\jmath_{\nu}^{a}(x) \equiv (-\,g)\,\bar\psi(x)\,\gamma^{\rho}\,\boldsymbol{\tau}_F^{\,a}\,\psi(x)\,g_{\nu\rho}$$

does satisfy the covariant continuity equation, which means

$$\nabla_\mu^{ab}\, \jmath^{\mu b}(x) = 0 \tag{2.85}$$

As a matter of fact, from the commutator (2.82) and the Yang Mills field equations (2.84) we immediately obtain

$$\begin{aligned} \nabla_\lambda^{ab} \jmath^{\lambda b}(x) &= \nabla_\nu^{ab}\, \nabla_\mu^{bc}\, F^{\mu\nu c}(x) = \tfrac{1}{2}\, [\, \nabla_\nu^{ab}\, ,\, \nabla_\mu^{bc}\,]\, F^{\mu\nu c}(x) \\ &= \tfrac{1}{2}\, g\, f^{\,acd}\, F^{\mu\nu c}(x)\, F_{\mu\nu}^{\,d}(x) \equiv 0 \end{aligned}$$

owing to the complete anti-symmetry of the group structure constant. It is worthwhile to remark that in the non-Abelian case the classical equations of motion can also be rewritten as

$$\partial_x^\mu\, F_{\mu\nu}^{\,a}(x) = \jmath_\nu^{\,a}(x) - g f^{\,abc}\, A_\mu^{\,b}(x)\, F_{\lambda\nu}^{\,c}(x)\, g^{\,\mu\lambda} \tag{2.86}$$

in such a manner that one can always define a new non-Abelian current

$$J_\nu^{\,a}(x) \;\equiv\; (-\,g)\, \left\{ \bar\psi(x)\, \gamma^{\,\rho}\, \boldsymbol{\tau}_F^{\,a}\, \psi(x)\, g_{\,\nu\rho} + f^{\,abc}\, A_\mu^{\,b}(x)\, F_{\lambda\nu}^{\,c}(x)\, g^{\,\mu\lambda} \right\} \tag{2.87}$$

that actually involves both matter and gauge fields and satisfies the usual continuity equation

$$\partial^{\,\mu} J_\mu^{\,a}(x) = 0 \tag{2.88}$$

which drives *via* the Nöther theorem to the multiplet of conserved charges

$$Q^{\,a} \;\equiv\; \int \mathrm{d}\mathbf{x}\; J_0^{\,a}(t, \mathbf{x}) \qquad (\, a = 1, 2, \ldots, N^2 - 1\,) \tag{2.89}$$

`Proof`: the classical Yang-Mills Lagrangian (2.83) is invariant under non-Abelian gauge transformation so that, *a fortiori*, it turns out to be invariant under global, *i.e.* space-time point independent, transformations of the unitary SU(N) Lie group. It follows that, from the classical Nöther theorem, we can extract the corresponding $N^2 - 1$ Nöther vector currents, their number being equal to the dimension of the symmetry Lie group: namely,

$$J_\mu^{\,a}(x)\, \delta\omega^{\,a} = [\, \delta\mathcal{L}_{\,\mathrm{YM}}/\delta\partial^{\,\mu} A_\nu^{\,a}(x)\,]\; \Delta\, A_\nu^{\,a}(x) + [\, \delta\mathcal{L}_{\,\mathrm{YM}}/\delta\partial_\mu\psi(x)\,]\; \Delta\, \psi(x)$$

Now, owing to the space-time point independence of the internal symmetry group SU(N) the infinitesimal global variations do coincide with the local ones, *i.e.*

$$\begin{aligned} \delta\psi\,(x) &= ig\, \boldsymbol{\tau}_F^{\,a}\, \psi\,(x)\, \delta\,\omega^{\,a} \\ \delta A_\mu^{\,a}(x) &= g f^{abc}\, A_\mu^{\,b}(x)\, \delta\,\omega^{\,c} \\ \delta F_{\mu\nu}^{\,a}(x) &= g f^{abc}\, F_{\mu\nu}^{\,b}(x)\, \delta\omega^{\,c} \end{aligned}$$

and thereby we get

$$J_\mu^{\,a}(x) \,= (-\,g)\, \left\{ f^{abc}\, g^{\,\nu\rho}\, F_{\mu\rho}^{\,b}(x)\, A_\nu^{\,c}(x) + \overline{\psi}(x)\, \gamma^{\,\mu}\, \boldsymbol{\tau}_F^{\,a}\, \psi\,(x) \right\}$$

which entails

$$\Delta S_{\rm YM} = \Delta \int {\rm d}^4x\, \mathcal{L}_{\rm YM}(x) = 0 \quad \Longleftrightarrow \quad \partial^\mu J^a_\mu(x) = 0 \qquad (\,a = 1, 2, \ldots, N^2 - 1\,)$$

that proves the above statement □

The matrix-valued field strength dual tensor keeps the very same definition also in the non-Abelian case: in the adjoint representation, for example, from the identity (2.81) we can write

$$F_*^{\mu\nu} \equiv \tfrac12\, \epsilon^{\mu\nu\rho\sigma}\, I^{\,a}\, F^a_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\,[\,\nabla_\rho\,,\nabla_\sigma\,]\; g^{-1} \tag{2.90}$$

so that

$$F_*^{01} = -\,E_x^* = \tfrac12\,\epsilon^{0123}\, F^{23} = -\,B_x$$

$$\mathbf{E}^a_*(t,\mathbf{x}) = \mathbf{B}^a(t,\mathbf{x}) \qquad \mathbf{B}^a_*(t,\mathbf{x}) = \mathbf{E}^a(t,\mathbf{x}) \qquad (\,a = 1, 2, \ldots, N^2 - 1\,)$$

From the very definition of the matrix-valued field strength dual tensor it immediately follows that the non-Abelian generalization of the first couple of Maxwell equations holds true: namely,

$$g\,\nabla_\mu\, F_*^{\mu\nu}(x) = \epsilon^{\mu\nu\rho\sigma}\,\nabla_\mu\,[\,\nabla_\rho\,,\nabla_\sigma\,] \equiv 0 \tag{2.91}$$

The above differential geometric identity, which is called the Bianchi identity and does not change even in the presence of matter sources, is nothing but the non-Abelian generalization of the Faraday-Lenz law and the absence of magnetic monopoles in the electromagnetic theory. However, owing to the presence of the covariant derivative, the non-Abelian generalization of the first pair of Maxwell equations is highly nontrivial. Moreover it is possible to show that

$$\mathrm{tr}\, F_*^{\mu\nu} F_{\mu\nu} = \partial_\lambda K^{\,\lambda} \tag{2.92}$$

where

$$K^{\,\lambda} = \epsilon^{\,\lambda\nu\rho\sigma}\,\mathrm{tr}\left(F_{\nu\rho}\, A_\sigma + \tfrac23\, i\, A_\nu\, A_\rho\, A_\sigma \right) \tag{2.93}$$

`Proof.` For the sake of simplicity and without loss of generality, let us normalize the generators in the adjoint representation according to Shur's lemma

$$\mathrm{tr}\,(I^{\,a} I^{\,b}) = g^{\,ab}\,\mathrm{tr}\,\mathbb{I}_A \equiv 2\varkappa\,\delta^{\,ab}$$

where the Cartan-Killing metric is provided by

$$g_{ab} = f_{adc} f_{bcd} \qquad g_{ab}\, g^{\,ab} \equiv n = N^2 - 1$$

for SU(N), while $\varkappa$ is a suitable normalization constant. Then we find

$$
\begin{aligned}
&\varkappa^{-1}\,\mathrm{tr}\,F_*^{\mu\nu}F_{\mu\nu} = \epsilon^{\,\mu\nu\rho\sigma}\,F_{\mu\nu}^{\,a}\,F_{\rho\sigma}^{\,a} = \epsilon^{\,\mu\nu\rho\sigma}\Big(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b\,A_\nu^c\Big)F_{\rho\sigma}^{\,a}\\
= \;\;& 2\epsilon^{\,\mu\nu\rho\sigma}\,\partial_\mu\left(A_\nu^a\,F_{\rho\sigma}^{\,a}\right) - 2\epsilon^{\,\mu\nu\rho\sigma}\,A_\nu^a\,\partial_\mu F_{\rho\sigma}^{\,a} + gf^{abc}\epsilon^{\,\mu\nu\rho\sigma}\,A_\mu^b\,A_\nu^c\,F_{\rho\sigma}^{\,a}\\
= \;\;& 2\epsilon^{\,\mu\nu\rho\sigma}\,\partial_\mu\left(A_\nu^a\,F_{\rho\sigma}^{\,a}\right) - 2gf^{ade}\epsilon^{\,\mu\nu\rho\sigma}\,A_\nu^a\,\partial_\mu\left(A_\rho^d\,A_\sigma^e\right) + gf^{abc}\epsilon^{\,\mu\nu\rho\sigma}\,A_\mu^b\,A_\nu^c\,F_{\rho\sigma}^{\,a}\\
= \;\;& \epsilon^{\,\mu\nu\rho\sigma}\,\partial_\mu\,\mathrm{tr}\,(A_\nu\,F_{\rho\sigma}) - 2gf^{ade}\epsilon^{\,\mu\nu\rho\sigma}\,\partial_\mu\left(A_\nu^a\,A_\rho^d\,A_\sigma^e\right) + 2gf^{ade}\epsilon^{\,\mu\nu\rho\sigma}(\partial_\mu\,A_\nu^a)A_\rho^d\,A_\sigma^e\\
+ \;\;& 2gf^{abc}\epsilon^{\,\mu\nu\rho\sigma}\,A_\mu^b\,A_\nu^c\,\partial_\rho A_\sigma^{\,a} + g^2f^{abc}\,f^{ade}\,\epsilon^{\,\mu\nu\rho\sigma}\,A_\mu^b\,A_\nu^c\,A_\rho^d\,A_\sigma^e
\end{aligned}
$$

Now the very last term of the last equality identically vanishes owing to symmetry. As a matter of fact, consider the anti-Hermitean infinitesimal operators of the $n-$dimensional irreducible adjoint representation with $n = N^2 - 1$. We find

$$
I_a\,I_b = \tfrac{1}{2}\,\{I_a\,,I_b\} + \tfrac{1}{2}\,[\,I_a\,,I_b\,] \qquad\qquad \mathrm{tr}\,(I_a\,I_b) = g_{ab} = \tfrac{1}{2}\,\mathrm{tr}\,\{I_a\,,I_b\}
$$

where g_{ab} is the non-singular Cartan-Killing metric of the semi-simple group SU(N), so that we can write

$$
I_a\,I_b = n^{-1}g_{ab}\,\mathbb{I} + \tfrac{1}{2}\,f^{abc}I_c
$$

with matrix elements

$$
f^{adc}f^{\,bce} = n^{-1}g_{ab}\,\delta_{de} + \tfrac{1}{2}\,f^{abc}f^{ced} = n^{-1}g_{ab}\,\delta_{de} + \tfrac{1}{2}\left(f^{\,bce}f^{adc} - f^{eac}f^{cbd}\right)
$$

or even

$$
f^{\,bce}f^{\,adc} + f^{\,ace}f^{\,bdc} = \frac{2}{n}\,g_{ab}\,\delta_{de}
$$

Now the RHS of the above relation is manifestly symmetric under the exchange of a and b indexes, for the Cartan-Killing metric is symmetric, so that we can definitely write

$$
f^{\,ace}f^{\,bdce} \;=\; -\,n^{-1}\,g_{ab}\,\delta_{cd}
$$

and thereby

$$
f^{abc}\,f^{ade}\,\epsilon^{\,\mu\nu\rho\sigma}\,A_\mu^b\,A_\nu^c\,A_\rho^d\,A_\sigma^e \equiv 0
$$

Furthermore one gets, for instance,

$$
f^{ade}\,\epsilon^{\,\mu\nu\rho\sigma}(\partial_\mu\,A_\nu^a)A_\rho^d\,A_\sigma^e = \tfrac{1}{3}\,f^{ade}\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu\left(A_\nu^a\,A_\rho^d\,A_\sigma^e\right)
$$

Hence we definitely obtain

$$
\begin{aligned}
\mathrm{tr}\,F_*^{\mu\nu}F_{\mu\nu} \;&=\; \epsilon^{\,\mu\nu\rho\sigma}\,\partial_\mu\,\mathrm{tr}\,(A_\nu\,F_{\rho\sigma}) - \tfrac{2}{3}\,gf^{ade}\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu\left(A_\nu^a\,A_\rho^d\,A_\sigma^e\right)\\
&=\; \epsilon^{\,\lambda\nu\rho\sigma}\,\partial_\lambda\,\mathrm{tr}\,\Big(F_{\nu\rho}\,A_\sigma + \tfrac{2}{3}\,i\,A_\nu\,A_\rho\,A_\sigma\Big)
\end{aligned}
$$

which is the result because

$$
\mathrm{tr}\;(A_\nu\,A_\rho\,A_\sigma) = \tfrac{1}{2}\,\mathrm{tr}\;(A_\nu\,[\,A_\rho\,,A_\sigma\,]) = \tfrac{1}{2}\,i\,f^{abc}A_\rho^b\,A_\sigma^c\,\mathrm{tr}\;(A_\nu\,\boldsymbol{\tau}_F^a) = i\,f^{abc}A_\nu^a\,A_\rho^b\,A_\sigma^c
$$

as claimed. □

The Yang-Mills Lagrangian can be split into the free quadratic part and the cubic and fourth-order interaction Lagrangian, *i.e.*

$$\begin{aligned}
\mathcal{L}_{\rm YM} &= \mathcal{L}_0 + \mathcal{L}_{\rm int} && (2.94)\\
\mathcal{L}_0 &= -\tfrac{1}{2}\, g^{\mu\rho}\, g^{\nu\sigma} \Big(\partial_\mu\, A^a_\nu(x) - \partial_\nu\, A^a_\mu(x)\Big)\, \partial_\rho\, A^a_\sigma(x) \\
&+ \bar\psi(x)\,(i\partial\!\!\!/ - M)\,\psi(x) && (2.95)\\
\mathcal{L}_{\rm int} &= g\,\bar\psi(x)\,\gamma^\mu\, \boldsymbol{\tau}^a_F\, \psi(x)\, A^a_\mu(x) \\
&- g\, f^{abc}\, g^{\mu\rho}\, g^{\nu\sigma}\, A^b_\rho(x)\, A^c_\sigma(x)\, \partial_\mu\, A^a_\nu(x) \\
&- \tfrac{1}{4}\, g^2\, f^{abc}\, f^{ade}\, g^{\mu\rho}\, g^{\nu\sigma}\, A^b_\mu(x)\, A^c_\nu(x)\, A^d_\rho(x)\, A^e_\sigma(x) && (2.96)
\end{aligned}$$

Once again, as in the Abelian case, in order to invert the kinetic term and to obtain the free propagator for the non-Abelian vector potential, one has to suitably introduce a gauge fixing Lagrangian

$$\mathcal{L}_{\rm g.f.} = g^{\mu\nu}\, A^a_\mu(x)\, \partial_\nu\, B^a(x) \, + \tfrac{1}{2}\, \xi\, B^a(x)\, B^a(x)$$

where $B(x) = B^a(x)\, \boldsymbol{\tau}^a_F$ ($a = 1, 2, \ldots, N^2 - 1$) is a multiplet of auxiliary, non-physical, ghost-like scalar fields which transform in accordance with the adjoint representation of the gauge symmetry group

$$B(x) \quad\longmapsto\quad B_\omega(x) = U_\omega(x) B(x) U^\dagger_\omega(x) \qquad \delta\, B^a(x) = g f^{abc}\, B^b(x)\, \omega^c(x)$$

while the dimensionless parameter $\xi \in \mathbb{R}$ is again the gauge fixing parameter. Then the field equations become

$$\begin{cases} g^{\mu\lambda}\, \partial_\mu\, F^a_{\lambda\nu}(x) = J^a_\nu(x) - \partial_\nu\, B^a(x) \\ (iD\!\!\!\!/\, - M)\, \psi(x) = 0 \\ \partial^\mu\, A^a_\mu(x) = \xi\, B^a(x) \end{cases} \qquad (2.97)$$

Notice however that the non-Abelian vector currents $J^a_\mu(x)$ no longer fulfill the continuity equation, because the gauge fixing Lagrangian gives rise to an extra term in the Nöther theorem for the global SU(N) transformations, that is

$$[\, \delta\mathcal{L}_{\rm g.f.}/\delta\partial^\mu B^a(x)\,]\ \Delta B^a(x) = A^a_\mu(x)\, g f^{abc}\, B^b(x)\, \delta\,\omega^c$$

in such a manner that we can write

$$\overline{J}^a_\mu(x) = J^a_\mu(x) - g f^{abc}\, A^b_\mu(x)\, B^c(x)$$

whence

$$\partial^\mu \overline{J}^a_\mu(x) = 0 \quad\Longleftrightarrow\quad \partial^\mu J^a_\mu(x) = g f^{abc}\, A^b_\mu(x)\, \partial^\mu B^c(x)$$

where use has been made of the last field equation (2.97). Thus, taking the divergence of the first field equation (2.97) we get

$$\nabla_\mu^{ab}\, \partial^\mu B^b(x) = \Box B^a(x) + g f^{abc} A_\mu^b(x)\, \partial^\mu B^c(x) = 0 \tag{2.98}$$

whence it appears that the non-physical auxiliary fields are no longer free fields as it happened in the Abelian case – see eq. (2.65). On the contrary, the ghost auxiliary fields $B^a(x)$ are directly as well as indirectly coupled to the gauge potentials $A_\mu^a(x)$ and to the fermionic matter fields multiplet $\psi(x)$ through the field equations. Hence, the covariant quantization of the non-Abelian gauge theories will require some new tools: the introduction of Lorentz scalar but anti-commuting non-physical fields, the Faddeev Popov ghost-antighost fields, and the so called Becchi-Rouet-Stora-Tyutin (BRST) symmetry, an algebraic graded continuous symmetry involving the whole set of physical and non-physical quantized field.

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2. Carlo Maria Becchi, Alain Rouet & Raymond Stora (1974)
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   Preprint No. 39, FIAN AN SSSR Lebedev Physical Institute
   USSR Academy of Sciences, Moscow.

4. Taichiro Kugo and Izumi Ojima (1979)
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The covariant quantization and renormalization of the non-Abelian gauge theories are the cornerstones of the Standard Model of the Electroweak and Strong Interactions. According to this model the gauge symmetry group is the unitary group $SU(3) \times SU(2) \times U(1)$ of dimensions $8+3+1 = 12$, which is a non-semi-simple Lie group. Thus the fermion multiplets which undergo strong interactions will posses three degrees of freedom named colors (red, green and blue) associated to the $SU(3)$ symmetry governing the nuclear strong interactions, while the $SU(2)$ spinor multiplets do exhibit two weak isospin degrees of freedom named flavors, that will be attached to the nuclear weak interactions together with the $U(1)$ hypercharge that will be related to the electromagnetic interactions.

`Gauge fields and scalar Matter`. The coupling of the non-Abelian gauge fields with matter scalar fields runs along the very same steps of fermionic matter. To be general, consider a multiplet of N complex scalar fields

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_N(x) \end{pmatrix} \tag{2.99}$$

where $\phi_\imath : \mathcal{M} \longrightarrow \mathbb{C} \quad (\forall \imath = 1, 2, \ldots, N)$ and transform according to a fundamental representation of $SU(N)$

$$\Phi'(x) = U_\omega(x)\,\Phi(x) \tag{2.100}$$

The covariant derivative is the same

$$D_\mu \Phi(x) = \partial_\mu \Phi(x) - igA_\mu^a(x)\,\boldsymbol{\tau}_F^a\,\Phi(x) \tag{2.101}$$

and fulfills

$$D'_\mu = U_\omega(x)\,D_\mu\,U_\omega^{-1}(x)$$

so that the quantity

$$\begin{aligned} [\,D_\mu \Phi(x)\,]^\dagger D^\mu\,\Phi(x) \;&=\; \partial_\mu\,\Phi^\dagger(x)\,\partial^\mu\,\Phi(x) \\ -\quad ig\,[\,\partial_\mu\,\Phi^\dagger(x)\,]\,A_\mu(x)\,\Phi(x) &+ ig\,\Phi^\dagger(x) A_\mu(x)\,\partial^\mu\,\Phi(x) + g^2\,\Phi^\dagger(x)\,A_\mu(x)\,A^\mu(x)\,\Phi(x) \end{aligned}$$

is a Poincaré invariant and gauge invariant quantity. It follows that the most general Poincaré invariant and gauge invariant Lagrangian, which involves non-Abelian vector fields and complex scalar fields and which satisfies all the general requirements of the local quantum field theory, including power-counting renormalizability, is provided by

$$\mathcal{L} = -\,\tfrac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}\,F^{\,\mu\nu}) + (D_\mu \Phi)^\dagger D^{\,\mu}\,\Phi \mp m^2\,\Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2 \tag{2.102}$$

the lower sign in the mass term just corresponding to the spontaneous breaking of the $U(N)$ gauge symmetry. The Euler-Lagrange field equations can be easily found to be

$$\begin{cases} g^{\,\mu\lambda}\,\nabla_\mu^{ab}\,F_{\lambda\nu}^b = \jmath_\nu^a \\ \{(\,\partial^{\,\mu} + igA^\mu\,)D_\mu \pm m^2 + 2\lambda(\Phi^\dagger \Phi)\}\Phi = 0 \end{cases} \tag{2.103}$$

where the so called non-Abelian bosonic vector current reads

$$\jmath_\nu^a(x) \;\equiv\; g\,\Phi^\dagger(x)\,\boldsymbol{\tau}_F^a\,i\overleftrightarrow{\partial}_\nu \Phi(x) + 2g^2\Phi^\dagger(x)\boldsymbol{\tau}_F^a\,A_\nu(x)\Phi(x)$$

As a matter of fact we find

$$\begin{aligned} [\,\delta\mathcal{L}/\delta\partial_\mu A_\nu^a(x)\,] \;&=\; -\,F^{\mu\nu a}(x) \\ [\,\delta\mathcal{L}/\delta A_\nu^a(x)\,] \;&=\; -\,F^{\mu\nu d}(x)\,gf^{dba}A_\mu^b(x) - \jmath^{\,a\nu}(x) \\ &=\; gf^{abd}A_\mu^b(x)\,F^{\mu\nu d}(x) - \jmath^{\,a\nu}(x) \end{aligned}$$

so that consequently

$$\nabla_\mu^{ab} F^{\mu\nu b}(x) = \jmath^{\,a\nu}(x)$$

as claimed. Then it can be readily proved by symmetry reasons that the non-Abelian bosonic vector current does stil satisfy the covariant continuity equation, which means

$$\nabla_\mu^{ab}\, \jmath^{\,\mu b}(x) = 0$$

Actually, from the commutator (2.82) and the Yang Mills field equations (2.84) we get

$$\begin{aligned}\nabla_\lambda^{ab} \jmath^{\,\lambda b}(x) &= \nabla_\nu^{ab}\, \nabla_\mu^{bc}\, F^{\mu\nu c}(x) = \tfrac{1}{2}\,[\, \nabla_\nu^{ab}\, ,\, \nabla_\mu^{bc}\,]\, F^{\mu\nu c}(x) \\ &= \tfrac{1}{2}\, g\, f^{\,acd}\, F^{\mu\nu c}(x)\, F_{\mu\nu}^{\,d}(x) \equiv 0\end{aligned}$$

owing to the complete anti-symmetry of the group structure constant. It is worthwhile to remark that in the non-Abelian case the classical equations of motion can also be rewritten as

$$\partial_x^\mu\, F_{\mu\nu}^{\,a}(x) = \jmath_\nu^{\,a}(x) - g f^{\,abc}\, A_\mu^{\,b}(x)\, F_{\lambda\nu}^{\,c}(x)\, g^{\,\mu\lambda} \tag{2.104}$$

in such a manner that one can always define a new non-Abelian current

$$J_\nu^{\,a}(x) \;\equiv\; \jmath_\nu^{\,a}(x) - g f^{\,abc}\, A_\mu^{\,b}(x)\, F_{\lambda\nu}^{\,c}(x)\, g^{\,\mu\lambda} \tag{2.105}$$

that actually involves both matter and gauge fields and satisfies the usual continuity equation

$$\partial^{\,\mu} J_\mu^{\,a}(x) = 0 \tag{2.106}$$

which drives *via* the Nöther theorem to the multiplet of conserved charges

$$Q^{\,a} \;\equiv\; \int \mathrm{d}\mathbf{x}\; J_0^{\,a}(t, \mathbf{x}) \qquad (\, a = 1, 2, \ldots, N^2 - 1\,) \tag{2.107}$$

It turns out that the above classical Lagrangian (2.83) is invariant under non-Abelian gauge transformation as well as ordinary local phase transformations so that, *a fortiori*, it turns out to be invariant under global, *i.e.* space-time point independent, transformations of the unitary $U(N)$ Lie group. It follows that, from the classical Nöther theorem, we can extract the corresponding N^2 Nöther vector currents, their number being equal to the dimension of the symmetry Lie group.

2.6 Euclidean Field Theories

The Feynman rules for Euclidean field theories can be readily obtained in accordance with the main guidelines I have discussed in the framework of the quantum field theories in the Minkowski space. To be definite, let me consider the model of the Euclidean scalar and fermion fields interacting *à la* Yukawa, which is described by the Euclidean Action

$$\begin{aligned}S_E\,[\,\phi_E\,,\,\psi_E\,,\,\bar\psi_E\,] \;&=\; \int \mathrm{d}x_E\;\left(\tfrac{1}{2}\,\partial_\mu\phi_E\,\partial_\mu\phi_E + \tfrac{1}{2}\,m^2\,\phi_E^2 + \tfrac{\lambda}{4!}\,\phi_E^4\right) \\ &+\; \int \mathrm{d}x_E\;\bar\psi_E\,(\,\partial\!\!\!/_E + M + g\,\phi_E\,)\,\psi_E\end{aligned} \tag{2.108}$$

where as usual

$$x_{E\mu} = (\mathbf{x}\,,\, x_4 = i\,x^0\,) \qquad \partial_\mu = \frac{\partial}{\partial x_{E\mu}} \qquad x_{E\mu}y_{E\mu} = \mathbf{x}\cdot\mathbf{y} + x_4 y_4$$

$$i\,\partial\!\!\!/_E \;=\; \overline{\gamma}_\mu \frac{\partial}{\partial x_{E\mu}} \qquad \left\{\overline{\gamma}_\mu\,,\,\overline{\gamma}_\nu\right\} = 2\,\delta_{\mu\nu} \qquad \overline{\gamma}_\mu \;=\; \overline{\gamma}_\mu^{\dagger}$$

The generating functional for Euclidean disconnected correlation functions will be defined by

$$\begin{aligned} Z_E[\,J_E\,,\,\bar\zeta_E\,,\,\zeta_E\,] &= \exp\left\{-\,W_E[\,J_E\,,\,\zeta_E\,,\,\bar\zeta_E\,]\right\} \\ &= \exp\left\{-\,V_E[\delta/\delta J_E\,,\,\delta/\delta\zeta_E\,,\,\delta/\delta\bar\zeta_E]\right\} \\ &\times\;\; Z_E^0[\,J_E\,,\,\bar\zeta_E\,,\,\zeta_E\,] \\ &\stackrel{\text{def}}{=} \exp\left\{-\,V_E[\delta/\delta J_E\,,\,\delta/\delta\zeta_E\,,\,\delta/\delta\bar\zeta_E]\right\} \\ &\times\;\; \exp\left\{-\,W_E^0[\,J_E\,,\,\zeta_E\,,\,\bar\zeta_E\,]\right\} \end{aligned} \qquad (2.109)$$

where

$$\begin{aligned} Z_E^0[\,J_E\,,\,\bar\zeta_E\,,\,\zeta_E\,] &= \mathcal{N}\int \mathfrak{D}\phi_E\,\mathfrak{D}\psi_E\,\mathfrak{D}\bar\psi_E\;\exp\left\{-\,S_E^0[\,\phi_E\,,\,\psi_E\,,\,\bar\psi_E\,]\right\} \\ &\times\;\;\exp\int \mathrm{d}x_E\;\left(\phi_E\,J_E + \bar\psi_E\,\zeta_E + \bar\zeta_E\,\psi_E\right) \end{aligned} \qquad (2.110)$$

$$S_E^0[\,\phi_E\,,\,\psi_E\,,\,\bar\psi_E\,] \;=\; \int \mathrm{d}x_E\;\left[\,\tfrac12\,\partial_\mu\phi_E\,\partial_\mu\phi_E + \tfrac12\,m^2\,\phi_E^2 + \bar\psi_E\,(\,\partial\!\!\!/_E + M\,)\,\psi_E\,\right] \qquad (2.111)$$

$$W_E^0[\,J_E\,,\,\zeta_E\,,\,\bar\zeta_E\,] = \left\langle\,\tfrac12\,J_{Ex}\,D^E_{xy}\,J_{Ey} + \bar\zeta_{Ex}\,S^E_{xy}\,\zeta_{Ey}\,\right\rangle$$

in which

$$D_E\,(x_E) = \frac{1}{(2\pi)^4}\int \mathrm{d}\,k_E\;\frac{\exp\{i\,k_E\cdot x_E\}}{k_E^2 + m^2}$$

$$S^E_{\alpha\beta}\,(x_E) = \int \frac{\mathrm{d}\,k_E}{(2\pi)^4}\;\exp\{i\,k_E\cdot x_E\}\;\left(\frac{i}{-\,p\!\!\!/_E + iM}\right)_{\alpha\beta}$$

Then the Feynman rules immediately follows and read

euclidean scalar propagator : $\dfrac{1}{k_E^2 + m^2}$

euclidean spinor propagator : $\dfrac{i}{-\,p\!\!\!/_E + iM} = \dfrac{-\,i\,p\!\!\!/_E + M}{p_E^2 + M^2}$

$$\text{euclidean scalar vertex :} \qquad -\lambda \qquad \left(\bar{k}_1 + \bar{k}_2 + \bar{k}_3 + \bar{k}_4 \,=\, 0\right)$$
$$\text{euclidean Yukawa vertex :} \qquad -g \qquad \left(\bar{p} + \bar{q} + \bar{k} \,=\, 0\right)$$

where all momenta are supposed to be incoming [3] while $k_E \equiv \bar{k}$ and $p_E \equiv \bar{p}$

$$\text{euclidean scalar loop :} \qquad \int \frac{\mathrm{d}^4\bar{k}}{(2\pi)^4}$$
$$\text{euclidean spinor loop :} \qquad (-1)\int \frac{\mathrm{d}^4\bar{p}}{(2\pi)^4}$$

the symmetry factors being obviously the same as in Minkowski space. The Euclidean formulation of quantum electrodynamics is also achieved after a straightforward generalization of the above recipe. To be specific we have

$$\bar{A}_\mu = (\,\mathbf{A}\,,\,A_4 = -\,i\,A_0\,) \qquad \bar{F}_{\mu\nu} = \partial_\mu\,\bar{A}_\nu - \partial_\nu\,\bar{A}_\mu \tag{2.112}$$
$$i\,\gamma^\mu\,[\,\partial_\mu + iq\,A_\mu(x)\,] \;\mapsto\; \overline{\gamma}_\mu\left[\,-\,\partial_{E\,\mu} - iq\,\bar{A}_\mu\,(x_E)\,\right] \tag{2.113}$$

$$S_E\,[\,\bar{A}_\mu\,,\,B_E\,,\,\bar{\psi}_E\,,\,\psi_E\,] \;=\; \int \mathrm{d}\bar{x}\,\Big\{\tfrac{1}{4}\,\bar{F}_{\mu\nu}\,\bar{F}_{\mu\nu} - B_E\,\partial_\mu\,\bar{A}_\mu + \tfrac{1}{2}\,B_E^2 + \;\bar{\psi}_E\left(\partial\!\!\!/_E + M - ie\overline{\gamma}_\mu\bar{A}_\mu\right)\psi_E\Big\}$$

and thereby

$$\text{euclidean photon propagator :} \qquad \frac{\delta_{\mu\nu}}{k_E^2}$$
$$\text{euclidean photon} - \text{spinor vertex :} \qquad e\overline{\gamma}_\mu \qquad (p_E + k_E + q_E \,=\, 0\,)$$

The connected Euclidean Green's functions are named correlation functions or Schwinger's functions and are given by

$$G_E^{(n)}\,(\bar{x}_1\,,\,\ldots\,,\,\bar{x}_n\,) \;=\; -\;\delta^{(n)}\,W_E\,[\,J_E\,]\,/\,\delta\,J_E(\bar{x}_1)\;\cdots\;\delta\,J_E(\bar{x}_n)\,\big\rfloor_{J_E=0} \tag{2.114}$$

Now it is very instructive to compare the Feynman rules for quantum field theories in the Minkowski and Euclidean spaces. In the case of the self-interacting scalar field we find

$$\frac{1}{k_E^2 + m^2} \qquad \text{propagator} \qquad \frac{i}{k^2 - m^2 + i\varepsilon}$$

[3] We have to remember that ψ_E and $\bar{\psi}_E$ are truly independent.

$$
\begin{array}{ccc}
-\,\lambda & \text{vertex} & -\,i\lambda \\
\displaystyle\int \frac{\mathrm{d}^4\ell_E}{(2\pi)^4} & \text{loop integration} & \displaystyle\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}
\end{array}
$$

while for Yukawa neutral meson theory and quantum electrodynamics

$$
\begin{array}{ccc}
\dfrac{i}{-\not p_E + iM} & \text{propagator} & \dfrac{i}{\not p - M + i\varepsilon} \\[2ex]
\dfrac{\delta_{\mu\nu}}{k_E^2 + m^2} & \text{propagator} & \dfrac{-\,i\,g_{\mu\nu}}{k^2 - m^2 + i\varepsilon} \\[2ex]
-\,g\,\delta\,(p_E + k_E + q_E) & \text{vertex} & -\,ig\,\delta\,(p + k - q) \\
ie\,\overline{\gamma}_\mu \delta\,(p_E + k_E + q_E) & \text{vertex} & ie\,\gamma^\mu\,\delta\,(p + k - q) \\
(-1)\displaystyle\int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} & \text{spinor loop integration} & (-1)\displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4}
\end{array}
$$

As a consequence, the transition from a connected $n-$point Schwinger's function to a connected $n-$point Green's function in momentum space can be readily achieved. We shall see how to proceed in the forthcoming sections.

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2.7 Problems

Evaluate the fermion determinant in the presence of a background classical Abelian real vector field $A_\mu(x)$.

Solution. By proceeding in close analogy with the Yukawa case, we start from the functional integral representation

$$Z^{\rm F}[\,\zeta\,,\,\bar\zeta\,;A^\mu\,] \;=\; \mathcal{N}_A \int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi$$

$$\exp\left\{\, i\textstyle\int {\rm d}x\,\bar\psi(x)\,[\,\gamma^\mu\, i\partial_\mu - M - q\,\gamma^\nu A_\nu(x)\,]\,\psi(x)\,\right\}$$

$$\times\;\exp\left\{ i\textstyle\int {\rm d}x\;\left[\,\bar\zeta(x)\,\psi(x) + \bar\psi(x)\,\zeta(x)\,\right]\right\}$$

where q is the electric charge of the spinor particle while

$$Z^{\rm F}[\,0\,,\,0\,;A^\mu\,] \;=\; \mathcal{N}_A \int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi$$

$$\exp\left\{\, i\textstyle\int {\rm d}x\;\bar\psi\,(x)\,[\,\gamma^\mu\, i\partial_\mu - M - q\,\gamma^\nu A_\nu(x)\,]\,\psi\,(x)\,\right\}$$

$$\stackrel{\rm def}{=}\;\mathcal{N}_A\,\det\;\|\; i\,\partial\!\!\!/ - M - q\,A\!\!\!/\;\|$$

the latter definition being understood, as usual, up to the transition to the Euclidean formulation. Notice that the constant $\mathcal{N}_A$ is conveniently fixed by the requirement that in the limit of a pure gauge vector potential

$$A_\nu(x) \to \partial_\nu f(x)$$

we recover $Z^{\rm F}[\,0\,,\,0\,;0\,] \;=\; 1$. Actually it is very easy to recognize that the above fermionic determinant is gauge invariant. As a matter of fact, from the legitimate manipulations of the functional integral we get

$$Z^{\rm F}[\,0\,,\,0\,;A_\nu + \partial_\nu f\,] \;=\; \mathcal{N}_A \int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi$$

$$\exp\left\{ i\textstyle\int {\rm d}x\,\bar\psi(x)\,[\,\gamma^\mu\, i\partial_\mu - M - q\,\gamma^\nu A_\nu(x) - q\,\gamma^\mu\partial_\mu f(x)\,]\,\psi(x)\right\}$$

$$= \mathcal{N}_A \int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi\;\exp\left\{ i\textstyle\int {\rm d}x\,\bar\psi(x)\,e^{-iqf(x)}\right.$$

$$\left.\times\;[\,\gamma^\mu\, i\partial_\mu - M - q\,\gamma^\nu A_\nu(x)\,]\,{\rm e}^{iqf(x)}\,\psi(x)\right\}$$

Changing the functional integration variables according to

$$\psi\,'(x) = e^{\,iqf(x)}\,\psi(x) \qquad\qquad \bar\psi\,'(x) = e^{-iqf(x)}\,\bar\psi(x)$$

$$\int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi = \int \mathfrak{D}\psi\,' \int \mathfrak{D}\bar\psi\,'$$

we come to the conclusion that

$$Z^{\mathrm{F}}[\,0\,,\,0\,;A_\nu + \partial_\nu f\,] = Z^{\mathrm{F}}[\,0\,,\,0\,;A_\nu\,] \tag{2.115}$$

which amounts to the endorsement of gauge invariance. As a consequence, we eventually come to the symbolic equality

$$\begin{aligned}
\frac{\det \|\; i\,\partial\!\!\!/ - M - q\,A\!\!\!/ \;\|}{\det \|\; i\,\partial\!\!\!/ - M \;\|} &= \det \|\; \mathbb{I} - q\,(\,i\,\partial\!\!\!/ - M\,)^{-1} A\!\!\!/ \;\| \\
&= \exp\{\,\mathrm{Tr}\,\ln \|\; \mathbb{I} - q\,(\,i\,\partial\!\!\!/ - M\,)^{-1} A\!\!\!/ \;\|\,\} \\
&= \exp\left\{(-\,1)\sum_{n\,=\,1}^{\infty} \frac{1}{n}\;\mathrm{Tr}\,[\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,q\,A\!\!\!/\,]^{\,n}\right\}
\end{aligned}$$

in which we understand

$$\langle\, x\,|\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,|\,y\,\rangle \;=\; \frac{1}{i}\,S_F(\,x - y\,)$$

while the symbol [Tr] indicates the sum over spinor indexes and integration over space-time coordinates. After setting

$$X_A^n = (-1)\,\mathrm{Tr}\,[\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,q\,A\!\!\!/\,]^{\,n}$$

explicit evaluation for $n = 1$ yields

$$\begin{aligned}
X_A^1 &= (-1)\,\mathrm{Tr}\,[\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,q\,A\!\!\!/\,] \\
&= -\,q\int \mathrm{d}x_1\;\langle\, x_1\,|\,\mathrm{tr}\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,A\!\!\!/\,|\,x_1\,\rangle \\
&= -\,q\int \mathrm{d}x_1 \int \mathrm{d}y\;\mathrm{tr}\,\langle\, x_1\,|\,(\,i\,\partial\!\!\!/ - M\,)^{-1}\,|\,y\rangle\,\langle\, y\,|\,A\!\!\!/\,|\,x_1\,\rangle \\
&= iq\int \mathrm{d}x_1 \int \mathrm{d}y\;\mathrm{tr}\,[\,A\!\!\!/(x_1)\,S_F(x_1 - y)\,]\,\delta(x_1 - y) \\
&= iq\int \mathrm{d}x_1\;\mathrm{tr}\,[\,A\!\!\!/(x_1)\,S_F(x_1 - x_1)\,] \\
&= iq\int \mathrm{d}x_1\;\mathrm{tr}\,[\,A\!\!\!/(x_1)\,S_F(0)\,] \;\stackrel{\mathrm{def}}{=}\; iq\;\mathrm{Tr}\,(A\!\!\!/\,S_F)
\end{aligned}$$

in which the symbol [tr] denotes the sum over spinor indexes. The next term can be handled in a quite similar way by making repeatedly use of the completeness relation

$$\int \mathrm{d}x\;|\,x\,\rangle\,\langle\, x\,| \;=\; \mathbb{I}$$

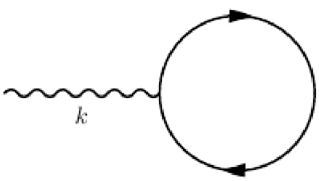

Figure 2.8: the tadpole graph X_A^1

Then we obtain

$$X_A^2 = q^2 \int \mathrm{d}x_1 \int \mathrm{d}x_2 \, \mathrm{tr}\,[\, S_F(x_1 - x_2)\, \not{A}(x_1)\, S_F(x_2 - x_1)\, \not{A}(x_2)\,]$$

It is convenient to introduce the center of mass and relative coordinates

$$\bar{x} = \frac{1}{2}\,(x_1 + x_2) \qquad\qquad x = x_1 - x_2 \qquad\qquad \frac{\partial(x_1, x_2)}{\partial(X, x)} = 1$$

so that

$$\begin{aligned}
X_A^2 &= q^2 \int \mathrm{d}\bar{x} \int \mathrm{d}x \;\mathrm{tr}\,[\, S_F(x)\, \not{A}(\bar{x} + x/2)\, S_F(-\,x)\, \not{A}(\bar{x} - x/2)\,] \\
&= q^2 \int \mathrm{d}\bar{x} \int \mathrm{d}x \int \frac{\mathrm{d}\ell}{(2\pi)^4} \int \frac{\mathrm{d}k}{(2\pi)^4} \int \frac{\mathrm{d}p}{(2\pi)^4} \int \frac{\mathrm{d}q}{(2\pi)^4} \\
&\times\; \exp\{-\, i\bar{x}\cdot(\ell + k) -\; ix\cdot(\ell - k)/2 + i(p - q)\cdot x\} \\
&\times\; \mathrm{tr}\left[\frac{i}{\not{p} - M + i\varepsilon}\,\widetilde{\not{A}}(\ell)\,\frac{i}{\not{q} - M + i\varepsilon}\,\widetilde{\not{A}}(k)\right] \\
&= -\, q^2 \int \frac{\mathrm{d}k}{(2\pi)^4}\;\widetilde{A}_\mu(k)\,\widetilde{A}_\nu(-\,k) \int \frac{\mathrm{d}p}{(2\pi)^4} \int \frac{\mathrm{d}q}{(2\pi)^4} \\
&\times\; \frac{\mathrm{tr}\,[\,(\,\not{p} + M\,)\gamma^\mu(\,\not{q} + M\,)\gamma^\nu\,]}{(\,p^2 - M^2 + i\varepsilon\,)(\,q^2 - M^2 + i\varepsilon\,)}\;(2\pi)^4\,\delta\,(k + p - q) \\
&= (-\,iq)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\;\widetilde{A}_\rho(k)\,\widetilde{A}_\sigma(-\,k) \\
&\times\; (-1) \int \frac{\mathrm{d}^4p}{(2\pi)^4}\;\mathrm{tr}\left[\frac{i}{\not{p} - M + i\varepsilon}\,\gamma^\rho\,\frac{i}{\not{p} + \not{k} - M + i\varepsilon}\,\gamma^\sigma\right]
\end{aligned}$$

the very last line corresponding to a fermion loop with two propagators and

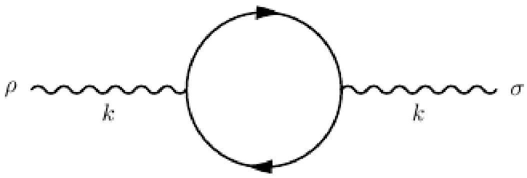

Figure 2.9: the self-energy graph X_A^2

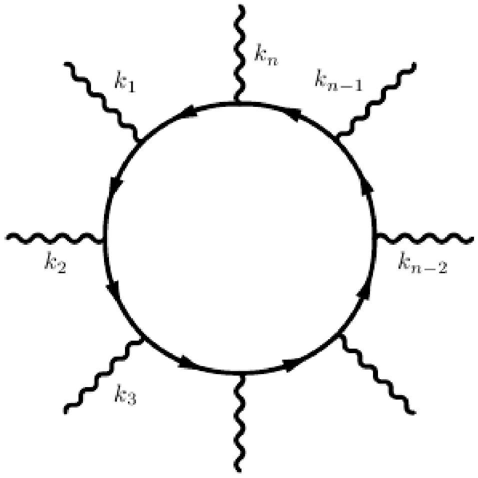

Figure 2.10: the $n-$point QED 1-loop box X_A^n

two vertexes. A little thought will convince the reader [4] that the iteration of the above machinery leads to the result

$$
\begin{aligned}
& X_A^n = (-\,iq)^n \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \cdots \int \frac{\mathrm{d}^4 k_{n-1}}{(2\pi)^4}\,(-1) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \\
\times \quad & \mathrm{tr}\,\Big[\, \widetilde{S}_F(p)\, \widetilde{\not{A}}(k_1)\, S_F(p+k_1)\, \widetilde{\not{A}}(k_2)\, S_F(p+k_1+k_2)\, \cdots \\
\cdots \quad & \widetilde{\not{A}}(k_{n-2})\, S_F(p+k_1+\cdots+k_{n-2})\, \widetilde{\not{A}}(k_{n-1})\, S_F(p+k_n)\, \widetilde{\not{A}}(k_n) \,\Big] \\
= \quad & (-1)\,(-\,iq)^n\ \mathrm{Tr}\,(S_F\,\not{A})^n \qquad\quad (\,k_n = k_1 + \cdots + k_{n-1}\,) \qquad (2.116)
\end{aligned}
$$

which corresponds to a fermion loop with $n-$external legs associated to the vector field vertexes. As a consequence, we can see by direct inspection that the symbolic equality (2.116) can be understood in a perturbative sense as a power series in the electric charge q the $n-$th coefficient of which does

[4] The trick is to introduce a change of variables to a new system of coordinates with the center of mass and $(n-1)$ relative coordinates with a unit Jacobian.

involve the 1-loop fermion boxes with $n-$external vector legs with momenta $(k_1, k_2, \ldots, k_n = k_1 + k_2 + \cdots + k_{n-1})$: namely,

$$\frac{\det \| \, i\not{\partial} - M - q\not{A} \, \|}{\det \| \, i\not{\partial} - M \, \|} = \exp\left\{(-1)\sum_{n=1}^{\infty} \frac{1}{n}\, (-iq)^n \, \mathrm{Tr}\, (\, S_F \not{A})^n \right\}$$

or even

$$\det \| \, i\not{\partial} - M - q\not{A} \, \| = \det \| \, i\not{\partial} - M \, \| \, \exp\{\mathrm{Tr}\, \ln(1 + iq\, S_F \not{A})\}$$

as naïvely expected by taking into account the suggestive symbolic relation $(\, i\not{\partial} - M\,)^{-1} = -iS_F$, as well as $\mathrm{Tr}\,(\not{A}\, S_F) = \mathrm{Tr}\,(S_F \not{A})$, *i.e.* the cyclic property of the Tr operation. Notice however that the first two coefficients of the perturbation expansion, that is

$$iq \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{tr}\left[\frac{i}{\not{p} - M + i\varepsilon}\, \gamma^{\mu}\right]$$

$$q^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{tr}\left[\frac{i}{\not{p} - M + i\varepsilon}\, \gamma^{\mu}\, \frac{i}{\not{p} + \not{k} - M + i\varepsilon}\, \gamma^{\nu}\right]$$

appear to be ultraviolet divergent. Hence, they call for some regularization method in order to be properly defined, in such a manner that we can write the formal expression

$$\begin{aligned} X_A^1 &= iq \int \frac{\mathrm{d}^4 k}{(2\pi)^8}\, \widetilde{A}_{\nu}(k) \int \mathrm{d}^4 p\, \frac{\mathrm{tr}\,[\,(\not{p} + M)\gamma^{\nu}\,]}{p^2 - M^2 + i\varepsilon} \\ &= iq \int \frac{\mathrm{d}^4 k}{(2\pi)^8}\, \widetilde{A}_{\nu}(k) \int \mathrm{d}^4 p\; p^{\nu}/(\, p^2 - M^2 + i\varepsilon\,) = 0 \end{aligned}$$

owing to symmetric integration, while

$$\begin{aligned} X_A^2 &= -q^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\; \widetilde{A}_{\mu}(k)\, \widetilde{A}_{\nu}(-k) \\ &\times \frac{\mathrm{tr}\,[\,(\,\not{p} + M\,)\,\gamma^{\mu}\,(\,\not{p} + \not{k} + M\,)\,\gamma^{\nu}\,]}{(\,p^2 - M^2 + i\varepsilon\,)[\,(\,p + k\,)^2 - M^2 + i\varepsilon\,]} \end{aligned} \tag{2.117}$$

Actually it can be shown that all 1-loop fermion cycles with an odd number of vector vertexes do vanish for symmetry reasons: this is known as the Furry theorem[5] of quantum electrodynamics. Thus we can eventually write the expansion

$$\ln\det \| \, i\not{\partial} - M - q\not{A} \, \| = \ln\det \| \, i\not{\partial} - M \, \| + \sum_{n=1}^{\infty} \frac{1}{2n}\, X_A^{2n}$$

[5]W. H. Furry, *A Symmetry Theorem in the Positron Theory*, The Physical Review, Vol. **51**, pp. 125-129 (1937)

which leads to the perturbative expansion of the 1-loop effective action for QED in the presence of a background real vector potential, *viz.*,

$$\Gamma_{\text{eff}}\,[A_\mu] = \frac{VM^4}{8\pi^2}\left(\ln\frac{M}{\mu}-\frac{3}{4}\right)+\sum_{n=1}^{\infty}\frac{i}{2n}\,(-\,iq)^{2n}\,\mathrm{Tr}\,(S_F\,\not{A})^{2n}$$

Notice that in the case of a constant gauge potential $A_\mu(x) = a_\mu =$ constant we find

$$\Gamma_{\text{eff}}(a_\mu) = \Gamma^{(0)}_{\text{eff}} = \frac{VM^4}{8\pi^2}\left(\ln\frac{M}{\mu}-\frac{3}{4}\right)$$

because a constant vector potential can always be removed by means of a gauge transformation with gauge function $f(x) = a \cdot x$.

Chapter 3

Scattering Operator

In this chapter I will analyze the basic concepts which lie at the ground of the perturbation theory for collisions. This is the most important topic of the Quantum Field Theory since it provides the bridge to connect theory and experimental High Energy Particle Physics.

3.1 The S-Matrix in Quantum Mechanics

In the framework of Non-Relativistic Quantum Mechanics one can rigorously formulate the so called asymptotic condition[1]. Consider an isolated system with a time independent self-adjoint Hamiltonian operator

$$H = H_0 + V$$

acting upon the Hilbert space $\mathfrak{H}$ of the system, and let $\psi \in \mathfrak{H}$ a proper state of this quantum mechanical system. Without loss of generality, the free Hamiltonian operator H_0 is supposed to be self-adjoint, time independent and endowed with a purely continuous spectrum. Moreover, the interaction potential is supposed to fall down to zero at large distances according to

$$V(\mathbf{r}) \stackrel{r\to\infty}{\sim} O\left(r^{-3/2-\epsilon}\right) \qquad (r = |\mathbf{r}| \quad \epsilon > 0)$$

which corresponds to sufficiently short range interactions. Then, the proper asymptotic states ψ_{as} exist, which are related to the proper states $\psi \in \mathfrak{H}$, characterized by the following behavior

$$\exp\{-(i/\hbar)\,H\,t\}\ |\psi\rangle \stackrel{|t|\to\infty}{\sim} \exp\{-(i/\hbar)\,H_0\,t\}\ |\psi_{\mathrm{as}}\rangle$$

[1]See for instance the textbook by John R. Taylor, *Scattering Theory: The Quantum Theory on Nonrelativistic Collisions*, John Wiley & Sons, New York (1972).

In other terms, the asymptotic in and out states are defined by

$$\mathrm{s}-\lim_{t\to\mp\infty}\exp\{(i/\hbar)H_0 t\}\exp\{-(i/\hbar)Ht\}\,|\psi\rangle \stackrel{\mathrm{def}}{=} \begin{cases} |\psi_{\mathrm{in}}\rangle & (t\to-\infty) \\ |\psi_{\mathrm{out}}\rangle & (t\to+\infty)\end{cases}$$

where the symbol $\mathrm{s}-\lim$ stands for the limit in the strong topology of the Hilbert space $\mathfrak{H}$, that is

$$\|\exp\{(i/\hbar)H_0 t\}\ \exp\{-(i/\hbar)H t\}\,\psi-\psi_{\mathrm{as}}\ \|\to\ 0 \qquad |\,t\,|\ \to\ \infty$$

Hence, in so doing, we can define the isometric asymptotic wave operators

$$\Omega_{\mathrm{as}}\ \stackrel{\mathrm{def}}{=}\ \mathrm{s}-\lim_{\,|\,t\,|\,\to\,\infty}\ \exp\{(i/\hbar)H_0\,t\}\ \exp\{-(i/\hbar)H\,t\} \tag{3.1}$$

where

$$\Omega_{\mathrm{as}}\ :\ \mathfrak{H}\ \longrightarrow\ \mathfrak{H}_{\mathrm{as}}\ \equiv\ \mathfrak{H}\setminus\mathfrak{B}$$

in which the subset $\mathfrak{B}\subset\mathfrak{H}$ of the Hilbert space is spanned by the bound states of the Hamiltonian operator H , *i.e.* the eigenstates belonging to the purely discrete part of the spectrum of H . Hence, in general, the asymptotic wave operators are not unitary but only isometric because

$$\|\ \psi_{\mathrm{as}}\ \| = \|\ \Omega_{\mathrm{as}}\,\psi\ \| = \|\ \psi\ \| \qquad \Omega_{\mathrm{as}}^{\dagger}\,\Omega_{\mathrm{as}} = \mathbb{I}$$

$$\Omega_{\mathrm{as}}\,|\,\psi\,\rangle = |\,\psi_{\mathrm{as}}\,\rangle\in\mathfrak{H}\setminus\mathfrak{B}$$

$$\Omega_{\mathrm{as}}\,|\,\psi_b\,\rangle = 0 \qquad \forall\,|\,\psi_b\,\rangle\in\mathfrak{B} \qquad H\,|\,\psi_b\,\rangle = E_b\,|\,\psi_b\,\rangle$$

Notice that, according to the fundamental theorem for self-adjoint operators, the complementary sub-spaces $\mathfrak{B}$ and $\mathfrak{H}\setminus\mathfrak{B}$ are mutually orthogonal.

The Scattering Operator or Collision Operator or $S-$matrix is a unitary operator $S\ :\ \mathfrak{H}_{\mathrm{in}}\ \longrightarrow\ \mathfrak{H}_{\mathrm{out}}$ which is defined by

$$\begin{aligned} S &\stackrel{\mathrm{def}}{=} \Omega_{\mathrm{out}}\,\Omega_{\mathrm{in}}^{\dagger} = U_{\mathrm{int}}\,(\,\infty\,,\,-\,\infty\,) \\ &= \mathrm{w}-\lim_{t\,\to\,+\,\infty}\ \lim_{t'\,\to\,-\,\infty}\ \exp\{(i/\hbar)H_0 t\}\ U\,(\,t\,,\,t'\,)\,\exp\{-(i/\hbar)H_0 t'\}\end{aligned}$$

where $U\,(\,t\,,\,t_0\,)\ \equiv\ \exp\{-(i/\hbar)H\,(t-t_0)\}$ is the time evolution operator, whereas the limits are now understood in the weak topology of the Hilbert space, that is

$$(\,\varphi\,|\,\psi\,) = \langle\,\varphi_{\mathrm{out}}\,|\,S\,\psi_{\mathrm{in}}\,\rangle \qquad \forall\,\varphi\,,\,\psi\in\mathfrak{H}$$

while the time evolution operator in the interaction picture is just provided by

$$U_{\mathrm{int}}\,(\,t_f\,,\,t_\imath\,) = \exp\{(i/\hbar)H_0\,t_f\}\ U\,(\,t_f\,,\,t_\imath\,)\ \exp\{-(i/\hbar)H_0\,t_\imath\}$$

In the interaction picture the time evolution of operators is governed by the free part H_0 of the complete self-adjoint Hamiltonian operator $H = H_0 + V$ of the quantum mechanical system, *i.e.*

$$A_{\text{int}}(t) = \exp\{(i/\hbar)H_0 t\}\, A(0)\, \exp\{-(i/\hbar)H_0 t\}$$

where A is any linear operator acting on the Hilbert space $\mathfrak{H}$, while the state vectors obey the evolution equation

$$i\hbar\,\partial_t\,|\,\psi_{\text{int}}(t)\,\rangle = V_{\text{int}}(t)\;|\,\psi_{\text{int}}(t)\,\rangle$$

where

$$|\psi(t)\rangle = \mathrm{e}^{-iHt/\hbar}\,|\psi(0)\rangle \qquad\qquad |\,\psi_{\text{int}}(t)\,\rangle = \mathrm{e}^{iH_0 t/\hbar}\,|\psi(t)\rangle$$

with the formal solution

$$|\,\psi_{\text{int}}(t)\,\rangle = U_{\text{int}}(t\,,\,t_0)\;|\,\psi_{\text{int}}(t_0)\,\rangle$$

$$U_{\text{int}}(t\,,\,t_0) = T\,\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}\tau\;V_{\text{int}}(\tau)\right\}$$

Hence, the scattering operator can be written in the very suggestive form

$$S = U_{\text{int}}(\infty\,,\,-\infty) = T\,\exp\left\{-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\;V_{\text{int}}(t)\right\} \tag{3.2}$$

which apparently maps the incoming states from $t \to -\infty$ into outgoing states at $t \to +\infty$, as naïvely expected.

3.2 S-Matrix in Quantum Field Theory

Let me consider, for the sake of simplicity but without loss of generality, the simplest interacting quantum field theory model, *i.e.* , the $\lambda\phi_4^4$ scalar field model described by the classical Lagrange density

$$\mathcal{L} = \tfrac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi(x)\,\partial_\nu\phi(x) - \tfrac{1}{2}\,(mc/\hbar)^2\,\phi^2(x) - \tfrac{1}{24}\,(\lambda/\hbar c)\,\phi^4(x)$$

leading to the conjugate momentum field

$$\Pi(x) = \frac{\delta\mathcal{L}}{\delta\dot\phi(x)} = \dot\phi(x)/c^2 = \partial_0\phi(x)/c$$

and to the classical functional

$$\begin{aligned}
\mathrm{H} &= \mathrm{H}_0 + \mathrm{V} \geq 0\\
\mathrm{H}_0 &= \int \mathrm{d}\mathbf{x}\;\tfrac{1}{2}\left[c^2\,\Pi^2(x) - \phi(x)\boldsymbol{\nabla}^2\phi(x) + (mc/\hbar)^2\phi^2(x)\right]\\
\mathrm{V}[\phi] &= \lambda\int \mathrm{d}\mathbf{x}\;\phi^4(t,\mathbf{x})/4!\hbar c \qquad\quad (\lambda > 0)
\end{aligned}$$

3.2.1 Fields in the Interaction Picture

To the aim of understanding the meaning of the interaction representation in the quantum field theory of the interacting fields, let me begin with the general Fourier representation of a real scalar self-interacting quantum field on the 3-dimensional ordinary Euclidean space at a given time $x_0 = ct = 0$:

$$\phi(0\,,\,\mathbf{x}) \;=\; \frac{c\sqrt{\hbar c}}{(2\pi)^3}\int\frac{\mathrm{d}\mathbf{k}}{2\omega_{\mathbf{k}}}\left[a(\mathbf{k})\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\;+\;a^{\dagger}(\mathbf{k})\,\mathrm{e}^{-\,i\mathbf{k}\cdot\mathbf{x}}\right]\;\equiv\;\phi(\mathbf{x}) \qquad (3.3)$$

where

$$[\,a\,(\mathbf{k})\,,\,a^{\dagger}(\mathbf{k}'\,)\,] = 2(2\pi)^3\;\frac{\omega_{\mathbf{k}}}{c}\;\delta\,(\,\mathbf{k}-\mathbf{k}'\,)$$

$$\omega_{\mathbf{k}} = c\,\sqrt{\mathbf{k}^2 + (\,mc/\hbar\,)^2}$$

all other commutators being null. Notice that in the physical C. G. S. system of units the creation and destruction operators have canonical engineering dimensions given by $[\,a(\mathbf{k})\,] = [\,a^{\dagger}(\mathbf{k})\,] = \mathrm{cm}$. Next I can introduce the field conjugate momentum in a similar way, by means of the equality

$$\Pi(\,0\,,\,\mathbf{x}\,) \;=\; \frac{\sqrt{\hbar/c}}{(2\pi)^3}\int\frac{\mathrm{d}\mathbf{k}}{2i}\left[a\,(\mathbf{k})\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\;-\;a^{\dagger}(\mathbf{k})\,\mathrm{e}^{-\,i\mathbf{k}\cdot\mathbf{x}}\right]\;\equiv\;\Pi(\,\mathbf{x}\,)$$

in such a manner to satisfy the canonical commutation relations

$$[\,\phi\,(\,\mathbf{x}\,)\,,\,\Pi\,(\,\mathbf{y}\,)\,] \;=\; i\hbar\,\delta\,(\,\mathbf{x}-\mathbf{y}\,)$$

all the other commutators vanishing. Then we can easily build up the free Hamiltonian quantum operator at $x^0 = t = 0$ that reads

$$\mathrm{H}_0 = \frac{1}{2}\int \mathrm{d}\mathbf{x}\;:c^2\,\Pi^2\,(\mathbf{x}) + |\boldsymbol{\nabla}\phi\,(\mathbf{x})|^2 + (mc/\hbar)^2\,\phi^2(\mathbf{x}):$$

where the normal ordering means here that, when I substitute the normal mode expansions for $\Pi\,(\,\mathbf{x}\,)$ and $\phi\,(\,\mathbf{x}\,)$ in the quadratic expression H_0 , the creation operators $a^{\dagger}(\mathbf{k})$ stand always to the left of the destruction operators $a\,(\mathbf{k})$, in such a manner that $\mathrm{H}_0\,|\,0\,\rangle = 0$. The perturbative $(\,\lambda = 0\,)$ vacuum state at $x^0 = t = 0$ is defined by $a\,(\mathbf{k})\,|\,0\,\rangle = 0 = \langle\,0\,|\,a^{\dagger}(\mathbf{k})\;(\forall\,\mathbf{k}\in\mathbb{R}^3\,)$.

We can define the Hermitean spin-less quantum field in the interaction representation by the evolution law

$$\phi_{\,\mathrm{int}}\,(x) \;=\; \mathrm{e}^{(i/\hbar)\,\mathrm{H}_0\,t}\,\phi\,(\,\mathbf{x}\,)\,\mathrm{e}^{-\,(i/\hbar)\,\mathrm{H}_0\,t} \qquad (3.4)$$

which entails the free field theory relationships

$$c\,\Pi_{\,\mathrm{int}}(x) = \partial_0\phi_{\,\mathrm{int}}(x) = \frac{i}{\hbar c}\,[\,\mathrm{H}_0\,,\,\phi_{\,\mathrm{int}}(x)\,] \qquad (3.5)$$

From the canonical commutation relation, it follows that the self-interacting real scalar quantum field operator in the interaction representation fulfills the Klein-Gordon equation

$$\left(\Box + m^2c^2/\hbar^2\right)\phi_{\,\mathrm{int}}\,(x) = 0$$

so that we can write at any time $x^0 = t$ its normal mode decomposition

$$\phi_{\,\mathrm{int}}\,(x) = \sqrt{\hbar c}\int Dk\,\left[\,a\,(k)\,e^{-\,ikx} + a^{\dagger}\,(k)\,e^{\,ikx}\,\right] = \phi_{\,\mathrm{int}}^{(-)}\,(x) + \phi_{\,\mathrm{int}}^{(+)}\,(x)$$

$$\begin{aligned}\int Dk \;&\stackrel{\mathrm{def}}{=}\; \int \frac{c\,\mathrm{d}\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}} = \int \frac{\mathrm{d}^4k}{(2\pi)^3}\;\theta\,(k_0)\,\delta\left(k^2 - m^2c^2/\hbar^2\right)\\ ck^0 \;&=\; \omega_{\mathbf{k}} \;=\; c\sqrt{\mathbf{k}^2 + m^2c^2/\hbar^2}\end{aligned}$$

$$\left[\,a\,(k)\,,\,a^{\dagger}\,(k^{\prime}\,)\,\right] = 2(2\pi)^3\,\frac{\omega_{\mathbf{k}}}{c}\;\delta\,(\,\mathbf{k} - \mathbf{k}^{\prime}\,)$$

$$[\,a\,(k)\,,\,a\,(k^{\prime}\,)\,] = 0 = \left[\,a^{\dagger}\,(k)\,,\,a^{\dagger}\,(k^{\prime}\,)\,\right]$$

In so arguing, the transition and the correspondence from the non-relativistic quantum mechanics to the relativistic quantum field theory seem to be quite natural and straightforward. Actually, a deeper inspection neatly shows that things are far more complicated, see the comments at the end of this section.

3.2.2 S-Matrix in Perturbation Theory

We can now use the suggestive relationship (3.2) in order to express the scattering operator for the self-interacting real scalar field model in terms of the creation and annihilation operators and, moreover, calculate its matrix elements between states containing scalar field quanta of sharply definite energy momentum. To the aim of calculating those matrix elements it is convenient to deal with normal ordered products, in the interaction picture, with all the creation parts $\phi_{\,\mathrm{int}}^{(+)}\,(x)$ of the scalar field operators standing to the left of the destruction parts $\phi_{\,\mathrm{int}}^{(-)}\,(x)$. This can be done by means of the Wick's theorem

> Giancarlo Wick (Torino, 15 October 1909 – Torino, 20 April 1992)
> *Evaluation of the collision matrix*
> Physical Review **80** (1950) 268

Wick's theorem for chronological products states that the $T-$ product of a system of n linear free field operators is equal to the sum of their normal products with all possible chronological pairings including the term with no pairings – each pairing of free field operators does correspond to the related Feynman propagator, *viz.*,

$$\overbrace{\phi_{\,\mathrm{int}}(x_\imath)\phi_{\,\mathrm{int}}(x_\jmath)} = D_F(x_\imath - x_\jmath)$$

When applied to the chronologically ordered product of n field operators, then Wick's theorem takes the well known form

$$\begin{aligned} T\,\phi_{\,\mathrm{int}}(x_1)\ldots\phi_{\,\mathrm{int}}(x_n) &= \;:\phi_{\,\mathrm{int}}(x_1)\ldots\phi_{\,\mathrm{int}}(x_n): \\ &+ \sum_{1\le\imath<\jmath\le n} D_F(x_\imath - x_\jmath)\;:\prod_{\kappa\ne\imath,\jmath}\phi_{\,\mathrm{int}}(x_\kappa):\;+\;\cdots \\ &+ D_F(x_1-x_2)\,D_F(x_3-x_4)\;:\phi_{\,\mathrm{int}}(x_5)\phi_{\,\mathrm{int}}(x_6)\cdots\phi_{\,\mathrm{int}}(x_n):\;+\;\cdots \\ &+ \sum_{1\le\imath_1<\jmath_1\le n}\ldots\sum_{1\le\imath_r<\jmath_r\le n} D_F(x_{\imath_1}-x_{\jmath_1})\cdots D_F(x_{\imath_r}-x_{\jmath_r}) \\ &\times\begin{cases} 1 & \text{for}\quad n=2\ell \qquad (\ell\in\mathbb{N}) \\ \phi_{\,\mathrm{int}}(x_s) & \text{for}\quad n=2\ell+1 \quad (s\ne\imath_1\ne\cdots\ne\jmath_r) \end{cases} \end{aligned} \tag{3.6}$$

Notice that for $n = 2\ell\,,\ \ell\in\mathbb{N}$, the very last term in the RHS contains the sum of $(2\ell-1)!!$ products of $n/2$ propagators, the arguments of which are always distinguished.

The above rather cumbersome formula can be readily checked by direct inspection for $n = 2$ and can be proved by induction in the general case. For a very detailed and exhaustive proof, which also includes the cases of the spinor and vector fields, see the classic textbook by N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience Publishers, New York, 1959, §16.2 pp. 159–168, §19.2 pp. 233–235.
`Check`. For $n = 2$ we find

$$\begin{aligned} T\,\phi_{\,\mathrm{int}}(x)\,\phi_{\,\mathrm{int}}(y) &= \theta(x_0-y_0)\Big(\phi^{(-)}_{\mathrm{int}}(x)+\phi^{(+)}_{\mathrm{int}}(x)\Big)\Big(\phi^{(-)}_{\mathrm{int}}(y)+\phi^{(+)}_{\mathrm{int}}(y)\Big) \\ &= \theta(y_0-x_0)\Big(\phi^{(-)}_{\mathrm{int}}(y)+\phi^{(+)}_{\mathrm{int}}(y)\Big)\Big(\phi^{(-)}_{\mathrm{int}}(x)+\phi^{(+)}_{\mathrm{int}}(x)\Big) \end{aligned}$$

and from the canonical commutation relations

$$[\,\phi^{(-)}_{\mathrm{int}}(x)\,,\,\phi^{(-)}_{\mathrm{int}}(y)\,]=0 \qquad [\,\phi^{(+)}_{\mathrm{int}}(x)\,,\,\phi^{(+)}_{\mathrm{int}}(y)\,]=0$$

$$[\,\phi^{(-)}_{\mathrm{int}}(x)\,,\,\phi^{(+)}_{\mathrm{int}}(y)\,]=(-\,i)\,D^{(-)}(x-y)$$

we obtain

$$\begin{aligned} T\,\phi_{\,\mathrm{int}}(x)\,\phi_{\,\mathrm{int}}(y) &= \Big\{\theta(x_0-y_0)\Big[\,\phi^{(-)}_{\mathrm{int}}(x)\phi^{(-)}_{\mathrm{int}}(y)+\phi^{(+)}_{\mathrm{int}}(x)\phi^{(+)}_{\mathrm{int}}(y)\Big] \\ &+\ \theta(x_0-y_0)\Big[\,\phi^{(+)}_{\mathrm{int}}(x)\phi^{(-)}_{\mathrm{int}}(y)+\phi^{(+)}_{\mathrm{int}}(y)\phi^{(-)}_{\mathrm{int}}(x)\Big] \\ &+\ \theta(x_0-y_0)(-\,i)\,D^{(-)}(x-y)\Big\}+\{\,x\ \leftrightarrows\ y\,\} \\ &= \;:\phi_{\,\mathrm{int}}(x)\,\phi_{\,\mathrm{int}}(y):+D_F(x-y) \end{aligned}$$

where use has been made of the well known relationships

$$: \phi_{\,\mathrm{int}}\,(x)\,\phi_{\,\mathrm{int}}\,(y) : \quad = \quad : \phi_{\,\mathrm{int}}\,(y)\,\phi_{\,\mathrm{int}}\,(x) :$$

$$\theta(x_0 - y_0)(-\,i\,)\,D^{\,(-)}(x-y) + \theta(y_0 - x_0)(-\,i\,)\,D^{\,(-)}(y-x) = D_F(x-y)$$

This completes our check. □

A much more compact and convenient functional expression is

$$T\,\mathrm{F}[\phi_{\,\mathrm{int}}] = \exp\left\{\tfrac{1}{2}\int \mathrm{d}x\mathrm{d}y D_F(x-y)\Big(\delta^{(2)}/\delta\phi_{\,\mathrm{int}}(x)\delta\phi_{\,\mathrm{int}}(y)\Big)\right\} : \mathrm{F}[\phi_{\,\mathrm{int}}] :$$

where $\mathrm{F}\,[\,\phi_{\,\mathrm{int}}\,]$ is any functional of the real scalar field in the interacting representation. Functional differentiation with respect to the operator $\phi_{\,\mathrm{int}}$ has been introduced here. It is a straightforward generalization of the usual functional differentiation, the only important difference being that one must keep in mind that the field operators at different times do not commute.

`Proof.` To start with consider for example

$$: \phi(x_1)\phi(x_2) : = : \phi(x_2)\phi(x_1) : = \tfrac{1}{2} : \phi(x_1)\phi(x_2) + \phi(x_2)\phi(x_1) :$$

where we understand in what follows the quantum scalar fields evolving in accordance with the interaction picture, which means that their evolution is governed by the Klein-Gordon differential equation, in such a manner to omit the label $_{\mathrm{int}}$ for the sake of simplicity. If we take the quadratic functional

$$: \mathrm{F}_2[\,\phi\,] : = \int \mathrm{d}x_1\mathrm{d}x_2\,\mathrm{F}_2(x_1, x_2)\; : \phi(x_1)\phi(x_2) :$$

where F_2 is any regular and symmetric function of the two variables (x_1, x_2), then we get

$$\begin{aligned}
& \exp\left\{\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,D_F(x-y)\left[\,\delta^{(2)}/\,\delta\phi(x)\,\delta\phi(y)\,\right]\right\} : \mathrm{F}_2[\,\phi\,] : \\
= \;& \int \mathrm{d}x_1\mathrm{d}x_2\,\mathrm{F}_2(x_1, x_2)\; : \phi(x_1)\phi(x_2) : \\
+ \;& \tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,D_F(x-y)\left[\,\delta^{(2)}/\delta\phi(x)\,\delta\phi(y)\,\right]\int \mathrm{d}x_1\mathrm{d}x_2\,\mathrm{F}_2(x_1, x_2)\; : \phi(x_1)\phi(x_2) : \\
= \;& : \mathrm{F}_2[\,\phi\,] : + \tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x_1\mathrm{d}x_2\,\mathrm{F}_2(x_1, x_2)\,D_F(x-y) \\
\times \;& [\,\delta(x-x_1)\,\delta(y-x_2) + \delta(x-x_2)\,\delta(y-x_1)\,] \\
= \;& \int \mathrm{d}x_1\mathrm{d}x_2\,\mathrm{F}_2(x_1, x_2)\,[\,D_F(x_1-x_2) + \; : \phi(x_1)\phi(x_2) :] \\
= \;& \int \mathrm{d}x_1\mathrm{d}x_2\,\mathrm{F}_2(x_1, x_2)\;T\,\phi(x_1)\phi(x_2) = T\,\mathrm{F}_2[\,\phi\,]
\end{aligned}$$

Consider now the cubic normal ordered functional

$$: \mathrm{F}_3[\,\phi\,] : = \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,\mathrm{F}_3(x_1, x_2, x_3)\; : \phi(x_1)\phi(x_2)\phi(x_3) :$$

so that we obtain

$$
\begin{aligned}
& \exp\left\{\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, D_F(x-y)\left[\,\delta^{(2)}/\,\delta\phi(x)\,\delta\phi(y)\,\right]\right\} : \mathrm{F}_3[\,\phi\,] : \\
= \;& : \mathrm{F}_3[\,\phi\,] : + \tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\, \mathrm{F}_3(x_1,x_2,x_3)\, D_F(x-y) \\
\times \;& \Big\{[\,\delta(x-x_1)\,\delta(y-x_2)+\delta(x-x_2)\,\delta(y-x_1)\,]\phi(x_3) \\
+ \;& [\,\delta(x-x_2)\,\delta(y-x_3)+\delta(x-x_3)\,\delta(y-x_2)\,]\phi(x_1) \\
+ \;& [\,\delta(x-x_3)\,\delta(y-x_1)+\delta(x-x_1)\,\delta(y-x_3)\,]\phi(x_2)\Big\} \\
= \;& \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\, \mathrm{F}_3(x_1,x_2,x_3)\Big[\, D_F(x_1-x_2)\,\phi(x_3) + D_F(x_2-x_3)\,\phi(x_1) \\
+ \;& D_F(x_3-x_1)\,\phi(x_2) + \;:\phi(x_1)\phi(x_2)\phi(x_3):\;\Big] \\
= \;& \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\, \mathrm{F}_3(x_1,x_2,x_3)\; T\,\phi(x_1)\phi(x_2)\phi(x_3) = T\,\mathrm{F}_3[\,\phi\,]
\end{aligned}
$$

Hence one can readily convince oneself and check that the very same manipulations lead to the very same formula for the $n-$th normal ordered functional: namely,

$$
\begin{aligned}
& : \mathrm{F}_n[\,\phi\,] : = \int \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\, \mathrm{F}_n(x_1,x_2,\ldots,x_n)\;\; :\phi(x_1)\phi(x_2)\cdots\phi(x_n): \\
& \exp\left\{\tfrac{1}{2}\int \mathrm{d}y\,\mathrm{d}z\, D_F(y-z)\left[\,\delta^{(2)}/\,\delta\phi(y)\,\delta\phi(z)\,\right]\right\} : \mathrm{F}_n[\,\phi\,] : \\
= \;& : \mathrm{F}_n[\,\phi\,] : + \sum_{\jmath=1}^{[\,n/2\,]} \frac{2^{-\jmath}}{\jmath\,!} \int \mathrm{d}y_1\,\mathrm{d}z_1\ldots\mathrm{d}y_\jmath\,\mathrm{d}z_\jmath\, D_F(y_\jmath - z_\jmath)\left[\,\delta^{(2)}/\,\delta\phi(y_\jmath)\,\delta\phi(z_\jmath)\,\right] \\
\times \;& \int \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\, \mathrm{F}_n(x_1,x_2,\ldots,x_n)\;\; :\phi(x_1)\phi(x_2)\cdots\phi(x_n): \\
= \;& \int \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\, \mathrm{F}_n(x_1,x_2,\ldots,x_n)\; T\phi(x_1)\phi(x_2)\cdots\phi(x_n) = T\,\mathrm{F}_n[\,\phi\,]
\end{aligned}
$$

where we have taken into account that the functional derivatives vanish for all $\jmath$ greater than the integer part of $n/2$. Thus, for any analytic functional of the form

$$
\mathrm{F}[\,\phi\,] = \mathrm{F}_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\,\mathrm{F}_n[\,\phi\,]
$$

the above relation holds true, *viz.*

$$
\exp\left\{\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, D_F(x-y)\left[\,\delta^{(2)}/\,\delta\phi(x)\,\delta\phi(y)\,\right]\right\} : \mathrm{F}[\,\phi\,] : = \; T\,\mathrm{F}[\,\phi\,]
$$

which endorses the most general functional expression of the Wick theorem. □

Now, in particular, if we take the functional

$$
\mathrm{F}[\,\phi_{\mathrm{int}}\,] = \exp\left\{\, i\int \mathrm{d}x\;\phi_{\mathrm{int}}(x)\,J(x)\,\right\}
$$

$J(x)$ being as usual some classical external source, we find

$$
\begin{aligned}
& T\,\exp\left\{\, i\int \mathrm{d}x\;\phi_{\mathrm{int}}(x)\,J(x)\,\right\} \\
= \;& \exp\left\{-\tfrac{1}{2}\int \mathrm{d}x\mathrm{d}y D_F(x-y)J(x)J(y)\right\} : \exp\left\{\, i\int \mathrm{d}x\;\phi_{\mathrm{int}}(x)\,J(x)\,\right\} : \\
= \;& : \exp\left\{\, i\int \mathrm{d}x\;\phi_{\mathrm{int}}(x)\,J(x)\,\right\} : Z_0[\,J\,] \qquad (3.7)
\end{aligned}
$$

where $Z_0[J]$ is the previously introduced generating functional of the Green's functions for the free field theory, whereas a repeated use has been made of the identity

$$\frac{\delta}{\delta\phi_{\text{int}}(x)} : \exp\left\{ i\int \mathrm{d}w\ \phi_{\text{int}}(w)\, J(w) \right\} : \\ = i\, J(x) : \exp\left\{ i\int \mathrm{d}w\ \phi_{\text{int}}(w)\, J(w) \right\} : \tag{3.8}$$

It is worthwhile to recall the Symanzik functional identity (1.6)

$$\hbar\, J(x)\, Z_0[J] = \mathcal{K}_x\ (\delta / i\,\delta\, J_x)\ Z_0[J] \tag{3.9}$$

in which I have denoted the Klein-Gordon differential operator as

$$\mathcal{K}_x \ \stackrel{\text{def}}{=}\ \left(\Box_x + m^2c^2/\hbar^2\right)$$

`Proof`: as a matter of fact, on the one hand we have

$$J(x)\, Z_0[J] = J(x)\ \exp\left\{-\tfrac{1}{2}\int \mathrm{d}y \int \mathrm{d}z D_F(y-z)J(y)J(z)\right\}$$

while on the other hand we get

$$(\delta / i\,\delta\, J(x))\ Z_0[J] = i\int \mathrm{d}y\, D_F(x-y)J(y)\, Z_0[J]$$

Application of the Klein-Gordon operator yields

$$\begin{aligned} \mathcal{K}_x\,(\delta / i\,\delta\, J(x))\ Z_0[J] &= i\,\mathcal{K}_x \int \mathrm{d}y\, D_F(x-y)J(y)\, Z_0[J] \\ &= i\int \mathrm{d}y\,(-i\hbar)\,\delta(x-y)\, J(y)\, Z_0[J] \\ &= \hbar\, J(x)\, Z_0[J] \end{aligned}$$

which completes our proof. □

Hence, we can rewrite the functional relationship (3.7) in the form

$$T\ \exp\left\{ i\int \mathrm{d}x\ \phi_{\text{int}}(x)\, J(x) \right\} \ = \\ : \exp\left\{ \frac{i}{\hbar}\int \mathrm{d}x\ \phi_{\text{int}}(x)\, \mathcal{K}_x\ (\delta / i\,\delta\, J_x) \right\} :\ Z_0[J] \tag{3.10}$$

Now, once again, we can take profit of the very same expedient which led us to the Feynman rules. Let me start from the basic definition (3.2) – which holds rigorously true in non-relativistic quantum mechanics – and let's try to understand it within the framework of perturbative quantum field theory. After turning to natural units we get

$$
\begin{aligned}
S &= U_{\rm int}(\infty, -\infty) = T\,\exp\left\{-i\int_{-\infty}^{\infty}\mathrm{d}t\,V_{\rm int}(t)\right\} \\
&\equiv T\,\exp\left\{-i\int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}\mathbf{x}\,\frac{\lambda}{4!}\,\phi^4_{\rm int}(t,\mathbf{x})\right\} \\
&= \exp\left\{-\frac{i\lambda}{4!}\int\mathrm{d}^4x\left(\frac{\delta}{\delta J_x}\right)^4\right\}\,T\,\exp\left\{i\int\mathrm{d}^4y\,\phi_{\rm int}(y)\,J(y)\right\}\Bigg|_{J=0}
\end{aligned}
\tag{3.11}
$$

It is worthwhile to stress that the very last line expression for the collision matrix turns out to be manifestly Lorentz invariant, at variance with the expressions in the first line of the above equality chain. As a consequence, we can eventually express the scattering operator (3.2) in the suggestive form

$$
\begin{aligned}
S &\equiv \exp\left\{(-i\lambda/4!)\textstyle\int\mathrm{d}^4z\,(\delta/i\,\delta J_z)^4\right\} \\
&\times\ :\exp\left\{i\textstyle\int\mathrm{d}^4x\,\phi_{\rm int}(x)\,\mathcal{K}_x\,(\delta/i\,\delta J_x)\right\}:\,Z_0\,\big\rfloor_{J=0} \\
&=\ :\exp\left\{i\textstyle\int\mathrm{d}^4x\,\phi_{\rm int}(x)\,\mathcal{K}_x\,(\delta/i\,\delta J_x)\right\}: \\
&\times\ \exp\left\{(-i\lambda/4!)\textstyle\int\mathrm{d}^4z\,(\delta/i\,\delta J_z)^4\right\}\,Z_0\,\big\rfloor_{J=0} \\
&\stackrel{\rm def}{=}\ :\exp\left\{i\int\mathrm{d}^4x\,\phi_{\rm int}(x)\,\mathcal{K}_x\,(\delta/i\,\delta J_x)\right\}:\ Z\,\Bigg\rfloor_{J=0}
\end{aligned}
\tag{3.12}
$$

in which I have used in the last step the formal definition (2.5) with $\nu = 1$ of the generating functional for the Green's functions of the real self-interacting scalar field theory.

As a matter of fact, it turns out that in perturbation theory one can safely identify the Fock spaces $\mathfrak{F}_{\rm in}$ and $\mathfrak{F}_{\rm out}$ of the asymptotic spaces for the self-interacting quantum scalar field to be isomorphic to the Fock space of the Klein-Gordon field. This means in turn that in perturbation theory the absence of bound states and the asymptotic completeness of the scattering states is always tacitly assumed. In the present context, owing to the repulsive nature of the $\lambda\phi_4^4$ contact interaction for $\lambda > 0$, the above assumption is admittedly legitimate and coherent. Hence, within the framework of the perturbation theory we can always safely and suitably assume that $|\,0\,\mathrm{in}\,\rangle = |\,0\,\mathrm{out}\,\rangle = |\,0\,\rangle$ and thereby

$$
\begin{aligned}
Z[\,J\,] &= \langle\,\mathrm{out}\,0\,|\,0\,\mathrm{in}\,\rangle_J = \langle\,0\,|\,T\,\exp\left\{i\textstyle\int\mathrm{d}x\,\phi(x)\,J(x)\right\}\,|\,0\,\rangle \\
&\equiv \exp\left\{(-i\lambda/4!)\textstyle\int\mathrm{d}^4z\,(\delta/i\,\delta J_z)^4\right\}\,Z_0[\,J\,]
\end{aligned}
$$

Notice that the above form of the generating functional differs from the previous one (2.5) only in the (divergent) normalization constant, which appears to be irrelevant in the calculation of the relevant connected $n-$point Green's functions.

From the exponential Taylor's expansion (1.1)

$$\begin{aligned} Z[\,J\,] \;&=\; \left\langle T \exp\left\{ i\int \mathrm{d}x\ \phi(x)\,J(x)\right\}\right\rangle_0 \\ &\stackrel{\text{def}}{=}\; \sum_{n=0}^{\infty} \frac{i^n}{n!}\int \mathrm{d}x_1\, J(x_1)\cdots\int \mathrm{d}x_n\, J(x_n) \\ &\times\; \langle 0\,|\,T\,\phi\,(x_1)\cdots\phi\,(x_n)\,|\,0\,\rangle \end{aligned} \tag{3.13}$$

we eventually come to the fundamental and celebrated formula that relates the scattering operator, which describes spin-less massive particles collisions, to the $n-$point Green's functions of the self-interacting neutral spin-less field

$$\begin{aligned} S \;&=\; \sum_{n=0}^{\infty} \frac{i^n}{n!}\int \mathrm{d}x_1\int \mathrm{d}x_2\ \ldots\ \int \mathrm{d}x_n \\ &:\phi_{\text{int}}\,(x_1)\,\phi_{\text{int}}\,(x_2)\ \ldots\ \phi_{\text{int}}\,(x_n): \\ &\mathcal{K}\,(x_1)\,\mathcal{K}\,(x_2)\ \ldots\ \mathcal{K}\,(x_n) \\ &\langle\, 0\,|\,T\,\phi\,(x_1)\,\phi\,(x_2)\ \ldots\ \phi\,(x_n)\,|\,0\,\rangle \end{aligned} \tag{3.14}$$

in which the symbol $\mathcal{K}\,(x_\jmath\,) = (\,\Box_\jmath + m^2\,) \quad (\,\jmath = 1,2,\ldots,n\,)$ denotes the kinetic differential operator, *i.e.* specifically the Klein-Gordon operator. As a final remark it is worthwhile to stress that the above definition (3.14) of the scattering or collision operator is merely formal, since in perturbation theory the Green's functions as derived from the Feynman rules out of the classical Lagrangian are plagued by severe ultraviolet divergences and mathematically ill-defined. It turns out that only after the procedure of the removal of the ultraviolet divergences from the Green's functions in perturbation theory it will be possible to set up a renormalized $S-$matrix, the matrix elements of which could be eventually and actually compared with the experimental data.

3.2.3 LSZ Reduction Formulas

The calculation of the matrix elements of the scattering operator is now straightforward. Let the initial state involve N identical spin-less massive particles while the final state N' of such a kind of particles: then we have

$$|\,K_i\,\rangle = |\,k_1\,k_2\,\cdots\,k_N\,\rangle = (\,N\,!\,)^{-1/2}\,\prod_{\imath=1}^{N}\,a^\dagger\,(\,k_\imath\,)\,|\,0\,\rangle$$

$$\langle\,K_f'\,| = \langle\,k_1'\,k_2'\,\cdots\,k_{N'}'\,| = \langle\,0\,|\,\prod_{\jmath=1}^{N'}\,a\,(\,k_\jmath'\,)\,(\,N'\,!\,)^{-1/2}$$

If we suppose that $k_\imath \neq k'_\jmath$ for all pairs[2] $(\imath\,,\,\jmath)$, then solely the term in the series with

$$:\phi_{\rm int}(x'_1)\,\phi_{\rm int}(x'_2)\ \ldots\ \phi_{\rm int}(x'_{N'})\,\phi_{\rm int}(x_1)\,\phi_{\rm int}(x_2)\ \ldots\ \phi_{\rm int}(x_N):$$

will indeed contribute – see Exercise here below. Among the $(N+N')$ field operators in the interaction representation, N will act with their destruction parts

$$\phi^{(-)}_{\rm int}(x_\imath) \qquad (\imath = 1,2,\cdots,N)$$

whilst N' with their creation parts

$$\phi^{(+)}_{\rm int}(x'_\jmath) \qquad (\jmath = 1,2,\cdots,N')$$

Owing to the identical nature of the scalar neutral bosons attending the scattering process, the number of term which is produced in the reduction of the matrix element

$$\langle\, K'_f \,|:\phi_{\rm int}(x'_1)\ \ldots\ \phi_{\rm int}(x'_{N'})\,\phi_{\rm int}(x_1)\ldots\ \phi_{\rm int}(x_N):|\,K_i\,\rangle$$

is just $(\,N+N'\,)\,!$. As a matter of fact, the result is that the only non-vanishing matrix element is provided by

$$\begin{aligned}&\langle\, k'_1\,\cdots\,k'_{N'}\,|\,\phi^{(+)}_{\rm int}(x'_1)\ \ldots\ \phi^{(+)}_{\rm int}(x'_{N'})\phi^{(-)}_{\rm int}(x_1)\ldots\ \phi^{(-)}_{\rm int}(x_N)\,|\,k_1\,\cdots\,k_N\,\rangle\\ =\ &\exp\{-\,ik_1\cdot x_1\ \cdots\ +\ ik'_{N'}\cdot x'_{N'}\}\ +\ \text{permutations} \qquad (3.15)\end{aligned}$$

all the others being equal to zero, the total number of terms being equal to $(\,N+N'\,)\,!$ as expected. Hence, turning back to the expression (3.14), we can write

$$\begin{aligned}&\langle\, k'_1\,k'_2\,\cdots\,k'_{N'}\,|\,S\,|\,k_1\,k_2\,\cdots\,k_N\,\rangle\ =\\ &i^{\,N+N'}\,(\,N\,!\,N'\,!\,)^{-1/2}\int {\rm d}x'_1\ \ldots\int {\rm d}x'_{N'}\int {\rm d}x_1\ \ldots\int {\rm d}x_N\\ &\exp\{-\,ik_1\cdot x_1\ -\ \cdots\ -\ ik_N\cdot x_N\ +\ ik'_1\cdot x'_1\ +\ \cdots\ +\ ik'_{N'}\cdot x'_{N'}\}\\ &\mathcal{K}(x'_1)\ \ldots\ \mathcal{K}(x'_{N'})\,\mathcal{K}(x_1)\ \ldots\ \mathcal{K}(x_N)\,G_{\,N'+N}\,(\,x'_1,\ \ldots\ ,x'_{N'}\,;\,x_1,\ \ldots\ ,x_{\,N}\,)\end{aligned} \qquad (3.16)$$

in which all the temporal components of the incoming and outgoing momenta are understood to be on the mass shell, *i.e.* ,

$$k^0_\imath = \omega\,(\,\mathbf{k}_\imath\,) \qquad (\,\imath = 1,2,\ldots,N\,)$$

[2]This means that we disregard the case in which any of the incident particles is not scattered at all.

$$k'^{\,0}_{\jmath} = \omega(\mathbf{k}'_{\jmath}) \qquad (\jmath = 1, 2, \ldots, N')$$

It is now convenient to introduce the Green's functions in momentum space according to the standard definition

$$\begin{aligned}
&\langle 0 \,|\, T\,\phi(x_1)\,\phi(x_2)\ \ldots\ \phi(x_n)\,|\,0\,\rangle = \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_2}{(2\pi)^4} \cdots \int \frac{\mathrm{d}^4 p_n}{(2\pi)^4} \\
&(2\pi)^4\,\delta(P)\,\widetilde{G}_n(p_1, p_2 \ldots, p_n) \prod_{\jmath=1}^{n} \exp\{\, i\, p_\jmath \cdot x_\jmath \} \\
&= \ G_n(x_1, x_2 \ldots, x_n)
\end{aligned} \tag{3.17}$$

where the $\delta-$distribution of the total energy momentum

$$P \ \equiv\ p_1 + p_2 + \ \ldots\ + p_n$$

does vindicate the translation invariance of the $n-$point Green's functions in the configuration space. Notice that the canonical engineering dimensions of the Green's functions in natural units are

$$[\,G_n(x_1 \ldots, x_n)\,] = \mathrm{eV}^{\,n}$$

$$[\,\widetilde{G}_n(p_1 \ldots, p_n)\,] = \mathrm{cm}^{3n-4}$$

Then, by substituting the Fourier transform (3.17), it is straightforward to recast the above equation (3.16) into the final form

$$\begin{aligned}
&\langle\, k'_1\, k'_2 \cdots k'_{N'} \,|\, S \,|\, k_1\, k_2 \cdots k_N \,\rangle \ = \\
&i^{\,N+N'}\ (N!\,N'!)^{-1/2}\ (2\pi)^4\ \delta(K_i - K'_f)\ \times \\
&\prod_{\jmath=1}^{N'} \lim_{k'^{\,2}_{\jmath} \to m^2} \left(m^2 - k'^{\,2}_{\jmath}\right) \prod_{\imath=1}^{N} \lim_{k^2_\imath \to m^2} \left(m^2 - k^2_\imath\right) \\
&\times\ \widetilde{G}_{N'+N}(k_1, \ldots, k_N; -k'_1, \ldots, -k'_{N'})
\end{aligned} \tag{3.18}$$

This remarkable formula is known as the LSZ reduction formula, the acronym being associated to the names of Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann who originally derived that fundamental relation.

H. Lehmann, K. Symanzik and W. Zimmermann
Zur Formulierung Quantisierter Feldtheorien
Nuovo Cimento **1**, January 1955, pp. 205-225.

If the initial and final states do not truly correspond to sharply definite values of energy and momentum, then some normalized wave packets have to be suitably introduced according to

$$|\,N \text{ initial}\,\rangle \;=\; (\,N\,!\,)^{-1/2}\,\prod_{\imath=1}^{N}\int D\,k_\imath\;\widetilde{f}_\imath(\,k_\imath\,)\,a^\dagger(\,k_\imath\,)\;|\,0\,\rangle$$

$$\langle\,N' \text{ final}\,|\;=\;(\,N'\,!\,)^{-1/2}\,\prod_{\jmath=1}^{N'}\int Dk'_\jmath\;\widetilde{g}^{\,*}_\jmath(\,k'_\jmath\,)\;\langle\,0\,|\;a(\,k'_\jmath\,)$$

where I used the previously introduced notations

$$ck_0 = \omega\,(\mathbf{k}) = c\,\sqrt{\mathbf{k}^2 + m^2c^2/\hbar^2}$$

$$\begin{aligned} D\,k_\imath \;&=\; \frac{\mathrm{d}\mathbf{k}_\imath}{(2\pi)^3\,2\,k_\imath^{\,0}} \;=\; \frac{c\,\mathrm{d}\mathbf{k}_\imath}{(2\pi)^3\,2\omega(\mathbf{k}_\imath)} \\ D\,k'_\jmath \;&=\; \frac{\mathrm{d}\mathbf{k}'_\jmath}{(2\pi)^3\,2\,k'^{\,0}_\jmath} \;=\; \frac{c\,\mathrm{d}\mathbf{k}'_\jmath}{(2\pi)^3\,2\,\omega(\mathbf{k}'_\jmath)} \end{aligned} \tag{3.19}$$

$$\big[\,a\,(k)\,,\;a^\dagger\,(k'\,)\,\big] = 2(2\pi)^3\,\frac{\omega_{\mathbf{k}}}{c}\,\delta\,(\,\mathbf{k}-\mathbf{k}'\,)\qquad\text{et cetera}$$

$$f(t,\mathbf{x}) = \int\frac{c\,\mathrm{d}\mathbf{k}}{(2\pi)^3\,2\,\omega(\mathbf{k})}\;\widetilde{f}(\mathbf{k})\;\exp\{-\,i\,t\,\omega\,(\mathbf{k}) +\,i\,\mathbf{k}\cdot\mathbf{x}\}$$

$$(\,f\,,\,f\,) = \int \mathrm{d}\mathbf{x}\;f^*(t,\mathbf{x})\,i\,\overset{\leftrightarrow}{\partial}_0\,f(t,\mathbf{x}) = \int Dk\;|\,\widetilde{f}(k)\,|^2 = 1$$

It is worthwhile to recall that, according to the above definitions, we have to assign the following canonical dimensions in physical units: namely,

$$[\,a(k)\,] = [\,a^\dagger(k)\,] = \mathrm{cm}\qquad\qquad [\,Dk\,] = \mathrm{cm}^{-2}$$

$$[\,f\,] = \mathrm{cm}^{-1}\qquad\qquad [\,\widetilde{f}\,] = \mathrm{cm}$$

$$[\,|\,k_1\,k_2\,\cdots\,k_N\,\rangle\,] = \mathrm{cm}^N\qquad\qquad [\,|\,N\text{ initial}\,\rangle\,] = \text{dimensionless}$$

$$[\,G_n\,] = \Big(\sqrt{\mathrm{eV/cm}}\Big)^n\qquad\qquad \Big[\,\widetilde{G}_n\,\Big] = \mathrm{cm}^{-4+7n/2}\;\mathrm{eV}^{\,n/2}$$

As a consequence, the *LSZ* reduction formula for the dimensionless matrix elements of the perturbative scattering operator between normalizable states and in natural units becomes

$$\langle\,N'\text{ final}\,|\;S\;|\,N\text{ initial}\,\rangle \;=\; \frac{i^{\,N+N'}}{\sqrt{\,(\,N\,!\,N'!\,)}}\;\times \tag{3.20}$$

$$\prod_{\jmath=1}^{N'}\int Dk'_\jmath\;\widetilde{g}^{\,*}_\jmath\,(k'_\jmath\,)\;\lim_{k'^{\,2}_\jmath\,\to\,m^2}\;\big(m^2 - k'^{\,2}_\jmath\,\big)$$

$$
\prod_{\imath=1}^{N} \int Dk_\imath \, \widetilde{f}_\imath(k_\imath) \lim_{k_\imath^2 \to m^2} \left(m^2 - k_\imath^2 \right)
$$
$$
\times \, \widetilde{G}_{N+N'} \, (k_1, \, \ldots \, , k_N \, ; - k_1', \, \ldots \, , - k_{N'}') \; (2\pi)^4 \, \delta(\, K_i - K_f' \,)
$$

The disconnected $n-$point Green's functions do involve also trivial parts, that correspond to the absence of any scattering process. Hence, what we are really interested for is the reduction formula for the connected Green's functions, that means, the truly interacting part which actually contribute to the scattering amplitudes. For example, in the 4-point Green's function we find terms which are related to the products of two 2-point Green's functions, *i.e.* two full propagators: such terms do not describe neither scattering nor interaction. To see this, I first decompose the 4-point Green's function into disconnected and connected parts as shown graphically in the figure. The first three graphs represent the non-scattered or straight through or even forward propagation of the particles, albeit with fully interacting or dressed propagators, *i.e.* 2-point Green's functions that include all order radiative corrections which describe emission and absorption of virtual particles, in accordance with the energy-time uncertainty relation of quantum mechanics.

The final graph represents the processes that give rise to the scattering, once we have again removed the four dressed propagator factors to define an amplitude which is named truncated or amputated 4-point Green's function. In conclusion, from the reduction formulæ we have learned that the basic ingredients we have to build up in perturbation theory by means of the Feynman rules, in the aim of computing the scattering cross sections to be compared with the experimental data, are the connected, truncated, on shell $n-$point Green's functions in momentum space.

One of the fundamental inadequacies of the previously discussed and presently known as the customary perturbative approach to the quantum field theory of truly interacting field is the necessity to introduce into the formulation fictitious non-interacting particles, states and fields and to treat the interaction as some additional small perturbation, which slightly modifies the dynamical quantum system and which may be switched on or switched off *ad hoc* and *ad libitum.* At first glance it might appear that this procedure does not give rise to any basis for criticism of the theory. Indeed we know that the elementary particles interact intensively with each other only if they are extremely close, typically at a relative distance of few fm. Therefore, it would appear that at large distances among the particles, where large might have the realistic size of few μm, the field interaction could be disregarded and in a certain reliable approximation it is legitimate to neglect it and to regard the particles realistically as being free.

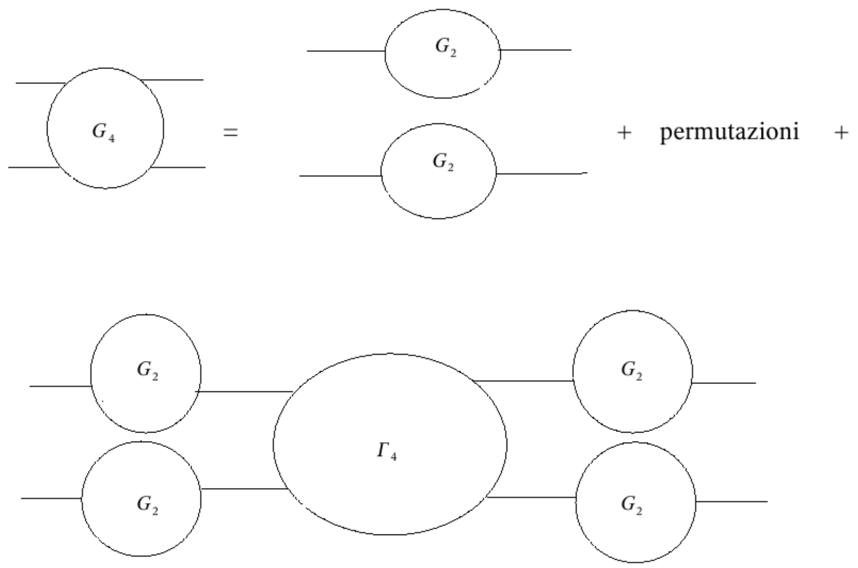

Figure 3.1: the full 4-point Green's function in the ϕ_4^4 theory

However, by arguing in this way we omit from consideration the crucial fact that the particles continuously interact with the vacuum, as it were a sort of a material medium through which the particles move. This is a typical quantum mechanical and relativistic effect, an unavoidable consequence of the Heisenberg uncertainty relations and of the equivalence between mass and energy in the Special Theory of Relativity.

It appears therefore to be rather longing for a development of the theory to deal from the outset with real interacting particles and to avoid carefully the introduction of such a kind of artificial concepts like the fictitious free particles, fields and corresponding quantum states. As a matter of fact, the separation between the free and interacting parts of the total Hamiltonian, as well as the very existence of a well defined total Hamiltonian operator, are non-covariant and frame dependent assumptions. The free Hamiltonian H_0 would be an ill defined part of the total energy-momentum vector – the true generator of space-time translations – and it won't be neither conserved in time nor referable to any observable quantity. Hence, owing to these reasons, the interaction picture in the quantum field theory is merely a poorly defined fictitious device to recover the collision matrix and the reduction formulas. Actually, it has been rigorously proved by Rudolf Haag [3] that the interaction

[3] R. Haag, *On quantum field theories*, Det Kongelige Danske Videnskabernes Selskab

picture does not exist at all in Quantum Field Theory, once a few very basic and general features are postulated about the nature of the interacting fields, *viz.* covariance, locality, micro-causality and spectrum conditions. The above sketched serious criticisms have led to some important developments such as *e.g.* an axiomatic non-perturbative formulation of the theory of collisions for massive interacting fields, the main pillars of which are the Lehmann-Symanzik-Zimmermann asymptotic conditions and the adiabatic switching of the interaction. Nonetheless, admittedly, the ultimate reason why all the above considerations have to be trusted stands in the comparison with the experimental data.

Exercise. Evaluate the quantity

$$\langle\, a(k_1')a(k_2') : \phi(x_2')\phi(x_1')\phi(x_1)\phi(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0$$

where $\phi(x) = \phi^{(-)}(x) + \phi^{(+)}(x)$ is a real scalar free field with

$$\phi^{(-)}(x) = \int Dk\, a(k)\,\mathrm{e}^{-ikx} \qquad \phi^{(+)}(x) = \int Dk\, a^\dagger(k)\,\mathrm{e}^{ikx}$$

$$k_0 = \omega_{\mathbf{k}} \qquad Dk = \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} = \frac{\mathrm{d}^4 k}{(2\pi)^3}\,\theta(k_0)\,\delta(k^2 - m^2)$$

$$[\,a(k)\,,\, a^\dagger(p)\,] = (2\pi)^3\, 2\omega_{\mathbf{k}}\,\delta(\mathbf{k} - \mathbf{p}) \qquad [\,a(k)\,,\, a(p)\,] = 0$$

Solution

Let us first calculate the simpler quantity

$$\begin{aligned}\langle\, a(k_1') : \phi(x_1')\phi(x_1) : a^\dagger(k_1)\,\rangle_0 \;&= \\ \langle\, a(k_1')\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_1)\,\rangle_0 &+ \langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(+)}(x_1)a^\dagger(k_1)\,\rangle_0 \\ +\, \langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_1)\,\rangle_0 &+ \langle\, a(k_1')\phi^{(+)}(x_1)\phi^{(-)}(x_1')a^\dagger(k_1)\,\rangle_0\end{aligned}$$

Now we have the non-vanishing commutation relations

$$[\,\phi^{(-)}(x), a^\dagger(k)\,] = \mathrm{e}^{-ikx} \qquad [\,a(k), \phi^{(+)}(x)\,] = \mathrm{e}^{ikx}$$

whence we readily obtain that the first line in the right hand side of the previous equality does vanish, while the very last line yields

$$\begin{aligned}\langle\, a(k_1') : \phi(x_1')\phi(x_1) : a^\dagger(k_1)\,\rangle_0 \;&=\; \exp\{-ik_1\cdot x_1 + ik_1'\cdot x_1'\} \\ &+\; \exp\{-ik_1\cdot x_1' + ik_1'\cdot x_1\}\end{aligned}$$

Next we find

$$\begin{aligned}&\langle\, a(k_1') : \phi(x_1')\phi(x_1)\phi(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;= \\ &\langle\, a(k_1')\phi^{(-)}(x_1')\phi^{(-)}(x_1)\phi^{(-)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\ &\langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(-)}(x_1)\phi^{(-)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+\end{aligned}$$

Matematisk-Fysiske Meddelelser **29** (1955) nr. 12, 1-37

$$
\begin{aligned}
&\langle\, a(k_1')\phi^{(+)}(x_1)\phi^{(-)}(x_2)\phi^{(-)}(x_1')a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+\\
&\langle\, a(k_1')\phi^{(+)}(x_2)\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+\\
&\langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(+)}(x_1)\phi^{(-)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+\\
&\langle\, a(k_1')\phi^{(+)}(x_1)\phi^{(+)}(x_2)\phi^{(-)}(x_1')a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+\\
&\langle\, a(k_1')\phi^{(+)}(x_2)\phi^{(+)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+\\
&\langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(+)}(x_1)\phi^{(+)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0
\end{aligned}
$$

The last four lines evidently vanish so that we are left with

$$
\begin{aligned}
&\langle\, a(k_1') : \phi(x_1')\phi(x_1)\phi(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0\\
=\;& \langle\, a(k_1')\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_2)\,\rangle_0 \;\exp\{-ik_1\cdot x_2\}\\
+\;& \langle\, a(k_1')\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_1)\,\rangle_0 \;\exp\{-ik_2\cdot x_2\}\\
+\;& \exp\{ik_1'\cdot x_1'\}\;\langle\,\phi^{(-)}(x_1)\phi^{(-)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0\\
+\;& \exp\{ik_1'\cdot x_1\}\;\langle\,\phi^{(-)}(x_2)\phi^{(-)}(x_1')a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0\\
+\;& \exp\{ik_1'\cdot x_2\}\;\langle\,\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0
\end{aligned}
$$

Again, the first two lines in the right hand side of the above equality do vanish and going on with the process of reduction we obtain

$$
\begin{aligned}
&\langle\, a(k_1') : \phi(x_1')\phi(x_1)\phi(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0\\
=\;& \exp\{ik_1'\cdot x_1'\}\;\langle\,\phi^{(-)}(x_1)a^\dagger(k_2)\phi^{(-)}(x_2)a^\dagger(k_1)\,\rangle_0 \;+\; k_1 \leftrightarrow k_2\\
+\;& \text{cyclic permutations of}\quad x_1', x_1, x_2\\
=\;& \exp\{ik_1'\cdot x_1' - ik_1\cdot x_1 - ik_2\cdot x_2\} \;+\; k_1 \leftrightarrow k_2\\
+\;& \text{cyclic permutations of}\quad x_1', x_1, x_2 \qquad (\,3!\ \text{terms}\,)
\end{aligned}
$$

Turning now to the evaluation of the quantity

$$
\langle\, a(k_1')a(k_2') : \phi(x_2')\phi(x_1')\phi(x_1)\phi(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0
$$

the iteration of the above described process of reduction clearly shows that the only non-vanishing contributions read

$$
\begin{aligned}
&\langle\, a(k_1')a(k_2') : \phi(x_2')\phi(x_1')\phi(x_1)\phi(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0\\
=\;& \langle\, a(k_1')a(k_2') : \phi^{(+)}(x_2')\phi^{(+)}(x_1')\phi^{(-)}(x_1)\phi^{(-)}(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0\\
+\;& \text{permutations} \qquad (\,4!\ \text{terms}\,)\\
=\;& \exp\{ik_1'\cdot x_1' + ik_2'\cdot x_2' - ik_1\cdot x_1 - ik_2\cdot x_2\} + \text{permutations}
\end{aligned}
$$

3.2.4 Feynman Rules for External Legs

The fundamental LSZ reduction formula for the scattering operator matrix elements actually entails the formulation of the so called Feynman rules for the external legs. The latter ones are essentially determined by the process of reduction of the matrix elements of the type (3.15) that appear indeed in the generalization of the master formula (3.18) to the case of fields with arbitrary spin and internal quantum numbers, such as U(1) electric charge, SU(2) flavor, SU(3) color, U(1) lepton number, U(1) baryon number and so on. To this concern, it is very convenient to write the most general covariant normal mode decomposition for a generic quantum field, which transforms according to some given representation of the Poincaré group IO(1,3) and of a general internal symmetry group G: namely,

$$\Phi_\imath(x) = \int Dk \, \left[A_\sigma(k)\, u_{\imath\sigma}(k)\, \mathrm{e}^{-ikx} + B_\sigma^\dagger(k)\, \upsilon_{\imath\sigma}(k)\, \mathrm{e}^{ikx} \right]$$
$$\Phi_\imath^\dagger(x) = \int Dk \, \left[B_\sigma(k)\, \upsilon_{\imath\sigma}^*(k)\, \mathrm{e}^{-ikx} + A_\sigma^\dagger(k)\, u_{\imath\sigma}^*(k)\, \mathrm{e}^{ikx} \right]$$

in which repeated indexes are summed over and I have set as usual

$$\int Dk = \frac{1}{(2\pi)^3} \int \mathrm{d}k \; \theta(k_0)\; \delta\left(k^2 - m^2\right) = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}\mathbf{k}}{2\omega_{\mathbf{k}}}$$

Here the quantum field multiplet index $\imath \in \mathfrak{I}$, where $\mathfrak{I}$ is some discrete and finite set, does label the number of components of the charged quantum field multiplet. For example, a charged scalar field has $\imath = 1$, a Weyl left spinor field has $\imath = 1L, 2L$, while any flavor doublet of Dirac spinors in the fundamental representation of $\mathrm{SU}(2)_{\text{flavour}}$ has $\{4 = (1L, 2L, 1R, 2R)\} \times \{2 = (\text{up}, \text{down})\} = 8$ components, in such a way that $\imath = 1, 2 \ldots, 8$. A gluon vector field

$$\{A_\mu^a(x) \,|\, \mu = 0, 1, 2, 3, \;\; a = 1, \ldots, 8\}$$

is a set of eight real vector fields that transform according to the adjoint representation of $\mathrm{SU}(3)_{\text{color}}$ so that $\imath = \{\mu, a\} = (1, \ldots, 32)$.

Conversely, the inner quantum number $\sigma = 1, 2, \ldots, n$ does label the number of polarization states, or helicity eigenstates. It is worthwhile to stress that this number is in general gauge dependent, *i.e.* , it depends upon the subsidiary condition which is imposed in order to define the perturbative expansion for the gauge dependent vector potential. For instance, a mass-less left Weyl spinor, *e.g.* a mass-less neutrino, has only one single left polarization state $\sigma = L$ of negative helicity $\mathrm{h} = -\frac{1}{2}$, a massive bispinor, *e.g.* an electron, has two polarization states $\sigma = 1, 2$. In the case of the massive vector boson triplet $V_\mu^a(x)$ ($a = 1, 2, 3$) which mediates weak interactions with

$$W_\mu^\pm(x) \equiv V_\mu^1(x) \pm i\, V_\mu^2(x) \qquad\qquad Z_\mu^0(x) \equiv V_\mu^3(x)$$

the inner quantum number takes the values $\sigma = 1, 2, 3$ because we have three polarization states for each component – the two massive charged $W_\mu^\pm(x)$ and the massive neutral $Z^0_\mu(x)$ – gauge boson triplet, while the index $\imath = \{\mu, a\}$ takes 12 possible values.

The generic quantum field $\Phi_\imath(x)$ is supposed to evolve in the interaction picture, so that it satisfies a homogeneous free field equation

$$\mathcal{K}_\imath^\jmath(x)\, \Phi_\jmath(x) = 0$$

where $\mathcal{K}_\imath^\jmath(x)$ is a general matrix-like differential operator such as *e.g.* the Klein-Gordon or Dirac operators, while canonical commutation relations $(-)$ or canonical anti-commutation relations $(+)$ are assumed to hold true

$$[\, A_\sigma(k)\,,\, A^\dagger_{\sigma'}(k')\,]_\pm = (2\pi)^3\, 2\omega_{\mathbf{k}}\, \delta_{\sigma\sigma'}\, \delta(\mathbf{k}-\mathbf{k}') \tag{3.21}$$

$$[\, B_\sigma(k)\,,\, B^\dagger_{\sigma'}(k')\,]_\pm = (2\pi)^3\, 2\omega_{\mathbf{k}}\, \delta_{\sigma\sigma'}\, \delta(\mathbf{k}-\mathbf{k}') \tag{3.22}$$

all the other commutators or anti-commutators being null. Finally, in the case of self-conjugated Hermitean fields there is only one type of creation and destruction operators, because the particles and antiparticles actually coincide.

The smeared normalizable 1-particle polarized states of a general quantum field are defined by

$$|\, f_\sigma \,\rangle = A^\dagger_\sigma(f)\,|\,0\,\rangle = \int Dk\; f(k)\, A^\dagger_\sigma(k)\,|\,0\,\rangle \tag{3.23}$$

$$|\, g_\sigma \,\rangle = B^\dagger_\sigma(g)\,|\,0\,\rangle = \int Dk\; g(k)\, B^\dagger_\sigma(k)\,|\,0\,\rangle \tag{3.24}$$

$$\langle\, 0\,|\,0\,\rangle = 1 \quad \Leftrightarrow \quad \langle\, f_\sigma\,|\, f_{\sigma'}\,\rangle = \langle\, g_\sigma\,|\, g_{\sigma'}\,\rangle = \delta_{\sigma\sigma'} \tag{3.25}$$

The normalizable wave functions or wave packets read

$$\begin{aligned} f_{\imath\sigma}(x) &= \langle\, 0\,|\,\Phi_\imath(x)\,|\, f_\sigma\,\rangle \\ &= \Big\langle\, [\,\Phi_\imath(x)\,,\, A^\dagger_\sigma(f)\,]_\pm \Big\rangle_0 \\ &= \int Dk\; f(k)\, u_{\imath\sigma}(k)\, \mathrm{e}^{-ikx} \\ g_{\imath\sigma}(x) &= \langle\, 0\,|\,\Phi^\dagger_\imath(x)\,|\, g_\sigma\,\rangle \\ &= \Big\langle\, [\,\Phi^\dagger_\imath(x)\,,\, B^\dagger_\sigma(g)\,]_\pm \Big\rangle_0 \\ &= \int Dk\; g(k)\, v^{\top *}_{\imath\sigma}(k)\, \mathrm{e}^{-ikx} \end{aligned}$$

where, as usual, $k_0 = \omega_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}^2}$. Notice that, by construction, the normalizable wave packets are solutions of the very same free field equation satisfied by the general quantum field $\Phi_\imath(x)$ in the interaction picture, *viz.*,

$$\mathcal{K}_\imath^\jmath(x)\, f_{\jmath\sigma}(x) = \mathcal{K}_\imath^\jmath(x)\, g_{\jmath\sigma}(x) = 0 \qquad (\,\forall\,\sigma = 1,\ldots,n, \quad \imath \in \mathfrak{I}\,)$$

In the case of M different particles and N different antiparticles it is necessary to consider the states

$$|\, f_{\sigma_1}, \dots\; f_{\sigma_M}\,;\, g_{\sigma_1}, \dots\; g_{\sigma_N}\,\rangle = \prod_{a=1}^{M}\;\prod_{c=1}^{N}\; A^{\dagger}_{\sigma_a}\,(\,f_a\,)\,B^{\dagger}_{\sigma_c}\,(\,g_c\,)\,|\,0\,\rangle$$

If the 1-particle wave packets are normalized to unit and if the particles are indeed different – which is necessarily true in the case of anti-commuting fields – then the many-particle states are also normalized to one.

In the case of quantum fields with integer spin, *i.e.* boson fields that satisfy canonical commutation relations, we have to treat separately the case in which among the inner quantum numbers $\sigma_1, \sigma_2\,, \dots, \sigma_N$ there are some identical ones, *i.e.* when among the N particles there are several groups of identical particles. In the case of several groups $\nu_1, \nu_2\,, \dots, \nu_a$ of identical particles we shall correspondingly obtain

$$|\, f_1 f_2\, \dots\, f_M\,\rangle = \Big(\prod\,[\,\nu\,!\,]\Big)^{-\frac{1}{2}}\;\prod_{i=1}^{M}\; A^{\dagger}_{\rm as}\,(\,f_i\,)\;|\,0\,\rangle \tag{3.26}$$

where the following notation has been used

$$\prod\,[\,\nu\,!\,]\;\;=\;\;\nu_1!\,\nu_2!\;\cdots\;\nu_a! \tag{3.27}$$

and a quite analogous formula evidently holds true for antiparticles.

`Spinor Examples.` To understand how to handle the reduction of the matrix elements of normal ordered products of quantum fields in the interaction representation, let me consider some simple enlightening examples. Let us first calculate the simplest quantity involving a pair of Dirac bispinor fields, *viz.,*

$$\begin{aligned}
& \langle\, 0\,|\, c_{r'}\,(p'\,) : \bar\psi(x'\,)\,\psi(x) : c^{\dagger}_{r}\,(p)\,|\,0\,\rangle \\
= \;& \langle\, 0\,|\, c_{r'}\,(p'\,) : \Big(\bar\psi^{\,(-)}(x'\,) + \bar\psi^{\,(+)}(x'\,)\Big)\,\Big(\psi^{\,(-)}(x) + \psi^{\,(+)}(x)\Big) : c_{r}\,(p)\,|\,0\,\rangle \\
= \;& \langle\, 0\,|\, c_{r'}\,(p'\,)\,\bar\psi^{\,(-)}(x'\,)\,\psi^{\,(-)}(x)\,c^{\dagger}_{r}\,(p)\,|\,0\,\rangle \\
+ \;& \langle\, 0\,|\, c_{r'}\,(p'\,)\,\bar\psi^{\,(+)}(x'\,)\,\psi^{\,(+)}(x)\,c^{\dagger}_{r}\,(p)\,|\,0\,\rangle \\
- \;& \langle\, 0\,|\, c_{r'}\,(p'\,)\,\psi^{\,(+)}(x'\,)\,\bar\psi^{\,(-)}(x)\,c^{\dagger}_{r}\,(p)\,|\,0\,\rangle \\
+ \;& \langle\, 0\,|\, c_{r'}\,(p'\,)\,\bar\psi^{\,(+)}(x'\,)\,\psi^{\,(-)}(x)\,c^{\dagger}_{r}\,(p)\,|\,0\,\rangle
\end{aligned} \tag{3.28}$$

where of course

$$\psi^{\,(-)}(x) = \int Dq\; c_{\,s}\,(q)\,u_{\,s}\,(q)\,{\rm e}^{-\,iqx}$$

$$\psi^{\,(+)}(x) = \int Dq\; d^{\dagger}_{s}\,(q)\,v_{\,s}\,(q)\,{\rm e}^{\,iqx}$$

$$\overline{\psi}^{\,(-)}(x) = \int Dq\; d_{\,s}\,(q)\,\bar v_{\,s}\,(q)\,{\rm e}^{-\,iqx}$$

$$\overline{\psi}^{\,(+)}(x) = \int Dq\; c_s^\dagger(q)\,\bar{u}_s(q)\,\mathrm{e}^{\,iqx}$$

with

$$\{\psi_\alpha(x)\,,\,\bar{\psi}_\beta(y)\} = S_{\alpha\beta}(x-y) \qquad (\,\alpha,\beta = 1L, 2L, 1R, 2R\,)$$

$$\{c_{r'}(p')\,,\,c_r^\dagger(p)\} = \{d_{r'}(p')\,,\,d_r^\dagger(p)\} = (2\pi)^3\,2\omega_{\mathbf{p}}\,\delta(\mathbf{p}-\mathbf{p}')\,\delta_{rr'}$$

all the other anti-commutators being null. A little though is sufficient to gather that only the very last term in the right hand side of equation (3.28) does contribute and gives

$$\begin{aligned}
&\langle\,0\,|\,\underbrace{c_{r'}(p') : \bar{\psi}(x')}\;\underbrace{\psi(x) : c_r^\dagger(p)}\,|\,0\,\rangle \\
\equiv\;& \langle\,0\,|\,\{c_{r'}(p')\,,\,\bar{\psi}^{(+)}(x')\}\,\{\psi^{(-)}(x)\,,\,c_r^\dagger(p)\}\,|\,0\,\rangle \\
=\;& \bar{u}_{r'}(p')\,u_r(p)\,\exp\{-\,ip\cdot x + ip'\cdot x'\}
\end{aligned} \qquad (3.29)$$

A quite similar calculation for antiparticle states immediately yields

$$\begin{aligned}
&(-1)\quad \langle\,0\,|\,\underbrace{d_{r'}(p') : \psi(x')}\;\underbrace{\bar{\psi}(x) : d_r^\dagger(p)}\,|\,0\,\rangle \\
\equiv\;& (-1)\langle\,0\,|\,\{d_{r'}(p')\,,\,\psi^{(+)}(x')\}\,\{\bar{\psi}^{(-)}(x)\,,\,d_r^\dagger(p)\}\,|\,0\,\rangle \\
=\;& (-1)\,\bar{v}_r(p)\,v_{r'}(p')\,\exp\{\,-\,ip\cdot x + ip'\cdot x'\}
\end{aligned} \qquad (3.30)$$

the minus sign being due to the anti-commutation of the spinor field operators. Of course, had we employed normalizable wave packets it is immediate to obtain

$$\begin{aligned}
&\langle\,0\,|\,\underbrace{c_{r'}(f') : \bar{\psi}(x')}\;\underbrace{\psi(x) : c_r^\dagger(f)}\,|\,0\,\rangle \\
\equiv\;& \langle\,0\,|\,\{c_{r'}(f')\,,\,\bar{\psi}^{(+)}(x')\}\,\{\psi^{(-)}(x)\,,\,c_r^\dagger(f)\}\,|\,0\,\rangle \\
=\;& \bar{u}_{r'}(p')\,u_r(p)\,\exp\{-\,ip\cdot x + ip'\cdot x'\}\,[\,\widetilde{f}'(p')\,]^*\,\widetilde{f}(p)
\end{aligned} \qquad (3.31)$$

together with

$$\begin{aligned}
&(-1)\quad \langle\,0\,|\,\underbrace{d_{r'}(g') : \psi(x')}\;\underbrace{\bar{\psi}(x) : d_r^\dagger(g)}\,|\,0\,\rangle \\
\equiv\;& (-1)\langle\,0\,|\,\{d_{r'}(g')\,,\,\psi^{(+)}(x')\}\,\{\bar{\psi}^{(-)}(x)\,,\,d_r^\dagger(g)\}\,|\,0\,\rangle \\
=\;& (-1)\,\bar{v}_r(p)\,v_{r'}(p')\,\exp\{\,-\,ip\cdot x + ip'\cdot x'\}\,\widetilde{g}^*(p)\,\widetilde{g}'(p')
\end{aligned} \qquad (3.32)$$

Hence we end up with the following momentum space Feynman rules for the external legs of a Dirac bispinor quantum field: namely,

$$\underbrace{\psi\,|\,-\,r\,p}\rangle \quad\leftrightarrow\quad u_r(p) \qquad \text{incoming particle} \qquad (3.33)$$

$$\underbrace{\bar{\psi}\,|\,+\,r\,p}\rangle \quad\leftrightarrow\quad \bar{v}_r(p) \qquad \text{incoming antiparticle} \qquad (3.34)$$

$$\langle\underbrace{p\,r\,-\,|\,\bar{\psi}} \quad\leftrightarrow\quad \bar{u}_r(p) \qquad \text{outgoing particle} \qquad (3.35)$$

$$\langle\underbrace{p\,r\,+\,|\,\psi} \quad\leftrightarrow\quad v_r(p) \qquad \text{outgoing antiparticle} \qquad (3.36)$$

and the corresponding ones for normalizable 1-particle and 1-antiparticle states, *viz.,*

$$\underbrace{\psi\,|\,-\,r\,f}\rangle \quad\leftrightarrow\quad u_r(p)\,\widetilde{f}(p) \qquad \text{incoming particle} \qquad (3.37)$$

$$\underbrace{\bar{\psi}\,|\,+\,r\,g\,\rangle} \quad \leftrightarrow \quad \bar{v}_r\,(p)\,\widetilde{g}^*(p) \qquad \text{incoming antiparticle} \tag{3.38}$$

$$\langle\,\underbrace{f\,r\,-\,|\,\bar{\psi}} \quad \leftrightarrow \quad \bar{u}_r\,(p)\,\widetilde{f}^*(p) \qquad \text{outgoing particle} \tag{3.39}$$

$$\langle\,\underbrace{g\,r\,+\,|\,\psi} \quad \leftrightarrow \quad v_r\,(p)\,\widetilde{g}(p) \qquad \text{outgoing antiparticle} \tag{3.40}$$

Furthermore, the number of anti-commutations necessary to embed all the field operators at the right place will settle the overall sign of any matrix elements of normal ordered products involving anti-commuting spinor fields in the interaction representation.

A straightforward extension of the above described procedure, leading to the Feynman rules for the so called external legs of a Dirac bispinor field, can be simply implemented to treat the general field $\Phi_\imath(x)$ and yields

$$\underbrace{\Phi_\imath\,|\,-\,\sigma\,f\,\rangle} \quad \leftrightarrow \quad u_{\imath\sigma}\,(p)\,\widetilde{f}(p) \qquad \text{incoming particle} \tag{3.41}$$

$$\underbrace{\Phi_\imath^\dagger\,|\,+\,\sigma\,g\,\rangle} \quad \leftrightarrow \quad v_{\imath\sigma}^{\top *}(p)\,\widetilde{g}^*(p) \qquad \text{incoming antiparticle} \tag{3.42}$$

$$\langle\,\underbrace{f\,\sigma\,-\,|\,\Phi_\varkappa^\dagger} \quad \leftrightarrow \quad u_{\varkappa\sigma}^{\top *}(p)\,\widetilde{f}^*(p) \qquad \text{outgoing particle} \tag{3.43}$$

$$\langle\,\underbrace{g\,\sigma\,+\,|\,\Phi_\varkappa} \quad \leftrightarrow \quad v_{\varkappa\sigma}(p)\,\widetilde{g}\,(p) \qquad \text{outgoing antiparticle} \tag{3.44}$$

For example, the gluons are massless quanta of a real vector field multiplet, which transforms according to the adjoint representation of the non-Abelian group $SU(3)_{\text{colour}}$. It turns out that the exchange of virtual gluons between quarks, antiquarks fermions and gluons themselves does actually produce, according to the gauge theory of Quantum ChromoDynamics (QCD), the strong interaction and eventually the nuclear forces. In this case we find $\Phi_\imath(x) = \{A_\mu^a(x)\,|\,a = 1, 2, \ldots, 8\}$ with

$$A_\mu^a(x) = \int Dk\;\varepsilon_{\mu A}^{\,a}(k)\,a_A(k)\,\mathrm{e}^{-\,ikx} \;+\; \text{h.c.} \tag{3.45}$$

where the index $A = (1, 2, L, S)$ runs over the two transverse and physical gluon polarization, as well as over the longitudinal and scalar non-physical ones. Moreover, in the Feynman gauge we have

$$[\,a_A(k)\,,\,a_B^\dagger(k'\,)\,] = \eta_{AB}\,(2\pi)^3\,2|\mathbf{k}|\,\delta(\mathbf{k}-\mathbf{k}'\,)$$

$$\underbrace{A_\mu^a\,|\,B\,f\,\rangle} \quad \leftrightarrow \quad \varepsilon_{\mu B}^{\,a}(k)\,\widetilde{f}(k) \qquad \text{incoming gluon} \tag{3.46}$$

$$\langle\,\underbrace{f\,B\,|\,A_\mu^a} \quad \leftrightarrow \quad \varepsilon_{\mu B}^{\,a*}(k)\,\widetilde{f}^*(k) \qquad \text{outgoing gluon} \tag{3.47}$$

with

$$\eta_{AB}\,\varepsilon^{a\,*}_{\mu A}(k)\,\varepsilon^{\,b}_{\nu B}(k) + \text{c.c.} = -\,g_{\mu\nu}\,\delta^{ab} \tag{3.48}$$

$$-\,g^{\mu\nu}\,\varepsilon^{a\,*}_{\mu A}(k)\,\varepsilon^{\,b}_{\nu B}(k) + \text{c.c.} = \eta_{AB}\,\delta^{ab} \tag{3.49}$$

As a final remark, it is worthwhile to note that when a field theory involves both kinds of fields, *i.e.* quantum free fields in the interaction picture, which satisfy canonical equal time commutation relation

$$[\,\Phi_\imath(t,\mathbf{x})\,,\Phi_\jmath(t,\mathbf{y})\,] = 0$$

as well as canonical equal time anti-commutation relations

$$\{\Psi_\varkappa(t,\mathbf{x})\,,\Psi_\ell(t,\mathbf{y})\} = 0$$

then it is assumed that they always commute at any times: namely,

$$[\,\Phi_\imath(x)\,,\;\Psi_\jmath(y)\,] = [\,\Phi_\imath(x)\,,\;\Psi^\dagger_\jmath(y)\,] = 0$$

3.2.5 Yukawa Potential

Soon after the discovery of the neutron particle by James Chadwick in 1932 at the Cavendish Laboratory in Cambridge (UK) it was Werner Heisenberg Who immediately put forward the idea the proton and neutrons were two different states of one single particle called Nucleon.

Heisenberg, W. (1932) *Uber den Bau der Atomkerne. I*, Z. Phys. **77**, pp. 1-11; *Über den Bau der Atomkerne. II*, Z. Phys. **78**, pp. 156-164.

As a matter of fact, the mass of the neutron (symbol n) and the proton (symbol p) are almost identical: they are nearly degenerate, and both are thus often called Nucleons. Experimentally one finds

$$(m_n - m_p)/(m_n + m_p) \simeq 0.7\times 10^{-3}$$

so that we can set $m_p \approx m_n \approx M \simeq 939\ \text{MeV}/c^2$. Although the proton has a positive electric charge while the neutron is neutral, they are almost identical in all other aspects. The strength of the attractive and very strong interaction between any pair of Nucleons is the same, independent of whether they are interacting as protons or as neutrons. According to this phenomenological framework, the two kinds of Nucleons can be supposed to be point-like and arranged into a doublet of Dirac fields

$$\Psi(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix}$$

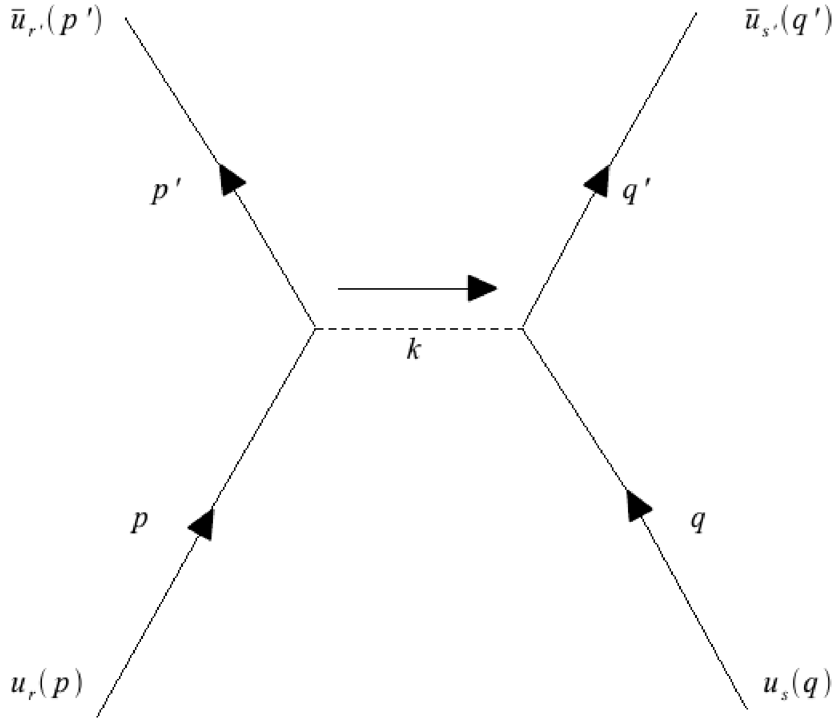

Figure 3.2: The lowest order diagram corresponding to Nucleon scattering in the Yukawa theory

transforming according to one of the fundamental representations of SU(2) that is called the Isotopic Spin or Isospin internal symmetry group, according to the original Heisenberg title to indicate this new quantum number. The point-like Nucleons are supposed to interact through the Yukava force carried by a spin-less isoscalar real meson field $\pi^0(x)$, so that the classical Lagrangian of the present Heisenberg-Yukawa model model for nuclear matter reads

$$\begin{aligned} \mathcal{L} &= \Psi^\dagger(x)\gamma_0(i\,\partial\!\!\!/ - M)\,\Psi(x) + \tfrac{1}{2}\partial_\mu\pi^0(x)\partial^\mu\pi^0(x) - \tfrac{1}{2}m^2[\,\pi^0(x)\,]^2 \\ &- \; y\,\pi^0(x)\Psi^\dagger(x)\gamma_0\Psi(x) \qquad (\,y \in \mathbb{R}\,) \end{aligned}$$

The Lagrangian is invariant under the full Lorentz group, under the charge conjugation symmetry, the SU(2) isospin transformations on the spinor fields

$$\Psi(x) \quad \longmapsto \quad \Psi'(x) = \exp\left\{\tfrac{1}{2}\,i\sigma_a\,\theta_a\right\}\Psi(x) \qquad (\,a = 1,2,3\,)$$

and the overall phase transformation on the SU(2) spinor doublet

$$\Psi(x) \quad \longmapsto \quad \Psi'(x) = \mathrm{e}^{i\varphi}\,\Psi(x)$$

where σ_a are the Pauli matrices while

$$0 \le \theta < 2\pi \qquad 0 \le \varphi < 2\pi \qquad \theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}$$

are the canonical coordinates of the internal symmetry group SU(2)×U(1). The invariance under the Abelian group of the phase transformations leads to conservation of the barion number B. Thus, if we measure the charge Q in units of the proton charge e, then we can write the relation $Q = T_3 + \frac{1}{2}B$ where

$$\begin{aligned} T_3 &= \int \mathrm{d}\mathbf{x}\,\Psi^\dagger(t,\mathbf{x})\,\tfrac{1}{2}\,\sigma_3\Psi(t,\mathbf{x}) \\ &= \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\,[\,p^\dagger(t,\mathbf{x})\,p(t,\mathbf{x}) - n^\dagger(t,\mathbf{x})\,n(t,\mathbf{x})\,] \\ B &= \int \mathrm{d}\mathbf{x}\,[\,p^\dagger(t,\mathbf{x})\,p(t,\mathbf{x}) + n^\dagger(t,\mathbf{x})\,n(t,\mathbf{x})\,] \\ \frac{Q}{e} &= \int \mathrm{d}\mathbf{x}\;p^\dagger(t,\mathbf{x})\,p(t,\mathbf{x}) \end{aligned}$$

The momentum space Feynman rules are the very same for both kinds of Nucleons as well as the Feynman rules for the incoming and outgoing particles and antiparticles: namely

- scalar propagator: $D_F(k) = i\,[\,k^2 - m^2 + i\varepsilon\,]^{-1}$

- spinor propagator: $S^F_{\alpha\beta}(p) = i(\not p + M)_{\alpha\beta}\,(p^2 - M^2 + i\varepsilon)^{-1}$
- meson-Nucleon-Nucleon vertex: $-\,iy \qquad (\,p_1 + k - p_2 = 0\,)$
- for each loop of internal line labeled by ℓ : $\int \mathrm{d}^4\ell/(2\pi)^4$
- a factor $(-\,1)$ for each fermion loop
- incoming Nucleon: $u_r(p_1)\,,\ u_s(p_2)$
- outgoing Nucleon: $\bar{u}_{r'}(p_1')\,,\ \overline{u}_{s'}(p_2')$
- incoming anti-nucleon: $\bar{v}_r(p_1)\,,\ \overline{v}_s(p_2)$
- outgoing anti-nucleon: $v_{r'}(p_1')\,,\ v_{s'}(p_2')$

Let us now consider, for the sake of pedagogical simplicity, the *pn* collision for incident Nucleons momenta much below Mc, *i.e.* in the non-relativistic approximation. In such a circumstance, by comparing the amplitude for this process – up to the lowest order in the Yukawa coupling y – to the scattering amplitude of non-relativistic quantum mechanics in the Born approximation, we can extract the potential $V(r)$ created by the Yukawa field theory model.

As the two colliding Nucleons are distinguishable, only the diagram of Fig. 3.2 does contribute to the lowest order y^2. Actually we understand the incoming particles as free spinor particles of given energy momentum and polarization (p,r) and (q,s), while the outgoing free particles will carry the energy momentum and polarization labels (p',r') and (q',s') respectively. Hence, the application of the basic formula (3.2) to the Yukawa interaction with two kinds of spinor fields yields

$$\begin{aligned} S &= \mathbb{I} - iy \int \mathrm{d}x\, \pi^0_{\rm int}(x)[\,\overline{p}_{\rm int}(x)\, p_{\rm int}(x) + \overline{n}_{\rm int}(x)\, n_{\rm int}(x)\,] \\ &- \ y^2 \int \mathrm{d}x \int \mathrm{d}x'\, T\, \pi^0_{\rm int}(x)\, \overline{p}_{\rm int}(x)\, p_{\rm int}(x) \pi^0_{\rm int}(x')\, \overline{n}_{\rm int}(x')\, n_{\rm int}(x') \\ &+ \ \cdots\cdots \end{aligned}$$

and the further application of the Wick's theorem to this scattering operator matrix element gives rise to only a single non-vanishing term, *viz.,*

$$\begin{aligned} & (-iy)^2\, \langle\, 0\,|\, c_{r'}(p')\, C_{s'}(q')\, : \bar{n}^{(+)}_{x'}\, n^{(-)}_{x'}\, \overline{p}^{(+)}_x\, p^{(-)}_x : C^\dagger_s(q)\, c^\dagger_r(p)\,|\,0\,\rangle\, D_{xx'} \\ = \ & (-1)(-iy)^2\, \langle\, 0\,|\, c_{r'}(p')\, C_{s'}(q')\, \bar{n}^{(+)}_{x'}\, \underbrace{n^{(-)}_{x'}\, c^\dagger_r(p)}\, \overline{p}^{(+)}_x\, \underbrace{p^{(-)}_x\, C^\dagger_s(q)}\,|\,0\,\rangle\, D_{xx'} \\ = \ & (+1)\,(-iy)^2\, \underbrace{c_{r'}(p')\, \bar{n}_{x'}}\, \underbrace{n_{x'}\, c^\dagger_r(p)}\ D_{xx'}\ \underbrace{C_{s'}(q')\, \overline{p}_x}\, \underbrace{p_x\, C^\dagger_s(q)} \\ \Rightarrow \ & \bar{u}_{r'}(p')\, u_r(p)\, \frac{-\,iy^2}{(p'-p)^2 - m^2}\, \overline{u}_{s'}(q')\, u_s(q) \qquad (\,p + q = p' + q'\,) \end{aligned}$$

where we have indicated with small and capital letters the creation and destruction operators of the neutron and proton particles respectively. Here the Dirac bispinor indexes have been always understood, to avoid too heavy notations while, of course, we have $r, s, r', s' = 1, 2$, $p^2 = q^2 = p'^2 = q'^2 = M^2$. In the non-relativistic limit we can approximate as follows:

$$\begin{aligned} \hbar p &\approx (Mc, \hbar\mathbf{p}) & \hbar q &\approx (Mc, \hbar\mathbf{q}) \\ \hbar p' &\approx (Mc, \hbar\mathbf{p}') & \hbar q' &\approx (Mc, \hbar\mathbf{q}') \\ (p - p')^2 &\approx -|\,\mathbf{p} - \mathbf{p}'\,|^2 & u_r(p) &\approx \xi_r \sqrt{\frac{Mc}{\hbar}} \quad \text{et cetera} \\ \bar{u}_{r'}(p')\,u_r(p) &\approx 2\,\frac{Mc}{\hbar}\,\delta_{rr'} & \bar{u}_{s'}(q')\,u_s(q) &\approx 2\,\frac{Mc}{\hbar}\,\delta_{ss'} \end{aligned}$$

in such a manner that the particle spin is conserved in the non-relativistic regime. Putting all pieces together we find from the Feynman graph the dimensionless transition amplitude

$$\overline{u}_{s'}(q')\,u_s(q)\,\frac{-\,iy^2\hbar^2}{\hbar^2k^2 - m^2c^2}\,\bar{u}_{r'}(p')\,u_r(p) \qquad (\,p + q = p' + q'\,) \tag{3.50}$$

where $k = p - p' = q' - q$ is the so called transferred four momentum; thus, in the non-relativistic approximation, we are left with

$$\frac{iy^2\hbar^2}{\hbar^2\mathbf{k}^2 + m^2c^2}\,\left(\frac{2Mc}{\hbar}\right)^2\,\delta_{rr'}\,\delta_{ss'} \equiv 4\pi i\,\mathrm{T}_{\mathbf{k}}\,\frac{2Mc^2}{\hbar c}\,\delta_{rr'}\,\delta_{ss'}$$

with $\mathbf{p} + \mathbf{q} = \mathbf{p}' + \mathbf{q}'$, in which I have suitably factorized the non-relativistic center of mass energy $2Mc^2$, as well as the conversion factor $\hbar c$, so that the transition matrix for relative motion in the non-relativistic approximation reads

$$\frac{\hbar}{8\pi Mc}\,\mathcal{M}_{\mathbf{p}\to\mathbf{p}'} \equiv \mathrm{T}_{\mathbf{k}} = \frac{y^2}{4\pi}\cdot\frac{2M\hbar c}{\hbar^2|\,\mathbf{p} - \mathbf{p}'\,|^2 + m^2c^2} \tag{3.51}$$

which has the dimensions of a length in physical units. In non-relativistic quantum mechanics, the scattering amplitude in the Born approximation for a spin-less particle in a time independent potential $V(\mathbf{r})$ is given by [4]

$$\begin{aligned} f(\theta) &= -\,\frac{M}{2\pi\hbar^2}\,\langle\,\mathbf{p}'\,|\,V\,|\,\mathbf{p}\,\rangle = -\,\frac{M}{2\pi\hbar^2}\,\widetilde{V}(\mathbf{p} - \mathbf{p}') \\ &= -\,\frac{M}{2\pi\hbar^2}\int \mathrm{d}\mathbf{r}\ \exp\{\,-i\,(\mathbf{p} - \mathbf{p}'\,)\cdot\mathbf{r}\}\ V(\mathbf{r}) \end{aligned}$$

[4] See *e.g.* Eugene Merzbacher, *Quantum Mechanics*, John Wiley & Sons, New York (1970) §**5** p.497

where M is the reduced nucleon mass, while θ is the scattering angle for the elastic process with

$$\mathbf{p}\cdot\mathbf{p}' = \mathrm{p}\,\mathrm{p}'\cos\theta \qquad\qquad \mathrm{p}^2 = \mathrm{p}'^2 = 2ME\,\hbar^{-2}$$

Hence, the differential cross-section for a scattering process by a static fixed target potential is simply provided by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |\,f(\theta)\,|^2 \qquad\qquad \mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Consider the attractive Yukawa central potential

$$V(r) = \begin{cases} +\infty & \text{for } 0 \le r < a \\ -\,V_0(r_0/r)\,e^{-r/r_0} & \text{for } \ r \ge a \end{cases}$$

in which a is the hard core repulsive barrier of nucleon impenetrability, while the characteristic length $r_0 = \hbar/\mu c$ may be identified with the range of the potential. Thus we get

$$\begin{aligned} f(\theta) &= \frac{M}{\hbar^2}\,V_0 r_0 \int_{-1}^{1}\mathrm{d}\tau \int_a^\infty r\,\mathrm{d}r\ \exp\{-\,(r/r_0) + i\,\tau\,r\,|\,\mathbf{p}-\mathbf{p}'\,|\} \\ &= 2MV_0 r_0 \int_a^\infty \mathrm{d}r\ \frac{\sin(\,|\,\mathbf{p}-\mathbf{p}'\,|\,r\,)}{\hbar^2|\,\mathbf{p}-\mathbf{p}'\,|}\ e^{-\,r/r_0} \\ &\overset{a\,\to\,0}{\sim} \frac{2MV_0 r_0}{\hbar^2|\,\mathbf{p}-\mathbf{p}'\,|^2 + \mu^2c^2} = \frac{2MV_0 r_0}{4\hbar^2\,\mathrm{p}^2\sin^2(\,\theta/2\,) + \mu^2c^2} \end{aligned}$$

because of the relation $\mathrm{p} = \mathrm{p}'$, which holds true in the non-relativistic limit, that implies

$$|\,\mathbf{p}-\mathbf{p}'\,|^2 = \mathrm{p}^2 + \mathrm{p}'^2 - 2\,\mathrm{pp}'\cos\theta = 4\,\mathrm{p}^2\sin^2(\,\theta/2\,)$$

Now it is apparent that we have to identify

$$\mu = m \qquad\qquad \frac{V_0 r_0}{\hbar c} = \frac{y^2}{4\pi} \qquad\qquad f(\theta) = \mathrm{T}_{\,\mathbf{p}\to\mathbf{p}'}$$

$$V(r) = -\,\hbar c\,\frac{y^2}{4\pi r}\,e^{-\,mcr/\hbar} \qquad\qquad \text{for} \quad r \ge a$$

in such a manner that we can eventually write

$$-\,\frac{2\pi\hbar^2}{M}\,\mathrm{T}_{\mathbf{k}} = \widetilde{V}(\,\mathbf{p}-\mathbf{p}'\,) = -\,\hbar c\left(\frac{\hbar c}{2Mc^2}\right)^2 \mathcal{M}_{\,\mathbf{p}\to\mathbf{p}'} \tag{3.52}$$

$$\widetilde{V}(\,\mathbf{p}-\mathbf{p}'\,) = \left(-\,\frac{2\pi\hbar^2}{M}\right)\cdot\frac{\hbar c\,(y^2/4\pi)\,2Mc}{|\,\mathbf{p}-\mathbf{p}'\,|^2 + m^2c^2/\hbar^2} \tag{3.53}$$

and thereby

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\left(\frac{y^{2}}{4\pi}\cdot\frac{2Mc\hbar}{4\hbar^{2}\mathrm{p}^{2}\,\sin^{2}(\,\theta/2\,)+m^{2}c^{2}}\right)^{2}$$

Hence, under the assumption of a nuclear force characteristic range of the order

$$r_0\;=\;\frac{\hbar}{mc}\;\sim 1\;\mathrm{fm}=10^{-15}\;\mathrm{m}$$

we eventually find a Yukawa meson mass

$$mc^2=\frac{\hbar c}{1\;\mathrm{fm}}\qquad\Leftrightarrow\qquad m\,\sim\,197\;\mathrm{MeV}/c^2$$

which is not too far from the neutral pion mass $m_{\pi^0}\,=\,134.9766\pm0.0006$ MeV/c^2 keeping in mind the crudeness of the approximation. Furthermore, from the phenomenological evidence that the nuclear force overcomes the Coulomb force at a distance of 1 fm of a factor thousand, we get the orders of magnitude of the depth of the nuclear potential well

$$V_0\approx e\times 10^{3}\times\hbar c\times\frac{\alpha}{r_0}\approx 2.720\,\frac{\hbar c\alpha}{1\;\mathrm{fm}}\simeq 4\times 10^{3}\,m_{\,p}\,c^{2}\approx 4\;\mathrm{GeV}\tag{3.54}$$

and of the Yukawa coupling

$$\frac{y^2}{4\pi}\approx 10^{3}\;\times\,2,72\;\alpha\,\simeq\,20\quad\Longleftrightarrow\quad y\simeq 16$$

which, unfortunately, is very unsuitable for a good perturbative expansion. Thus the Yukawa $\phi-$meson exchange model can be used and trusted at most as an effective description of the nuclear force valid at sufficiently low energies and momenta. Conversely, the related process of nucleon off anti-nucleon scattering leads to an amplitude with an opposite overall sign. As a matter of fact we find

$$\begin{aligned}
&(-iy)^2\,\langle\,0\,|\,c_{\,r^{\,\prime}}(p^{\,\prime}\,)\,d_{\,s^{\,\prime}}(q^{\,\prime}\,):\overline{\psi}_x^{\,(+)}\,\psi_x^{\,(-)}\,D_{xx'}\,\overline{\Psi}_{x'}^{\,(-)}\,\Psi_{x'}^{\,(+)}:d_s^{\dagger}(q)\,c_r^{\dagger}(p)\,|\,0\,\rangle\\
=\;&(-1)(-iy)^2\,\langle\,0\,|\,c_{\,r^{\,\prime}}(p^{\,\prime}\,)\,\underbrace{d_{\,s^{\,\prime}}(q^{\,\prime}\,)\,\Psi_{x'}^{\,(+)}}\,\overline{\psi}_x^{\,(+)}\,\psi_x^{\,(-)}\,\underbrace{\overline{\Psi}_{x'}^{\,(-)}\,d_s^{\dagger}(q)}\,c_r^{\dagger}(p)\,|\,0\,\rangle\,D_{xx'}\\
=\;&(-1)\,(-iy)^2\,\langle\,0\,|\,\underbrace{c_{\,r^{\,\prime}}(p^{\,\prime}\,)\,\overline{\psi}_x^{\,(+)}}\,\underbrace{d_{\,s^{\,\prime}}(q^{\,\prime}\,)\,\Psi_{x'}^{\,(+)}}\,\underbrace{\overline{\Psi}_{x'}^{\,(-)}\,d_s^{\dagger}(q)}\,\underbrace{\psi_x^{\,(-)}\,c_r^{\dagger}(p)}\,|\,0\,\rangle\,D_{xy}\\
=\;&(-1)\,(-iy)^2\,\underbrace{c_{\,r^{\,\prime}}(p^{\,\prime}\,)\,\overline{\psi}_x^{\,(+)}}\,\underbrace{\psi_x^{\,(-)}\,c_r^{\dagger}(p)}\,D_{xx'}\,\underbrace{\overline{\Psi}_{x'}^{\,(-)}\,d_s^{\dagger}(q)}\,\underbrace{d_{\,s^{\,\prime}}(q^{\,\prime}\,)\,\Psi_{x'}^{\,(+)}}\\
\Rightarrow\;&\bar{u}_{\,r^{\,\prime}}(p^{\,\prime}\,)\,u_{\,r}(p)\;\frac{-\,iy^2}{(p^{\,\prime}-p)^2-m^2}\;\bar{v}_{\,s}(\,q\,)\,v_{\,s^{\,\prime}}(\,q^{\,\prime}\,)\,(-1)\qquad\quad(\,p+q=p^{\,\prime}+q^{\,\prime}\,)
\end{aligned}$$

It is precisely this change of the overall sign that ensures, in the non-relativistic limit, the charge independence of the Yukawa nuclear force for

$$\bar{v}_{\,s^{\,\prime}}(\,q^{\,\prime}\,)\,v_{\,s}(\,q\,)\;\approx\;-\,2\,\frac{Mc}{\hbar}\,\delta_{\,s\,s^{\,\prime}}$$

Another quite useful example of application of the LSZ reduction formulæ is the nucleon-anti-nucleon annihilation and production by a virtual neutral meson $N+\overline{N} \longrightarrow \pi^0_* \longrightarrow \overline{N}+N$ that yields

$$\begin{aligned} & (-iy)^2 \langle 0 \,|\, d_{r'}(p')\, c_{s'}(q')\, \bar{\psi}^{(+)}_{x'}\, \psi^{(+)}_{x'}\, D_{xx'}\, \bar{\psi}^{(-)}_{x}\, \psi^{(-)}_{x}\, c^\dagger_s(q)\, d^\dagger_r(p) \,|\, 0 \rangle \\ = \;& (-1)^4 (-iy)^2 \langle 0 \,|\, \underbrace{c_{s'}(q')\, \overline{\psi}^{(+)}_{x'}}\; \underbrace{d_{r'}(p')\, \psi^{(+)}_{x'}}\; \underbrace{\overline{\psi}^{(-)}_{x}\, d^\dagger_r(p)}\; \underbrace{\psi^{(-)}_{x}\, c^\dagger_s(q)} \,|\, 0 \rangle \\ \Rightarrow \;& \bar{u}_{s'}(q')\, v_{r'}(p')\, \frac{-\,iy^2}{(p+q)^2 - m^2}\, \bar{v}_r(p)\, u_s(q) \qquad (\,p+q = p'+q'\,) \end{aligned}$$

with $\hbar^2(p+q)^2 = 4M_N^2c^2 \gg m^2_{\pi^0}c^2$ that endorses the virtual nature of the exchanged π^0 neutral meson.

3.2.6 Coulomb Potential

As a simple application of these Feynman rules, let me repeat the analysis of the non-relativistic scattering amplitude for quantum electrodynamics in natural units, in close analogy with what I did in the case of the nuclear Yukawa potential. The perturbative expansion of the collision matrix (3.2) for QED with two kinds of spinor field carrying equal masses though different electric charges reads

$$\begin{aligned} S \;=\; & \mathbb{I} + ie \int \mathrm{d}y\, A^\mu_{\mathrm{int}}(y) \overline{\psi}_{\mathrm{int}}(y) \gamma_\mu \psi_{\mathrm{int}}(y) \\ +\; & iZe \int \mathrm{d}x\, A^\nu_{\mathrm{int}}(x) \overline{\Psi}_{\mathrm{int}}(x) \gamma_\nu \Psi_{\mathrm{int}}(x) \\ -\; & Ze^2 \int \mathrm{d}x \int \mathrm{d}y\, T A^\mu_{\mathrm{int}}(x) \overline{\psi}_{\mathrm{int}}(x) \gamma_\mu \psi_{\mathrm{int}}(x) A^\nu_{\mathrm{int}}(y) \overline{\Psi}_{\mathrm{int}}(y) \gamma_\nu \Psi_{\mathrm{int}}(y) \\ +\; & \cdots\cdots \end{aligned}$$

To the first order in $e/\sqrt{2hc}$ some processes might occur in perturbation theory, in which three physical particles – one photon and two Dirac particles – would appear in the initial and final states on the mass shells. It can be readily seen, however, that those kinds of processes are impossible, owing to energy momentum conservation. If we denote by k^μ the photon momentum and by p^ν, q^ρ the Dirac particles momenta respectively, then the energy-momentum conservation is expressed by the equality $k = p \pm q$, the sign plus being related to a particle-antiparticle pair, the minus sign being instead referred to a 2-particles or a 2-antiparticles pair. The above equality is in fact impossible because $k^2 = 0$ while for *e.g.* $\mathbf{q} = 0$ we get

$$(\,p \pm q\,)^2 = 2(M^2 \pm p\cdot q) = 2(M^2 \pm p_0 q_0 \mp \mathbf{p}\cdot\mathbf{q}) = 2M(M \pm p_0)$$

and since $p_0 > M$ we find either $(p+q)^2 > 0$ or $(p-q)^2 < 0$. Hence, the first nontrivial term in the collision matrix becomes

$$\begin{aligned} S &= -Ze^2 \int \mathrm{d}x \int \mathrm{d}y \left(T\, A^{\mu}_{\mathrm{int}}(x)\, A^{\nu}_{\mathrm{int}}(y) \right) \\ &\times \left(T\, \overline{\psi}_{\mathrm{int}}(x)\gamma_\mu \psi_{\mathrm{int}}(x) \overline{\Psi}_{\mathrm{int}}(y)\gamma_\nu \Psi_{\mathrm{int}}(y) \right) + \cdots\cdots \end{aligned}$$

owing to the commutation between photon and Dirac field operators.

Thus, for elastic scattering, the Wick's theorem yields, up to the leading order and in natural units,

$$\begin{aligned} & -Ze^2 \langle 0 | c_s(q)\, C_{s'}(q') : \bar{\psi}^{(+)}_x \gamma_\mu \psi^{(-)}_x D^{\mu\nu}_{xy} \overline{\Psi}^{(+)}_y \gamma_\nu \Psi^{(-)}_y : C^\dagger_{r'}(p')\, c^\dagger_r(p) | 0 \rangle \\ = \; & -Ze^2 \langle 0 | c_s(q)\, \bar{\psi}^{(+)}_x \gamma_\mu \psi^{(-)}_x \underbrace{C_{s'}(q')\, \overline{\Psi}^{(+)}_y} \gamma_\nu \underbrace{\Psi^{(-)}_y C^\dagger_{r'}(p')}\, c^\dagger_r(p) | 0 \rangle\, D^{\mu\nu}_{xy} \\ = \; & -Ze^2 \langle 0 | \underbrace{c_s(q)\, \bar{\psi}^{(+)}_x} \gamma_\mu \underbrace{\psi^{(-)}_x c^\dagger_r(p)}\, D^{\mu\nu}_{xy} \underbrace{C_{s'}(q')\, \overline{\Psi}^{(+)}_y} \gamma_\nu \underbrace{\Psi^{(-)}_y C^\dagger_{r'}(p')}\, | 0 \rangle \\ \Rightarrow \; & -Ze^2\, \bar{u}_s(q)\, \gamma^\mu u_r(p)\, \frac{-i}{(p-p')^2}\, \overline{U}_{s'}(q')\, \gamma_\mu U_{r'}(p') \qquad (p+p' = q+q') \end{aligned}$$

where, of course, we have

$$r, s, r', s' = 1, 2\,,\;\; p^2 = q^2 = p'^2 = q'^2 = M^2$$

Putting all pieces together we find from the Feynman graph the dimensionless transition amplitude

$$\bar{u}_s(q)\,(ie\gamma^\mu)\, u_r(p)\, \frac{-i}{(p-q)^2}\, \overline{U}_{s'}(q')\,(iZe\gamma_\mu)\, U_{r'}(p') \tag{3.55}$$

Notice that the Feynman gauge photon propagator, which represents the electromagnetic interaction in the present lowest order amplitude, can be suitably rewritten in the very suggestive form

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{it}$$

where t is the Mandelstam variable which corresponds to the invariant norm of the transferred 4-momentum $k = p - q = q' - p'$, *viz.*,

$$t \equiv \hbar^2 k^2 = \hbar^2 (p-q)^2 = 2M^2c^2 \left[1 - \sqrt{1 + \frac{\hbar^2 \mathbf{q}^2}{M^2c^2}} \right] < 0$$

Here the rest frame of the incoming particle $p = (Mc, 0, 0, 0)$ has been used, without loss of generality thanks to the Lorentz invariance, to exhibit the space-like nature of the transferred momentum t. This means in turn that the

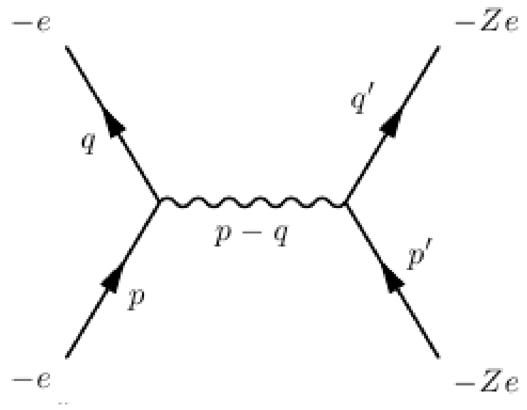

Figure 3.3: Coulomb scattering

exchanged photon, which mediates the electromagnetic interaction between the two charged Dirac spinor particles, is virtual and space-like, *i.e.* off its mass shell $k^2 = 0$, and that all the four kinds of polarization, physical and non-physical, do indeed carry the Coulomb interaction in the manifestly covariant Feynman gauge. In the non-relativistic limit we can approximate as follows:

$$\hbar p \;\approx\; (Mc, \hbar\mathbf{p}) \qquad \hbar q \;\approx\; (Mc, \hbar\mathbf{q})$$

$$\hbar p' \;\approx\; (Mc, \hbar\mathbf{p}') \qquad \hbar q' \;\approx\; (Mc, \hbar\mathbf{q}')$$

$$(p - p')^2 \;\approx\; -\,|\,\mathbf{p} - \mathbf{p}'\,|^2$$

$$u_1(p) \;\approx\; \sqrt{\frac{Mc}{\hbar}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad u_2(p) \;\approx\; \sqrt{\frac{Mc}{\hbar}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

so that

$$u_s^\dagger(q)\,u_r(p) \;\approx\; 2\,\frac{Mc}{\hbar}\,\delta_{rs} \qquad u_{s'}^\dagger(q')\,u_{r'}(q') \;\approx\; 2\,\frac{Mc}{\hbar}\,\delta_{r's'}$$

$$\bar{u}_s(q)\,\gamma^k\,u_r(p) = u_s^\dagger(q)\,\alpha^k\,u_r(p) \;\approx\; 0$$

$$\bar{u}_{s'}(q')\,\gamma_k\,u_{r'}(p') = u_{s'}^\dagger(q')\,\alpha_k\,u_{r'}(p') \;\approx\; 0$$

for $r, r', s, s' = 1, 2$, in such a manner that the particle spin is conserved in the non-relativistic regime. Then we eventually come to the non-relativistic approximation

$$\frac{-\,iZe^2c^2}{\hbar^2|\,\mathbf{p} - \mathbf{q}\,|^2}\;2M\,\delta_{rs}\;2M\,\delta_{r's'} = 4\pi i\,T_{\mathbf{p},\mathbf{q}}\;2\,\frac{Mc}{\hbar}\,\delta_{rs}\,\delta_{r's'}$$

and consequently

$$T_{\mathbf{p},\mathbf{q}} = f(\theta) = -\,\frac{2\,Mc\,Z\alpha}{\hbar|\,\mathbf{p}-\mathbf{q}\,|^2}$$

which corresponds to the repulsive Coulomb potential [5]

$$V(r) = \frac{Ze^2}{4\pi\, r} = Z\,\frac{\alpha}{r} \qquad \widetilde{V}(\,|\,\mathbf{p}-\mathbf{q}\,|\,) = \frac{Ze^2}{|\,\mathbf{p}-\mathbf{q}\,|^2}$$

so that

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \frac{(MZ\alpha\hbar c)^2}{4\,|\,\mathbf{p}\,|^4\,\sin^4(\theta/2\,)} = \frac{Z^2\,\alpha^2\hbar^2 c^2}{16E^2\,\sin^4(\theta/2\,)} \qquad (\,\mathbf{p}^2 = 2ME\,)$$

which is nothing but the celebrated Rutherford exact cross-section. For an antiparticle-particle scattering we have to make the replacement

$$\bar{u}_{\,s}\,(\,q\,)\,(\,ie\gamma^{\,\mu}\,)\,u_{\,r}\,(p) \quad\leftrightarrow\quad \bar{v}_{\,r}\,(p)\,(\,-\,ie\gamma^{\,\mu}\,)\,v_{\,s}\,(\,q\,)$$

and owing to

$$\bar{v}_{\,s}\,(\,q\,)\,\gamma^0\,v_{\,r}\,(p) \;\approx\; 2\,M\,\delta_{\,r\,s} \qquad \text{et cetera}$$

the sign of the non-relativistic Coulomb potential is opposite as it does. As a final remark I'd like to argue about gauge invariance. One is always free to replace the photon propagator in the Feynman gauge with the the most general expression in a Lorentz invariant non-homogeneous Lorenz gauge $\partial_\mu A^\mu(x) = \xi B(x)$ that yields

$$\widetilde{D}^{\,c}_{\lambda\mu}(k\,;\xi) = \frac{i\hbar c}{k^2 + i\varepsilon}\left(-\,g_{\,\lambda\mu} + \frac{1-\xi}{k^2+i\varepsilon}\,k_\lambda\,k_\mu\right)$$

with $k = p - p' = q' - q$. Now, if we recall the Dirac equations for the spin states, *viz.*,

$$(\not{p} - M)u_{\,r}(p) = 0 = \bar{u}_{\,r'}(p'\,)(\not{p}\,' - M)$$
$$(\not{q} - M)u_{\,s}(q) = 0 = \bar{u}_{\,s'}(q'\,)(\not{q}\,' - M)$$

then we obtain the matrix general element

$$\begin{aligned} &\bar{u}_{\,s}\,(\,q\,)\,(\,ie\gamma^{\,\mu}\,)\,u_{\,r}\,(p)\;\frac{-\,i}{(p-q\,)^2}\;\bar{u}_{\,s'}\,(\,q'\,)\,(\,iZe\gamma^{\,\nu}\,)\,u_{\,r'}\,(p'\,) \\ \times\;& \left[\,g_{\,\mu\nu} - \frac{1-\xi}{(p-q\,)^2 + i\varepsilon}\;(p-q)_\mu\,(\,q'-p'\,)_\nu\,\right] \\ =\;& \bar{u}_{\,s}\,(\,q\,)\,\gamma^{\,\mu}\,u_{\,r}\,(p)\;\frac{iZe^2}{(p-q\,)^2}\;\bar{u}_{\,s'}\,(\,q'\,)\,\gamma_\mu\,u_{\,r'}\,(p'\,) \end{aligned} \tag{3.56}$$

[5] The Coulomb potential is obtained as the limit of the Yukawa potential in which $m \to 0$ with $V_0 r_0 \to Z\alpha\hbar c$ – see *e.g.* Eugene Merzbacher, *Quantum Mechanics*, John Wiley & Sons, New York (1970).

which endorses gauge invariance, *i.e.* $\xi-$independence, of the lowest order scattering amplitude. However, it turns out that this fundamental feature holds true to any order, what corresponds to the so called Ward's identity.

3.3 Cross Section

3.3.1 Scattering Amplitude

In this section we shall consider, for the sake of simplicity, a self-interacting real scalar field describing spinless neutral particles without further internal structure. It is useful to express the scattering operator in the form

$$S = \mathbb{I} + i\,T \tag{3.57}$$

where the unit operator $\mathbb{I}$ is related to non-scattered, forward, straight-through particle propagation, while T is the transition matrix, the matrix elements of which do non-trivially depend upon the field interaction. Then the $S-$matrix elements are defined to be

$$S_{f\imath} = \langle\, f\,|\,S\,|\,\imath\,\rangle = \delta_{f\imath} + (2\pi)^4\, i\,\delta\left(P_f' - P_\imath\right)\,\mathcal{M}_{f\imath} \tag{3.58}$$

where the invariant $T-$matrix elements $\mathcal{M}_{f\imath}$ have been introduced for the scattering process $1 + 2 + \ldots + N \;\longmapsto\; 1' + 2' + \ldots + N'$: namely

$$\langle\, p_1'\, p_2' \,\ldots\, p_{N'}'\,|\,T\,|\,p_1\, p_2 \,\ldots\, p_N\,\rangle = (2\pi)^4\,\delta\left(P_f' - P_\imath\right)\,\mathcal{M}\,(\,p_\imath \mapsto p_f') \tag{3.59}$$

in which the energy momentum 1-particle eigenstates of all the particles are normalized according to the standard covariant convention

$$\langle\, q\,|\,p\,\rangle \;=\; 2(2\pi)^3\,\omega_{\mathbf{p}}\;\delta\,(\,\mathbf{p} - \mathbf{q}\,) \qquad\qquad p_0 = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$$

whilst

$$P_i = \sum_{j=1}^{N} p_j\;, \qquad P_f' = \sum_{k=1}^{N'} p_k'\;, \tag{3.60}$$

are the total momenta of the N incoming and N' outgoing particles. It is worthwhile to notice the canonical dimensions in physical units, *i.e.*

$$[\,S_{f\imath}\,] = [\,T_{f\imath}\,] = \mathrm{cm}^{\mathrm{N+N'}} \qquad\qquad [\,\mathcal{M}_{f\imath}\,] = \mathrm{cm}^{\mathrm{N+N'}-4}$$

It turns out that the scattering amplitude for the process $N_{\mathrm{in}} \;\longmapsto\; N_{\mathrm{out}}'$ will be given by the **dimensionless** complex quantity

$$\begin{aligned}
&\mathfrak{A}\,(f_1, f_2, \ldots, f_N \;\longmapsto\; g_1, g_2, \ldots, g_{N'}) \;=\\
&\prod_{j=1}^{N}\int\frac{\mathrm{d}\,\mathbf{p}_j}{(2\pi)^3\,2p_j^0}\;\prod_{k=1}^{N'}\int\frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\,2p_k^{0\,\prime}}\;(2\pi)^4\;i\,\delta\left(P_f' - P_i\right)\\
&\times\;\widetilde{f}_j(\mathbf{p}_j)\;\widetilde{g}_k^*(\mathbf{p}_k')\;\mathcal{M}(p_1, \ldots, p_N\,;\,p_1', \ldots, p_{N'}')
\end{aligned} \tag{3.61}$$

where $p_j^0 = \omega\,(\mathbf{p}_j)\,,\ p_k^{0\,\prime} = \omega\,(\mathbf{p}_k^{\prime})$ are the dispersion laws of the positive energy incoming and outgoing particles, whereas

$$
\begin{aligned}
f_\jmath(x) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}\mathbf{p}_\jmath}{2\omega\,(\mathbf{p}_\jmath)}\ \widetilde{f}_\jmath(\mathbf{p}_\jmath)\ \exp\{-\,i\,p_\jmath\cdot x\,\} \qquad \jmath = 1,\ldots,N\\
g_k(x) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}\mathbf{p}_k^{\prime}}{2\omega\,(\mathbf{p}_k^{\prime})}\ \widetilde{g}_k(\mathbf{p}_k^{\prime})\ \exp\{-\,i\,p_k^{\prime}\cdot x\,\} \qquad k = 1,\ldots,N^{\prime}
\end{aligned}
$$

are the particle wave functions that satisfy the Klein-Gordon wave equation and may be supposed to be real (*e.g.* Gaussian-like) without loss of generality.

Klein-Gordon probability current. We remind that for any complex solution $f(t,\mathbf{x})$ of the Klein-Gordon wave equation one can define the probability current density tetra-vector

$$
\jmath_\mu(t,\mathbf{x}) \stackrel{\mathrm{def}}{=} c\,f^{\,*}(t,\mathbf{x})\,i\,\overleftrightarrow{\partial}_\mu\,f(t,\mathbf{x}) \tag{3.62}
$$

which satisfies by construction the continuity equation

$$
\partial^{\,\mu}\,\jmath_\mu(t,\mathbf{x}) = 0
$$

in such a manner that its temporal component

$$
\varrho(t,\mathbf{x}) \stackrel{\mathrm{def}}{=} \jmath_0(t,\mathbf{x})/c = f^{\,*}(t,\mathbf{x})\,i\,\overleftrightarrow{\partial}_0\,f(t,\mathbf{x}) \qquad [\,\varrho\,] = \mathrm{cm}^{-3}
$$

will be interpreted as the probability density, *i.e.* the probability of finding a particle in the infinitesimal volume $\mathrm{d}\mathbf{x}$ at the position $\mathbf{x}$ and at the instant $x_0 = t$, according to the orthodox understanding of quantum mechanics. Notice that $\dot{\varrho}(t,\mathbf{x}) = 0$ owing to the Klein-Gordon wave equation, so that, by integrating over the whole space, we recover the conservation of the probability during the time evolution because

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\mathbf{x}\ \varrho(t,\mathbf{x}) = 0
$$

Note that in the limit where $\widetilde{f}(\mathbf{p})$ is very sharply peaked at some wave vector $\boldsymbol{\kappa}$ the probability density can be approximated by

$$
\begin{aligned}
c\varrho(t,\mathbf{x}) &= (2\pi)^{-6}\int \mathrm{d}\mathbf{p}\int \mathrm{d}\mathbf{p}^{\,\prime}\ \widetilde{f}^{\,*}(\mathbf{p}^{\,\prime})\,\widetilde{f}(\mathbf{p})\\
&\times\ \frac{\omega(\mathbf{p})+\omega(\mathbf{p}^{\,\prime})}{4\omega(\mathbf{p})\,\omega(\mathbf{p}^{\,\prime})}\ \exp\{\,i(\mathbf{p}-\mathbf{p}^{\,\prime})\cdot\mathbf{x} - it\,(\,\omega_{\mathbf{p}} - \omega_{\mathbf{p}^{\,\prime}}\,)\}\\
&\approx\ 2\,\omega_{\boldsymbol{\kappa}}\,|\,f(t,\mathbf{x})\,|^2
\end{aligned} \tag{3.63}
$$

In the same approximation the probability current density takes the form

$$
\boldsymbol{\jmath}\,(t,\mathbf{x}) \approx 2c\,\boldsymbol{\kappa}\,|\,f(t,\mathbf{x})\,|^2 \qquad [\,\jmath_\mu\,] = \mathrm{cm}^{-2}\ \mathrm{sec}^{-1}
$$

Let me now consider the scattering process

$$
1+2\ \longmapsto\ 1+2+\ldots+N \tag{3.64}
$$

so that the quantity

$$
\begin{aligned}
&\mathfrak{A}(f_1, f_2 \longmapsto g_1, g_2, \ldots, g_N) \ = \\
&\prod_{j=1}^{2} \int \frac{\mathrm{d}\mathbf{p}_j}{(2\pi)^3\, 2\,\omega\,(\mathbf{p}_j)}\ \widetilde{f}_j\,(\mathbf{p}_j\,)\ \prod_{k=1}^{N} \int D p_k'\ \widetilde{g}_k^{\,*}\,(\mathbf{p}_k'\,) \\
&\times\ (2\pi)^4\ i\,\delta\left(P_f' - p_1 - p_2\right)\ \mathcal{M}\ (p_1\,,\,p_2\,;\,p_1',\ldots,p_N'\,) \qquad (3.65)
\end{aligned}
$$

will represent the amplitude of the process in which there are N particles in the final state with wave packets $g_1, g_2, \ldots, g_N$ for two incoming particles with wave packets f_1 and f_2. Here, as usual, the invariant measure Dp_k' is provided by equation (3.19) with $P_f' = \sum_{k=1}^{N} p_k'$. Moreover, all the incoming and outgoing tetra-momenta $p_1, p_2; p_1', \ldots, p_N'$ are supposed to lie on the mass shell, as required by the LSZ reduction formula.

At first sight it would appear to be natural to take the square modulus of the above quantity in order to obtain the probability of the collision process. However, on the one hand, while we can control in detail the structure of the wave packets of the incident beams of particles, we have a very little and roughly approximate knowledge of the particle wave packets after the collision event. Hence, the above quantity and its square modulus can not be practically neither evaluated nor measured. On the other hand, had we attempted to consider the square of the related known and calculable part of the above density of probability amplitude, *i.e.,*

$$
\widetilde{f}_1\,(\mathbf{p}_1\,)\widetilde{f}_2\,(\mathbf{p}_2\,) \times\ (2\pi)^4\ i\,\delta\left(P_f' - p_1 - p_2\right)\ \mathcal{M}\ (p_1\,,\,p_2\,;\,p_1',\ldots,p_N'\,)
$$

then one has to face the problem of squaring a Dirac $\delta-$distribution, which turns out to be a mathematical nonsense. Hence, to circumvent this difficulty, it is convenient to introduce the quantity with the dimensions of an area, *viz.,*

$$
\begin{aligned}
F(P_f') \ &\equiv\ \prod_{j=1}^{2} \int \frac{\mathrm{d}\mathbf{p}_j\, \widetilde{f}_j\,(\mathbf{p}_j\,)}{(2\pi)^3\, 2\,\omega\,(\mathbf{p}_j)}\ (2\pi)^4\,\delta\,(P_f' - p_1 - p_2) \\
&=\ \prod_{j=1}^{2} \int \mathrm{d}\mathbf{p}_j\,[\,(2\pi)^3\, 2\,\omega\,(\mathbf{p}_j)\,]^{-1}\ \widetilde{f}_j\,(\mathbf{p}_j\,) \\
&\times\ \int \mathrm{d}^4x\ \ \exp\{\,i\,(P_f' - p_1 - p_2)\cdot x\,\} \\
&=\ \prod_{j=1}^{2}\ \int \mathrm{d}^4x\ \ \exp\{\,i\,x\cdot P_f'\,\} \\
&\times\ \int \mathrm{d}\mathbf{p}_j\,[\,(2\pi)^3\, 2\,\omega\,(\mathbf{p}_j)\,]^{-1}\ \widetilde{f}_j\,(\mathbf{p}_j\,)\ \exp\{-\,i\,p_j\cdot x\,\}
\end{aligned}
$$

$$= \int \mathrm{d}^4x \; f_1(x)\, f_2(x)\, \exp\{\, i\, P_f' \cdot x\,\} \tag{3.66}$$

It follows therefrom that, if the matrix elements of the transition matrix are smooth functions of the incident momenta, for two very narrow wave packets f_1 and f_2 centered around $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ respectively, it is possible to approximate the above introduced quantity in the form

$$\begin{aligned} & \mathfrak{A}(f_1, f_2 \longmapsto g_1, g_2, \ldots, g_N) \\ \approx \; & \prod_{k=1}^{N} \int Dp_k' \; \widetilde{g}_k^{\,*}(\mathbf{p}_k')\, F(P_f')\, i\, \mathcal{M}\; (\kappa_1\,,\, \kappa_2\,; p_1', \ldots, p_N') \end{aligned}$$

so that we can suitably **define** and write down the differential probability of the process in which there are N particles in the final state within the momentum space infinitesimal volume elements $\mathrm{d}\mathbf{p}_k'$ around $\mathbf{p}_k'$, which evidently reads

$$\begin{aligned} & \mathrm{d}W\,(f_1, f_2 \longmapsto p_1', p_2', \ldots, p_N'\,) \;\equiv \\ & \big|\, F(P_f'\,)\, \mathcal{M}\,(\kappa_1, \kappa_2\,;\, p_1', \ldots, p_N')\,\big|^{\,2} \; \prod_{k=1}^{N} \frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\, 2\omega\,(\mathbf{p}_k')} \end{aligned} \tag{3.67}$$

with $\kappa_j^0 = \omega(\boldsymbol{\kappa}_j)$. Notice that, for very narrow wave packets centered around $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ respectively, we can further approximate

$$\begin{aligned} & \frac{1}{(2\pi)^4} \int \mathrm{d}^4P\; |\, F(P)\,|^{\,2} \\ = \; & \frac{1}{(2\pi)^4} \int \mathrm{d}^4P \int \mathrm{d}^4x \int \mathrm{d}^4y \;\, f_1^*(x)\, f_2^*(x)\, f_1(y)\, f_2(y)\; \exp\{iP\cdot(x-y)\} \\ = \; & \int \mathrm{d}^4x\; |\, f_1(x)\, f_2\,(x)\,|^{\,2} \approx \; [\,4\,\omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)\,]^{-1} \int \mathrm{d}^4x \;\, \varrho_1(x)\, \varrho_2(x) \end{aligned}$$

so that we can approximately set

$$\begin{aligned} |\, F(P_f')\,|^{\,2} \;\; & \approx \;\; (2\pi)^4\, \delta\,(P_f' - \kappa_1 - \kappa_2) \\ & \times \;\; [\,4\,\omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)\,]^{-1} \int \mathrm{d}^4x \;\, \varrho_1(x)\varrho_2(x) \end{aligned}$$

because for any test function $\varphi \in \mathcal{S}(\mathbb{R})$ we approximately get

$$\frac{1}{(2\pi)^4} \int \mathrm{d}^4p\; |\, F(p)\,|^{\,2}\, \varphi(p) \approx \frac{\varphi(\kappa_1 + \kappa_2)}{4\,\omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)} \; \int \mathrm{d}^4x \;\, \varrho_1(x)\varrho_2(x)$$

As a consequence we eventually find that, for very narrow incoming wave packets centered around $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ respectively, we can safely write

$$\mathrm{d}\,W\,(f_1, f_2 \longmapsto p_1', p_2', \ldots, p_N') \approx \tag{3.68}$$

$$(2\pi)^4\,\delta\,(P_f - \kappa_1 - \kappa_2)\;|\,\mathcal{M}\,(\kappa_1, \kappa_2\,;\,p_1', \ldots, p_N')\,|^2$$

$$\times\;\; \tfrac{1}{4}\int \mathrm{d}^4x\;\frac{\varrho_1\,(x)}{\omega\,(\boldsymbol{\kappa}_1)}\cdot\frac{\varrho_2\,(x)}{\omega\,(\boldsymbol{\kappa}_2)}\prod_{k=1}^{N}\;\frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\,2\omega\,(\mathbf{p}_k')} \tag{3.69}$$

where the infinitesimal volume element in the $3N-$dimensional space of the wave vectors of the outgoing quanta is usually called the *phase space volume element of the final state* in the particle physicists jargon. It is important to gather that the quantity

$$\int \mathrm{d}^4x\;[\,\varrho_1(x)\,\varrho_2(x)/\,4\,\omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)\,] \tag{3.70}$$

is dimensionless. In order to compare different experiments, *e.g.* in a large high energy colliding machine, it is convenient to define a quantity which does not depend upon the details of the wave functions of the incoming particles: the differential cross-section

$$\begin{aligned}\mathrm{d}\sigma \;&\equiv\; \frac{4\omega\,(\boldsymbol{\kappa}_1)\,\omega\,(\boldsymbol{\kappa}_2)}{\int \mathrm{d}^4x\;\varrho_1(x)\;\varrho_2(x)}\;\mathrm{d}W\;\tfrac{1}{4}\,[\,(\kappa_1\cdot\kappa_2)^2 - (m_1\,m_2)^2\,]^{-1/2}\\ &=\; \tfrac{1}{4}\,[\,(\kappa_1\cdot\kappa_2)^2 - m_1^2 m_2^2\,]^{-1/2}\;|\,\mathcal{M}\,(\,\kappa_1, \kappa_2\,;\,p_1', \ldots, p_N'\,)\,|^2\\ \times&\;\;(2\pi)^4\,\delta\,(P_f - \kappa_1 - \kappa_2)\;\prod_{k=1}^{N}\;\frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\,2\,\omega\,(\mathbf{p}_k')}\end{aligned} \tag{3.71}$$

which has the dimensions of a surface area and turns out to be manifestly Lorentz invariant. This provides the Golden Rule for the High Energy Physics (HEP) and the Quantum Field Theory (QFT).

3.3.2 Luminosity

In an actual scattering experiment one has the situation in which two particle beams collide, or one beam scatters off of some fixed target. In those cases the densities $\varrho_1(x)$ and $\varrho_2(x)$ equal the particle densities in the beams and/or in the target, up to a normalization constant which accounts of the beams and/or target geometric structures. With such a kind of normalization, the proportionality factor in eq. (3.71) is then the integrated luminosity, which turns out to be Lorentz invariant: namely,

$$\begin{aligned}\textstyle\int_{-\infty}^{\infty}\mathrm{d}t\;\mathfrak{L} \;&\equiv\; \mathrm{d}W(f_1, f_2 \longmapsto 1, 2, \ldots, N\,)/\mathrm{d}\sigma\\ &=\; \sqrt{(\kappa_1\cdot\kappa_2)^2 - m_1^2 m_2^2}\;\int \mathrm{d}^4x\;\frac{\varrho_1(x)}{\omega\,(\boldsymbol{\kappa}_1)}\cdot\frac{\varrho_2(x)}{\omega\,(\boldsymbol{\kappa}_2)}\end{aligned} \tag{3.72}$$

for reactions with two incoming massive particles. In physical units we get $[\,\mathrm{d}\sigma\,] = \mathrm{barn}\,,\ [\,\mathfrak{L}\,] = \mathrm{barn}^{-1}\,\mathrm{s}^{-1}$, where 1 barn $= 10^{-24}\ \mathrm{cm}^2$. The luminosity is the quantity that gives a measure of the magnitude of scattering event differential rate $\mathrm{d}R$, within experimental settings described by two bunches of incident particles with the particle densities $\varrho_\imath(x)\ (\imath = 1, 2)$ owing to

$$\mathrm{d}\dot{W} \equiv \mathrm{d}R = \mathfrak{L}\,\mathrm{d}\sigma \tag{3.73}$$

For instance [1] the luminosity at LHC for proton-proton collisions during the years 2012-2015 has been designed to be $\mathfrak{L} \sim 10^{34}\ \mathrm{cm}^{-2}\,\mathrm{s}^{-1} = 10^{10}\ \mathrm{b}^{-1}\ \mathrm{s}^{-1}$ and it was first reached in June 2016. In general, present day high energy colliding storage ring machines (*collider* in jargon) reach a luminosity within the range $10^{28} \div 10^{34}\ \mathrm{cm}^{-2}\,\mathrm{s}^{-1}$. The luminosity is one of the most crucial parameters for a colliding beam storage ring accelerator machine. Cross-sections at the GeV scale for the center of momentum energy are usually of the order of [6] :

- 1 millibarn = 1 mb = $10^{-27}\ \mathrm{cm}^2$ for strong interactions
- 1 nanobarn = 1 nb = $10^{-33}\ \mathrm{cm}^2$ for electromagnetic interactions
- 1 femtobarn = 1 fb = $10^{-39}\ \mathrm{cm}^2$ for weak interactions

Typical event rates in these processes, assuming $\mathfrak{L} \sim 10^{34}\ \mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ are, therefore, of the order $10^7\ \mathrm{s}^{-1}$, $10\ \mathrm{s}^{-1}$, $10^{-5}\ \mathrm{s}^{-1} \simeq 26\ \mathrm{month}^{-1}$ respectively. These numbers clearly illustrate the difficulty to measure weak interaction effects in colliding beam experiments.

In order to understand the physical meaning of the cross-section and of the luminosity, let us consider for instance the case of a fixed target experiment in the target rest frame. We have a target of volume V_{target} placed in a particle beam. The particle densities ϱ_1 and ϱ_2 are supposed to be approximately homogeneous and the number of target particles is $N_{\mathrm{target}} = \varrho_2\, V_{\mathrm{target}}$, where ϱ_2 is the particle density of the target. Then we have

$$\int \mathrm{d}^4x\, \varrho_1(x)\, \varrho_2(x) = \int_{-\infty}^{\infty} \mathrm{d}t\, \varrho_1\, \varrho_2\, V_{\mathrm{target}} = \int_{-\infty}^{\infty} \mathrm{d}t\, \varrho_1\, N_{\mathrm{target}} \tag{3.74}$$

Let me now rewrite the kinematic factor $(\kappa_1 \cdot \kappa_2)^2 - m_1^2 m_2^2$. To this purpose we first evaluate the related quantity $\omega_1\, \boldsymbol{\kappa}_2 - \omega_2\, \boldsymbol{\kappa}_1$ with

$$\kappa_1^0 = E_1 = \sqrt{\boldsymbol{\kappa}_1^2 + m_1^2} = \omega(\boldsymbol{\kappa}_1) = \omega_1 \qquad \kappa_2^0 = E_2 = \sqrt{\boldsymbol{\kappa}_2^2 + m_2^2} = \omega(\boldsymbol{\kappa}_2) = \omega_2$$

---

[6] The standard unit of measure for the cross-sections is 1 barn=10^{-24} cm^2 and a typical paradigmatic quantity is the Thomson cross-section $\sigma_T = 8\pi r_e^2/3 = 0.665\,245\,873(13)$ barn. Moreover 1 unit of R at 1 GeV of center of mass energy is 0.3 nanobarns.

that yields

$$
\begin{aligned}
(E_1\,\boldsymbol{\kappa}_2 - E_2\,\boldsymbol{\kappa}_1)^2 &= E_2^2\boldsymbol{\kappa}_1^2 + E_1^2\boldsymbol{\kappa}_2^2 - 2E_1E_2\,\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_2 \\
&= E_2^2\boldsymbol{\kappa}_1^2 + E_1^2\boldsymbol{\kappa}_2^2 + (\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_2 - E_1E_2)^2 - (\,\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_2\,)^2 - E_1^2E_2^2 \\
&= (\kappa_1\cdot\kappa_2)^2 - (\,\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_2\,)^2 \\
&+ (\boldsymbol{\kappa}_2^2 + m_2^2)\boldsymbol{\kappa}_1^2 + (\boldsymbol{\kappa}_1^2 + m_1^2)\boldsymbol{\kappa}_2^2 - (\boldsymbol{\kappa}_1^2 + m_1^2)(\boldsymbol{\kappa}_2^2 + m_2^2) \\
&= (\kappa_1\cdot\kappa_2)^2 - (\,\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_2\,)^2 + \boldsymbol{\kappa}_1^2\,\boldsymbol{\kappa}_2^2 - m_1^2\,m_2^2 \qquad (3.75)
\end{aligned}
$$

in such a manner that we can eventually write

$$(\kappa_1\cdot\kappa_2)^2 - m_1^2m_2^2 = |\,\omega_1\boldsymbol{\kappa}_2 - \omega_2\boldsymbol{\kappa}_1\,|^2 + (\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_2)^2 - \boldsymbol{\kappa}_1^2\boldsymbol{\kappa}_2^2 \qquad (3.76)$$

The last two terms in the right hand side of the above relation can be dropped in the case of parallel momenta, which is the case in the fixed target rest frame or in the colliding storage ring in the center of mass rest frame, so that we can write

$$\sqrt{(\kappa_1\cdot\kappa_2)^2 - m_1^2m_2^2} = E_1E_2\,\left|\,\frac{\boldsymbol{\kappa}_1}{\omega_1} - \frac{\boldsymbol{\kappa}_2}{\omega_2}\,\right| = \omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)\,v_{\rm rel} \qquad (3.77)$$

where the relative velocity in physical units is provided by

$$\mathbf{v}_{\rm rel} = \frac{c^2\boldsymbol{\kappa}_1}{\omega_1(\boldsymbol{\kappa}_1)} - \frac{c^2\boldsymbol{\kappa}_2}{\omega_2(\boldsymbol{\kappa}_2)}$$

in such a manner that in the non-relativistic limit we recover the familiar formula

$$\mathbf{v}_{\rm rel} \approx \frac{\mathbf{p_1}}{m_1} - \frac{\mathbf{p_2}}{m_2} \qquad \mathbf{p}_\jmath = \hbar\boldsymbol{\kappa}_\jmath \quad \text{for} \quad |\,\boldsymbol{\kappa}_\jmath\,| \ll \frac{m_\jmath c}{\hbar} \qquad (\,\jmath = 1,2\,)$$

Combining eq.s (3.72),(3.74),(3.77) we eventually find that the luminosity is given by

$$\mathfrak{L} = \varrho_{\,\rm beam}\,v_{\rm rel}\,N_{\rm target} = \jmath_{\,\rm beam}\,N_{\rm target} \qquad (3.78)$$

where the beam current density $\jmath_{\,\rm beam}$ is obviously defined as the number of incoming particle passing through a unit area orthogonal the relative velocity vector *per* unit of time. The luminosity in a fixed target experiment is much higher than for colliding beams in a storage ring machine. Typical beam current densities, also called flux factors, are

- 10^{10} cm^{-2} s^{-1} for hadron beams
- 10^{8} cm^{-2} s^{-1} for electron beams
- 10^{6} cm^{-2} s^{-1} for neutrino beams

whereas a target contains $10^{26} \div 10^{35}$ protons. This explains why the huge number of protons in a target leads to event rates R much higher than those ones in colliding beam machines. Here, if two bunches containing N_1 and N_2 particles collide with frequency f, so that $v_{\text{rel}} = 2v_{\text{beam}} = 2f\, R_{\text{ring}}$ where R_{ring} is the storage ring mean radius, then the luminosity is roughly given by

$$\mathfrak{L} \;\approx\; v_{\text{rel}}\,\frac{N_1 N_2}{V_{\text{beam}}} \;=\; f\,\frac{N_1 N_2}{4\pi\sigma_h\sigma_v} \tag{3.79}$$

where σ_h and σ_v actually characterize the Gaussian transverse beam profile in the horizontal and vertical directions, in such a manner that the torus volume of the two bunches is approximately $V_{\text{beam}} \approx 4\sigma_h\sigma_v \cdot 2\pi R_{\text{ring}}$. More precisely, the luminosity of the LHC machine is defined by

$$\mathfrak{L} = f\sum_{\imath=1}^{k_b} \frac{N_{1\imath} N_{2\imath}}{4\pi\sigma_h\sigma_v} \tag{3.80}$$

where $f = 11.2$ kHz is the revolution frequency of the single bunch, $N_{n\imath} = 10^{11}$ p/bunch the number protons of the $n-$th beam, $k_b = 2808 \div 3564$ the number of bunches each $\Delta t = 25$ ns, while $\sigma_h = 375.2\ \mu$m and $\sigma_v = 16.7\ \mu$m.

3.3.3 Quasi-Elastic Scattering

To examine the kinematics further on, let me now consider a *quasi*-elastic scattering $1 + 2 \;\longmapsto\; 1' + 2'$ of two incident massive scalar particles with masses m_1, m_2 in two final massive scalar particles with masses m_1', m_2' and suppose that all masses are different. The space-time translations invariance entails

$$p_1 + p_2 = p_1' + p_2'$$

Then, in the center of momentum frame $\mathbf{p}_1 + \mathbf{p}_2 = 0$, we have that the total energy square is given by

$$s \equiv (p_1 + p_2)^2 = (p_1^0 + p_2^0)^2 = [\,\omega_1(\mathbf{p}_1) + \omega_2(-\mathbf{p}_1)\,]^2 \tag{3.81}$$

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = \tfrac{1}{4}\,[\,s - (m_1 - m_2)^2\,]\,[\,s - (m_1 + m_2)^2\,] \tag{3.82}$$

with $[s] = \text{eV}^2$. Notice that the last quantity vanishes, as it must, at the reaction threshold $s = (m_1 + m_2)^2$. Since the total 4-momentum is conserved in the scattering process we evidently obtain

$$s = (p_1 + p_2)^2 = (p_1' + p_2')^2 \tag{3.83}$$

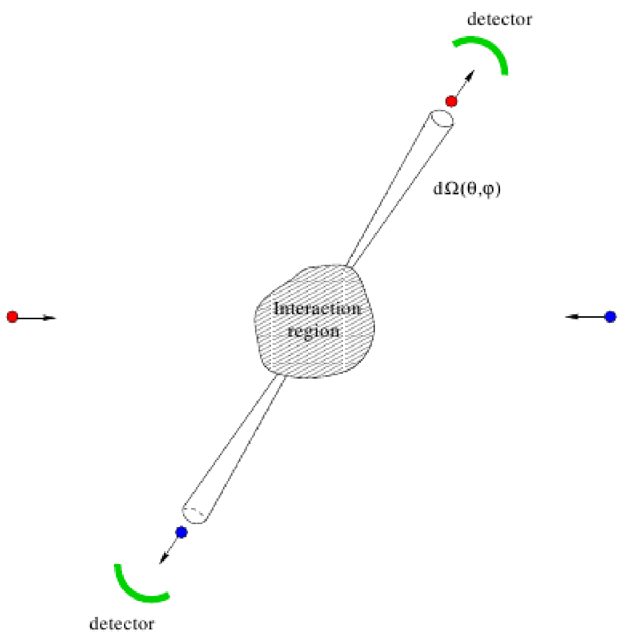

Figure 3.4: a schematic setup for a quasi-elastic scattering of two particles in the center of momentum frame

In addition to the square of the total energy in the center of mass frame, it is convenient to define the invariant 4-momentum transfer squared variable

$$t \equiv (p_1' - p_1)^2 = (p_2' - p_2)^2 \tag{3.84}$$

together with the invariant 4-momentum exchange squared variable

$$u \equiv (p_2' - p_1)^2 = (p_1' - p_2)^2 \tag{3.85}$$

A little algebra shows that

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2 \tag{3.86}$$

The kinematic relativistic invariant quantities s, t, u are the Mandelstam's variables [7]. For the special case of two particles in the final state, we can nicely simplify the general expression of eq. (3.71), by partially evaluating the so called phase space integrals in the center of momentum frame in which

$$\begin{aligned} &\mathbf{p}_1 + \mathbf{p}_2 = 0 = \mathbf{p}_1' + \mathbf{p}_2' \\ &\mathbf{p} = \mathbf{p}_1 = -\,\mathbf{p}_2\,, \quad \mathbf{p}' = \mathbf{p}_1' = -\,\mathbf{p}_2' \\ &E_1 + E_2 = \sqrt{s} = E_1' + E_2' \end{aligned}$$

so that the square energy factor becomes

$$\begin{aligned} 4I_{\mathrm{CM}} &\equiv 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2\, m_2^2} \\ &= 4\hbar c\,|\,E_1 \mathbf{p}_2 - E_2 \mathbf{p}_1\,| \;=\; 4\hbar c\,|\,\mathbf{p}\,|\,\sqrt{s} \end{aligned} \tag{3.87}$$

where use has been made of the previously derived relationship (3.76). On the other side one can rewrite the so called *phase space of the final state* in the center of momentum frame as follows: namely,

$$\begin{aligned} &\prod_{k=1}^{2} \int \frac{\mathrm{d}\mathbf{p}_k'\, \tilde{g}_k^*(\mathbf{p}_k')}{(2\pi)^3\, 2\omega(\mathbf{p}_k')}\, (2\pi)^4\, \delta\,(p_1' + p_2' - p_1 - p_2) \\ &= \int \frac{\mathrm{d}p'\, p'^2\, \mathrm{d}\Omega_{\mathrm{CM}}}{16\pi^2\, E_1'\, E_2'}\, \tilde{g}_1^*(\mathbf{p}')\, \tilde{g}_2^*(-\mathbf{p}') \\ &\times\; \delta\left(\sqrt{s} - E_1'(p') - E_2'(p')\right) \end{aligned} \tag{3.88}$$

where $\mathrm{d}\,\Omega_{\mathrm{CM}} = \mathrm{d}\,\varphi\, \sin\theta\, \mathrm{d}\,\theta$ is the solid angle element of the momentum $\mathbf{p}'$ in the center of momentum frame of the two outgoing particles, while $\tilde{g}_k^*(\mathbf{p}_k')$

[7] Stanley Mandelstam, *Determination of the Pion-Nucleon Scattering Amplitude from Dispersion Relations and Unitarity. General Theory*, Phys. Rev. **112** (1958) 1344

are their momentum space wave packets. Note that this integral vanishes unless $s > (\,m_1' + m_2'\,)^2$, *i.e.* the incoming energy in the collision must be actually enough to produce two physical particles at rest with masses $m_{1,2}'$.

In order to calculate the value of $p'(s)$ for which the argument of the $\delta-$distribution vanishes, we have to obtain the inversion formulas

$$\begin{aligned} s &= \left(\sqrt{\mathbf{p}'^2 + m_1'^2} + \sqrt{\mathbf{p}'^2 + m_2'^2}\,\right)^2 \\ &= 2\mathbf{p}'^2 + m_1'^2 + m_2'^2 + 2\,E_1'(\mathbf{p}')\,E_2'(\mathbf{p}') \end{aligned} \tag{3.89}$$

that is

$$\left(s - 2\mathbf{p}'^2 - m_1'^2 - m_2'^2\right)^2 = 4\left(\mathbf{p}'^2 + m_1'^2\right)\left(\mathbf{p}'^2 + m_2'^2\right) \tag{3.90}$$

which finally yields

$$\begin{aligned} |\,\mathbf{p}'\,|^2 &= \frac{1}{4s}\left[\,s^2 + (m_1'^2 - m_2'^2)^2 - 2s\,(m_1'^2 + m_2'^2)\,\right] \\ &\stackrel{\text{def}}{=} \left[\frac{1}{2\sqrt{s}}\,F(s, m_1', m_2')\right]^2 = [\,p'(s)\,]^2 \end{aligned} \tag{3.91}$$

$$\begin{aligned} |\,\mathbf{p}\,|^2 &= \frac{1}{4s}\left[\,s^2 + (m_1^2 - m_2^2)^2 - 2s\,(m_1^2 + m_2^2)\,\right] \\ &\stackrel{\text{def}}{=} \left[\frac{1}{2\sqrt{s}}\,F(s, m_1, m_2)\right]^2 = [\,p(s)\,]^2 \end{aligned} \tag{3.92}$$

In the case of equal masses, *i.e.* identical incoming or outgoing particles, the above inversion formulæ reduce to

$$|\,\mathbf{p}\,| = \tfrac{1}{2}\,\theta(\sqrt{s} - 2m)\,\sqrt{s - 4m^2} \tag{3.93}$$

$$|\,\mathbf{p}'\,| = \tfrac{1}{2}\,\theta(\sqrt{s} - 2m')\,\sqrt{s - 4m'^2} \tag{3.94}$$

where the so called reaction thresholds have been emphasized by the presence of the Heaviside step distributions. Now we have

$$\int_{-\infty}^{\infty} \mathrm{d}p'\;\theta(p')\,\varphi(p')\,\delta\left(f(p')\right) = \frac{\varphi(p_*)}{|\,f'(p_*)\,|} \qquad f\,(p_*) = 0 \tag{3.95}$$

and applying it to the integral (3.88) in the limit $\tilde{g}_1\,,\,\tilde{g}_2 \longrightarrow 1$ we get

$$\begin{aligned} &\frac{1}{16\pi^2}\int \frac{p'^2(s)\;\mathrm{d}\Omega_{\mathrm{CM}}}{\sqrt{\{\,[\,p'^2(s) + m_1'^2\,]\,[\,p'^2(s) + m_2'^2\,]\,\}}} \\ \times &\left(\frac{p'(s)}{\sqrt{[\,p'^2(s) + m_1'^2\,]}} + \frac{p'(s)}{\sqrt{[\,p'^2(s) + m_2'^2\,]}}\right)^{-1} \\ = &\frac{|\,\mathbf{p}'\,|}{16\pi^2\sqrt{s}}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}(-\,\cos\theta) \end{aligned} \tag{3.96}$$

where $|\,\mathbf{p}'\,| = p'(s)$ can be expressed as a function of the Mandelstam variable s by the inversion formula (3.91). Another equivalent *memento formula* for the two final particles phase space volume is given by

$$\int \frac{\mathrm{d}\,\mathbf{p}_1'}{(2\pi)^3\,2\omega(\mathbf{p}_1')} \int \frac{\mathrm{d}\,\mathbf{p}_2'}{(2\pi)^3\,2\omega(\mathbf{p}_2')}\,(2\pi)^4\,\delta\,(p_1'+p_2'-p_1-p_2) \;=$$
$$F(s,m_1'^{\,2},m_2'^{\,2})\int \frac{\mathrm{d}\,\Omega_{\,\mathrm{CM}}}{32\pi^2\,s} = \frac{|\,\mathbf{p}'(s)\,|}{16\pi^2\sqrt{s}}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\,(-\,\cos\theta) \qquad (3.97)$$

with

$$|\,\mathbf{p}'(s)\,| \;=\; \frac{1}{2\sqrt{s}}\,F(s,m_1',m_2') \qquad\qquad s=(p_1+p_2)^2=(p_1'+p_2')^2$$

Note that the 2-particle phase space volume is a Poincaré invariant and dimensionless quantity. Hence we eventually obtain the main formula for the differential cross-section for the quasi-elastic collision in the center of momentum frame: namely,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \left(\frac{\hbar c}{8\pi E_{\mathrm{CM}}}\right)^2\;|\,\mathcal{M}\,(\,s,t,u\,)\,|^2\,\frac{|\,\mathbf{p}'\,|}{|\,\mathbf{p}\,|} \qquad (3.98)$$
$$E_{\mathrm{CM}}=\sqrt{s}=c\,(p_1^0+p_2^0)=c\,(p_1^{0\prime}+p_2^{0\prime}) \qquad\qquad \mathbf{p}_1+\mathbf{p}_2=0=\mathbf{p}_1'+\mathbf{p}_2'$$

and taking also into account that the incoming current density or flux factor can be expressed by (3.87)

$$4I_{\mathrm{CM}}=4|\,\mathbf{p}\,|\,\sqrt{s} \;=\; 2F(s,m_1,m_2) \qquad (3.99)$$

the basic formula (3.71) for the two-particles quasi-elastic scattering in the center of momentum frame takes the form

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \;=\; \frac{1}{4I_{\mathrm{CM}}}\,|\,\mathcal{M}\,(s,t,u)\,|^2\,\frac{1}{32\pi^2\,s}\;F\left(s,m_1'^{\,2},m_2'^{\,2}\right)$$
$$=\; \frac{1}{64\pi^2 s}\,|\,\mathcal{M}\,(s,t,u)\,|^2\,\frac{F\,(\,s,m_1'^{\,2},m_2'^{\,2}\,)}{F(s,m_1^2,m_2^2)} \qquad (3.100)$$

which coincides with eq. (3.98). In the special limits of equal masses and elastic scattering we eventually come to the very suggestive formulæ

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \frac{|\,\mathcal{M}\,|^2}{64\pi^2 s} \qquad\qquad \sigma=\frac{|\,\mathcal{M}\,|^2}{16\pi s} \qquad (3.101)$$

It is useful to remark that in the case of equal masses the characteristic function $F(s,m_1^2,m_2^2)$ becomes

$$F(s,m^2)=\sqrt{s(s-4m^2)}\geq 0 \quad\Leftrightarrow\quad s\geq 4m^2 \qquad (3.102)$$

which is positive as it does only above the threshold, *i.e.* two equal massive particles at rest. By the way, in the case $m_1 = m_2 = m_e\,,\ m_1' = m_2' = m_\mu$ we have

$$\left[\, F\left(s, m_\mu^2\right) / F\left(s, m_e^2\right) \right] \ = \ \sqrt{\frac{s - 4m_\mu^2}{s - 4m_e^2}} \tag{3.103}$$

When $s \gtrsim 400\ \mathrm{GeV}^2$ and taking into account that $(m_e/m_\mu)^2 \ \approx\ 23 \times 10^{-6}$ we can safely approximate

$$\frac{F\left(s, m_\mu^2\right)}{F\left(s, m_e^2\right)} \simeq \sqrt{1 - \left(\frac{2m_\mu}{E_{\mathrm{CM}}}\right)^2} = \sqrt{1 - \left(\frac{m_\mu}{E}\right)^2} = \left(\frac{|\,\mathbf{p}\,|}{E}\right)_\mu \equiv \beta_\mu \tag{3.104}$$

where

$$E_{\mathrm{CM}} = 2\,\sqrt{\mathbf{p}^2 + m_\mu^2}$$

Another very important situation occurs for the collision of a fast and light particle with mass m off of a much more heavy particle of mass $M \gg m$ at rest in the laboratory frame. Then, if we disregard the recoil of the very heavy particle after collision, the kinematics reads

$$\mathbf{p}_1 = \mathbf{p} \qquad E_1 = \sqrt{\mathbf{p}^2 + m^2} = E_p \tag{3.105}$$

$$\mathbf{p}_2 = 0 \qquad E_2 = M \tag{3.106}$$

$$\mathbf{p}_2' \ = \ \mathbf{p}_1 - \mathbf{p}_1' \ \approx\ 0 \qquad E_2' \ \approx\ M \tag{3.107}$$

$$\mathbf{p}_1' \ = \ \mathbf{p}' \qquad E_1' \ = \ \sqrt{\mathbf{p}'^2 + m^2} \ = \ E_{p'}' \tag{3.108}$$

with $\ |\,\mathbf{p}\,| \equiv p \approx |\,\mathbf{p}'\,|$. Turning back to the main basic formula (3.71) for the differential cross-section, in the present case of a 2-particle final state with one very heavy particle we can use the above kinematics with $\mathbf{p}\!\cdot\!\mathbf{p}' \approx p^2 \cos\theta$, where θ is now the scattering angle of the light particle in the heavy particle rest frame, so that

$$\begin{aligned}
& 4\,I \equiv 4\,\sqrt{(p_1 \cdot p_2)^2 - m^2 M^2} = 4\,|\,\mathbf{p}\,|\,M \\
& \prod_{k=1}^{2} \int \frac{\mathrm{d}\mathbf{p}_k'\ \tilde{g}_k(\mathbf{p}_k')}{(2\pi)^3\, 2\omega(\mathbf{p}_k')}\ (2\pi)^4\, \delta\,(p_1' + p_2' - p_1 - p_2) \\
& \approx\ \tilde{g}_1(\mathbf{p}')\, \tilde{g}_2(\mathbf{p} - \mathbf{p}') \int \mathrm{d}\Omega\,(\phi,\theta) \int_0^\infty \frac{\mathrm{d}p\, p^2}{16\pi^2 M E\,(p)}\ \delta\left(E_{p'}' - E_p\right) \\
& \approx\ \tilde{g}_1(\mathbf{p}')\, \tilde{g}_2(0) \int \mathrm{d}\Omega\,(\phi,\theta) \int_0^\infty \frac{\mathrm{d}E\, p\,(E)}{16\pi^2 M}\ \delta\,(E' - E)
\end{aligned}$$

Thus, in the limit $\tilde{g}_1\,,\,\tilde{g}_2 \longrightarrow 1$ one eventually finds

$$\prod_{k=1}^{2}\int\frac{\mathrm{d}\,\mathbf{p}'_k}{(2\pi)^3\,2\omega(\,\mathbf{p}'_k\,)}\ (2\pi)^4\,\delta\,(p'_1+p'_2-p_1-p_2)$$
$$\approx\ \frac{|\,\mathbf{p}'\,|}{16\,\pi^2\,M}\int\mathrm{d}\,\Omega\,(\phi,\theta)\ \approx\ \frac{|\,\mathbf{p}\,|}{16\,\pi^2\,M}\int\mathrm{d}\,\Omega\,(\phi,\theta) \tag{3.109}$$

and according to the main formula (3.71) we end up with the remarkably simple expression

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{FT}}=\frac{|\,\mathcal{M}\,(s,t,u)\,|^{\,2}}{64\,\pi^{\,2}\,(Mc/\hbar)^2} \tag{3.110}$$

As a final important comment, I remark that the generalization of all the above formulas to the case of scattering of particles with spin is really straightforward. In such cases, in fact, the amplitudes have spinor and/or 4-vector indexes, which need thereby to be saturated with the corresponding suitable quantities describing the polarization states.

Specifically, for spin $\frac{1}{2}$ Dirac fermions $(\,r,s=1,2\,)$:

- $u_{\,r}(\,p\,)$ for an incoming particle
- $\bar{v}_{\,s}(\,q\,)$ for incoming antiparticles
- $\bar{u}_{\,r}(\,p\,)$ for outgoing particles
- $v_{\,s}(\,q\,)$ for an outgoing antiparticle

for spin 1 real vector bosons :

- $e^{r}_{\mu}(k)$ $(\,r=1,2,3\,)$ for incoming massive vector particles
- $\varepsilon^{A}_{\mu}(k)$ $(\,A=1,2,S\,)$ for incoming massless vector particles
- $[\,e^{s}_{\nu}(\,k'\,)\,]^{*}$ $(\,s=1,2,3\,)$ for outgoing massive vector particles
- $[\,\varepsilon^{A'}_{\mu}(\,k'\,)\,]^{*}$ $(\,A'=1,2,S\,)$ for outgoing massless vector particles

both in massive and mass-less cases, where the polarization vectors are real for linear polarization. Let me discuss quite a few enlightening examples.

3.4 Elementary Scattering Processes

Here below we will see some important collisions involving charged fermions and photons, to the lowest order in the fine structure constant

$$\alpha = \frac{e^2}{4\pi\hbar c} = 7.297\,352\,568(24)\times 10^{-3} \simeq \frac{1}{137}$$

These electromagnetic scattering processes play a key role in the development of Quantum Electrodynamics and of the Standard Model of Particle Physics.

3.4.1 $e^- e^+$ into $\mu^+\mu^-$ Pairs

The annihilation of an electron positron pair into a muon-antimuon pair is the simplest of all the quantum electrodynamics processes, but also one of the most important in high energy physics. It turns out to be fundamental for the understanding of all reactions which occur in e^+e^- colliders. As a matter of fact, it is used indeed to calibrate such a kind of machines. The related process of the electron positron pair annihilation into a quark anti-quark pair is extraordinarily useful and crucial to unravel elementary particle physics properties. Here below, the lowest order non-polarized cross-section will be obtained, up to the accuracy for the electron mass can be disregarded with respect to the muon mass – remember that $(m_e/m_\mu) \approx 0.5\%$. According to the Feynman rules and the LSZ reduction formulæ we obtain at once the lowest order $O(e^2)$ amplitude, see figure, *viz.*,

$$\begin{aligned}&(ie)^2\,\langle\,0\,|\,C_{q,s}\,D_{q',s'}\,\overline{\mu}_y^{(+)}\,\gamma^\nu\,\mu_y^{(+)}\,\overline{e}_x^{(-)}\,\gamma^\lambda\,e_x^{(-)}\,d^\dagger_{p',r'}\,c^\dagger_{p,r}\,|\,0\,\rangle\,\Delta^{xy}_{\lambda\nu} = \\ &-e^2\langle\,0\,|\,\underbrace{C_{q,s}\,\overline{\mu}_y^{(+)}}\,\gamma^\nu\;\underbrace{D_{q',s'}\,\mu_y^{(+)}}\;\underbrace{\overline{e}_x^{(-)}\,d^\dagger_{p',r'}}\,\gamma^\lambda\;\underbrace{e_x^{(-)}\,c^\dagger_{p,r}}\,|\,0\,\rangle\,\Delta^{xy}_{\lambda\nu}\end{aligned}$$

where $\mu(y)$ and $e(x)$ are the muon and electron spinor fields respectively, while the small and capital letters denote the e^-e^+ and $\mu^-\mu^+$ creation and destruction operators respectively; moreover $\Delta^{xy}_{\lambda\nu}$ does indicate the mass-less photon propagator in the Feynman gauge. Then we get

$$\begin{aligned}&\mathcal{M}_{rr'ss'}(p,p';q,q') \\ &= \;\bar{u}_s^{\mu^-}(q)\,\gamma^\rho\,v_{s'}^{\mu^+}(q')\,\frac{e^2}{k^2}\,\bar{v}_{r'}^{e^+}(p')\,\gamma_\rho\,u_r^{e^-}(p)\end{aligned} \tag{3.111}$$

which is dimensionless, where $p+p' = k = q+q'$ is the virtual (*i.e.* off shell) photon energy momentum such that

$$k^2 = (\,q+q'\,)^2 = s \geq 4m_\mu^2 > 0$$

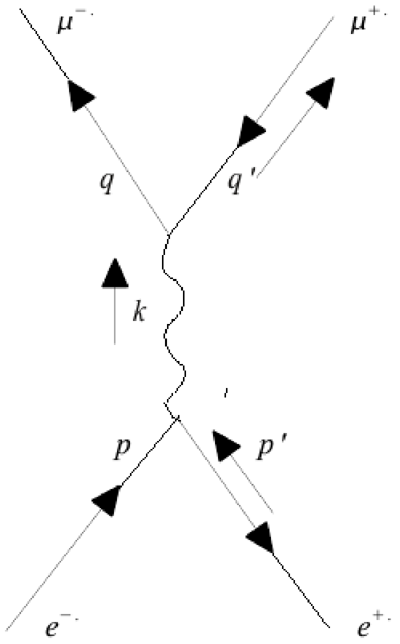

Figure 3.5: the lowest order e^+e^- annihilation into a $\mu^+\mu^-$ pair

the Mandelstam variable $s = (p_0 + p_0')^2 = 4(\mathbf{p}^2 + m_e^2)$ being the total energy in the center of momentum frame of *e.g.* the e^-e^+ pair. To compute the differential cross-section we need an expression for the square modulus of the above amplitude (3.111) : we find

$$\left(\bar{v}\,\gamma^\lambda\,u\right)^* = u^\dagger\,\gamma^{\lambda\dagger}\,\gamma^{0\dagger}\,v = u^\dagger\,\gamma^0\,\gamma^\lambda\,\left(\gamma^0\right)^2\,v = \bar{u}\,\gamma^\lambda\,v$$

that vindicates the great advantage of the adjoint spinor notation. Thus the squared matrix element becomes

$$\begin{aligned} |\,\mathcal{M}\,|^2 &= \frac{e^4}{s^2}\,g_{\mu\nu}\,g_{\rho\sigma} \\ &\times \Big(\bar{v}_{r'}(p')\,\gamma^\mu\,u_r(p)\,\bar{u}_r(p)\,\gamma^\rho\,v_{r'}(p')\Big) \\ &\times \Big(\bar{u}_s(q)\,\gamma^\nu\,v_{s'}(q')\,\bar{v}_{s'}(q')\,\gamma^\sigma\,u_s(q)\Big) \end{aligned} \tag{3.112}$$

In this expression any spin state of the involved four fermion Dirac particles is specified. However, in actual experiments it is very difficult or even not possible to keep polarization under control. For instance, one should prepare the initial state from accurately polarized materials and/or analyze the final state using *e.g.* spin dependent multiple scattering.

In most experiments the electron and positrons beams are non-polarized, in such a manner that the measured cross section is an average over the incoming electron and positron polarization r and r' respectively. On the other side, muon detectors are usually blind to polarization, so that the measured cross section is a sum over the negatively and positively charged muon spin indexes s and s' respectively.

In other words, I will be here mainly interested in the squared matrix element, which greatly simplifies when averaged over the initial electron and positron polarization and further summed over the final muon spins

$$\tfrac{1}{2}\sum_{r=1,2}\ \tfrac{1}{2}\sum_{r'=1,2}\ \sum_{s=1,2}\ \sum_{s'=1,2}\ |\,\mathcal{M}\,(r,r'\ \to\ s,s')\,|^2 \qquad (3.113)$$

By making use of the completeness relations

$$\sum_{r=1,2}\left\{\begin{array}{l} u_r(\mathbf{p})\otimes\bar{u}_r(\mathbf{p}) = \not{p} + m_e \\ v_r(\mathbf{p})\otimes\bar{v}_r(\mathbf{p}) = \not{p} - m_e \end{array}\right. \qquad \left(p_0 = \sqrt{\mathbf{p}^2 + m_e^2}\right)$$

$$\sum_{s=1,2}\left\{\begin{array}{l} u_s(\mathbf{q})\otimes\bar{u}_s(\mathbf{q}) = \not{q} + m_\mu \\ v_s(\mathbf{q})\otimes\bar{v}_s(\mathbf{q}) = \not{q} - m_\mu \end{array}\right. \qquad \left(q_0 = \sqrt{\mathbf{q}^2 + m_\mu^2}\right)$$

we readily arrive to

$$\begin{aligned} \tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}\ |\,\mathcal{M}\,|^2 \ &=\ \frac{e^4}{s^2}\, g_{\mu\nu}\, g_{\rho\sigma}\ \times \qquad (3.114) \\ \tfrac{1}{4}\,\mathrm{tr}\ [\,(\not{p}' - m_e)\,\gamma^\mu\,(\not{p} + m_e)\,\gamma^\rho\,]&\ \mathrm{tr}\ [\,(\not{q} + m_\mu)\,\gamma^\nu\,(\not{q}' - m_\mu)\,\gamma^\sigma\,] \end{aligned}$$

The general method of calculating traces consists of successive displacements of identical matrix four vector. In particular, the trace of an odd number of gamma matrices does vanish, while we easily find

$$\mathrm{tr}\ (\gamma^\mu\gamma^\nu)\ =\ g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I}\ =\ 4\,g^{\mu\nu} \qquad (3.115)$$

$$\mathrm{tr}\ \left(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\right)\ =\ 4\left(g^{\kappa\lambda}\,g^{\mu\nu}\ -\ g^{\kappa\mu}\,g^{\lambda\nu}\ +\ g^{\kappa\nu}\,g^{\lambda\mu}\right) \qquad (3.116)$$

Hence the e^+e^- trace is

$$4\,\left[\,p'^{\mu}p^{\rho} + p'^{\rho}p^{\mu} - g^{\mu\rho}\,\left(p\cdot p' + m_e^2\right)\,\right]$$

and similarly the muon pair trace yields

$$4\,\left[\,q'^{\sigma}q^{\nu} + q'^{\nu}q^{\sigma} - g^{\nu\sigma}\,\left(q\cdot q' + m_\mu^2\right)\,\right]$$

After contractions of the Lorentz indexes we come to the simple expression

$$\begin{aligned}\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \;&=\; \frac{8e^4}{s^2}\;\times\\ \Big[\,(p\cdot q)(p'\cdot q') + (p\cdot q')(p'\cdot q) &+ (p\cdot p')m_\mu^2\\ +\;(q\cdot q')m_e^2 + 2m_\mu^2 m_e^2\Big] \;&\equiv\; \big\langle\,|\,\mathcal{M}(s,t,u)\,|^2\,\big\rangle \qquad (3.117)\end{aligned}$$

Since we have $\mathbf{p}^2 + m_e^2 = \mathbf{q}^2 + m_\mu^2$ il follows that $p\cdot p' \gg q\cdot q' \gg m_e^2$ and thereby the last two addenda in the RHS of the above equality are absolutely negligible in respect to the former ones for high energy collisions. Hence, by neglecting the electron mass, in the center of momentum frame of the e^+e^- and $\mu^+\mu^-$ pairs we have for $m_e \approx 0$

$$\begin{aligned}&\text{electron : } \quad \mathbf{p} \qquad p_0 \;=\; \sqrt{\mathbf{p}^2 + m_e^2}\;\approx |\,\mathbf{p}\,|\\ &\text{positron : } \quad \mathbf{p}' = -\,\mathbf{p} \qquad p_0' = p_0 \approx |\,\mathbf{p}\,|\\ &\text{muon : } \quad \mathbf{q}\,, \quad q_0 = \sqrt{\mathbf{q}^2 + m_\mu^2} \equiv E\\ &\text{antimuon : } \quad \mathbf{q}' = -\,\mathbf{q} \quad q_0' = q_0 = p_0 = p_0' = E = \tfrac{1}{2}\sqrt{s}\end{aligned}$$

The Mandelstam's variables are

$$\begin{aligned}s &= (p+p')^2 = (q+q')^2 = 4q_0^2 = 2m_\mu^2 + 2q\cdot q' \approx 2p\cdot p'\\ t &= (p-q)^2 = (p'-q')^2 \approx m_\mu^2 - 2p\cdot q = m_\mu^2 - 2p'\cdot q'\\ u &= (p-q')^2 = (q-p')^2 \approx m_\mu^2 - 2p\cdot q' = m_\mu^2 - 2p'\cdot q\end{aligned}$$

with $s+t+u \approx 2m_\mu^2$. Then formulæ (3.100) and (3.104) yield

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \;=\; \frac{1}{(8\pi\sqrt{s}\,)^2}\,\big\langle\,|\,\mathcal{M}(s,t,u)\,|^2\,\big\rangle\cdot\frac{|\,\mathbf{q}\,|}{|\,\mathbf{p}\,|} \qquad (3.118)$$

Since we have

$$\begin{aligned}k^2 = s &= 4E^2\\ p\cdot p' \;=\; E^2 + |\,\mathbf{p}\,|^2 &\;\approx\; 2E^2\\ p\cdot q = p'\cdot q' &= E\,(E - |\,\mathbf{q}\,|\cos\theta)\\ p\cdot q' = p'\cdot q &= E\,(E + |\,\mathbf{q}\,|\cos\theta)\end{aligned}$$

where θ is the angle between the directions of the incident e^+e^- pair and the produced $\mu^+\mu^-$ pair in the center of momentum frame, we eventually obtain

$$\big\langle\,|\,\mathcal{M}(s,t,u)\,|^2\,\big\rangle \;=\; \frac{2e^4}{s}$$

$$
\begin{aligned}
&\times \left[(E - |\mathbf{q}| \cos\theta)^2 + (E + |\mathbf{q}| \cos\theta)^2 + 2\, m_\mu^2 \right] + O(m_e/m_\mu)^2 \\
&\approx \frac{4\,e^4}{s} \left(E^2 + E^2 \cos^2\theta - m_\mu^2 \cos^2\theta + m_\mu^2 \right)
\end{aligned}
$$

that leads to the differential cross-section

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{\mathrm{CM}} &= \frac{\alpha^2}{4s} \sqrt{1 - \frac{m_\mu^2}{E^2}} \\
&\times \left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) \cos^2\theta \right]
\end{aligned} \tag{3.119}
$$

and integrating over the solid angle we get the total cross-section

$$
\begin{aligned}
\sigma &= \frac{4\pi\,\alpha^2}{3s} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{m_\mu^2}{2\,E^2} \right) \\
&= \frac{4\pi\,\alpha^2}{3\,s} \sqrt{1 - \frac{4m_\mu^2}{s}} \left(1 + \frac{2m_\mu^2}{s} \right)
\end{aligned} \tag{3.120}
$$

The differential cross-section can also be written in the equivalent form [1]

$$
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{\mathrm{CM}} = \frac{\alpha^2}{4\,s}\, \beta\, \left[1 + \cos^2\theta + (1 - \beta^2) \sin^2\theta \right]
$$

where $\beta = v/c = |\mathbf{q}|\,/\,q_0$ is the muon velocity in the center of mass frame which, in a colliding machine, is the laboratory frame too. In the high energy limit $(\,q_0 \gg m_\mu\,)$ these formulæ reduces to

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{\mathrm{CM}} &\sim \frac{\alpha^2}{4\,s}\, (1 + \cos^2\theta) \qquad\qquad (\,s \gg 4m_\mu^2\,) \\
\sigma &\sim \frac{4\pi}{3} \left(\frac{\alpha}{\sqrt{s}} \right)^2 \left\{ 1 - \frac{3}{8} \left(\frac{2m_\mu}{\sqrt{s}} \right)^4 - \cdots \right\} \qquad (\,E_{\mathrm{CM}} \gg 4m_\mu^2\,)
\end{aligned}
$$

In the high energy regime the only dimensional quantity in the process is the energy scale $\sqrt{s} = E_{\mathrm{CM}}$, in such a manner that dimensional analysis requires $\sigma \propto E_{\mathrm{CM}}^{-2}$ and since we knew that $\sigma \propto \alpha^2$ the only remaining geometric factor to calculate is $4\pi/3$, the volume of the unit 2-sphere.

The remarkable energy dependence of the cross-section $\sigma\,(\,e^+e^- \to \mu^+\mu^-\,)$ sets the scale for all e^+e^- annihilation processes through a virtual photon and consequent spin $\frac{1}{2}$ point-like fermion-antifermion pairs production

$$
e^+e^- \;\to\; \gamma^* \;\to\; \bar{f}f
$$

At the center of mass square energy $E_{\rm CM} = \sqrt{s}$ it is given by

$$\begin{aligned}\sigma\left(e^+e^- \to \bar{f}f\right) &\stackrel{\beta\to 1}{\sim} N_c\,Q_f^2\,\frac{4\pi\alpha^2}{3s}\,(\hbar c)^2 \\ &= N_c\,Q_f^2\,\frac{86.8\ \text{nb}}{(\,E_{\rm CM}\ \text{in GeV}\,)^2} \\ &= N_c\,Q_f^2\cdot 1\ \text{unit of R}(s)\end{aligned} \tag{3.121}$$

where $e\,Q_f$ is the fermion charge while N_c is one for leptons and three for quarks, because each quark in the Standard Model appears in three colors. Experimentally, the easiest quantity to measure turns out to be the total rate for the production of all hadrons, the strongly interacting particles. The present understanding of strong interactions is provided by a field theory model named Quantum Chromo–Dynamics (QCD) that corresponds to a non-Abelian generalization of Quantum ElectroDynamics (QED) according to which all hadrons are composed of elementary Dirac fermions called quarks

Melinda Y. Han & Yoichiro Nambu
Three-Triplet Model with Double SU(3) Symmetry
The Physical Review **139**, B1006 - B1010 (1965) [Issue 4B – August 1965]
Harald Fritsch, Murray Gell-Mann & Heinrich Leutwyler
Advantages of the color octet gluon picture
Physics Letters **47B** (1973) 365

Quarks appear in a variety of six types, called flavors, with its own mass and fractional electric charge [1]

Quark	Mass	Charge
u	2.2 MeV	$\frac{2}{3}\,e$
d	4.7 MeV	$-\frac{1}{3}\,e$
s	96 MeV	$-\frac{1}{3}\,e$
c	1.28 GeV	$\frac{2}{3}\,e$
b	4.18 GeV	$-\frac{1}{3}\,e$
t	173.1 GeV	$\frac{2}{3}\,e$

Quarks also carry an additional internal quantum label, named color, taking three possible hues: conventionally, red, green and blue. Eventually, colour is the charge of the strong interaction. In Quantum Chromo-Dynamics (QCD) the simplest fundamental process which occurs inside all hadrons is

$$e^+e^- \;\to\; \gamma^* \;\to\; q\,\bar{q}$$

namely, the e^+e^- annihilation processes through a virtual photon with the consequent production of a quark anti-quark pair. Once they are created, the strong interaction among quark anti-quark pairs is such that the latter ones combine to form colourless mesons and baryons.

The astonishing feature predicted by QCD is that in the high energy limit the effects of the strong interaction on the quark production processes can be completely neglected: this amazing property is called the asymptotic freedom

Hugh David Politzer
Reliable Perturbative Results for Strong Interactions?
The Physical Review Letters **30** (1973) 1346–1349 [Issue 26 – June 1973]
David Jonathan Gross & Frank Anthony Wilczek
Asymptotically Free Gauge Theories
The Physical Review D **8**, 3633 - 3652 (1973) [Issue 10 – November 1973]

It is truly quite remarkable that the non-Abelian gauge field theories, based upon special unitary groups, turn out to be the only consistent local and renormalizable models which exhibit the property of the asymptotic freedom in four space-time dimensions. Asymptotically we expect

$$\frac{\sigma\,(e^+e^- \;\to\; \text{hadrons})}{\sigma\,(\,e^+e^- \to \mu^+\mu^-\,)} \;\stackrel{\beta\to 1}{\sim}\; 3\,\cdot\,\Big(\sum_{\text{flavors}}\;Q_f^2\Big) \tag{3.122}$$

where the sum runs over all quarks, the masses of which are smaller than $E_{\text{CM}}\,/\,2$. When the value of $E_{\text{CM}}\,/\,2$ is very close to one of the quark masses, then strong interaction cause large deviations from (3.122), the most striking effect being the appearance of bound states just below $E_{\text{CM}} = 2\,m_q$, endorsed by sharp spikes in the cross-section – see Fig. 3.7 and [1] for an up-to-date review Actually, experimental measurements between 2.5 and 45 GeV agree quite well with the naïve prediction (3.122) and, in particular, the factor 3 is a strong evidence for the existence of color

$$\sigma\,(\,e^+e^- \;\to\; \text{hadrons}\,) \;\stackrel{\beta\to 1}{\sim}\; 3\,\cdot\,\Big(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}+\frac{4}{9}+\frac{1}{9}\Big)\;\mathrm{R} = \frac{11}{3}\;\mathrm{R}$$

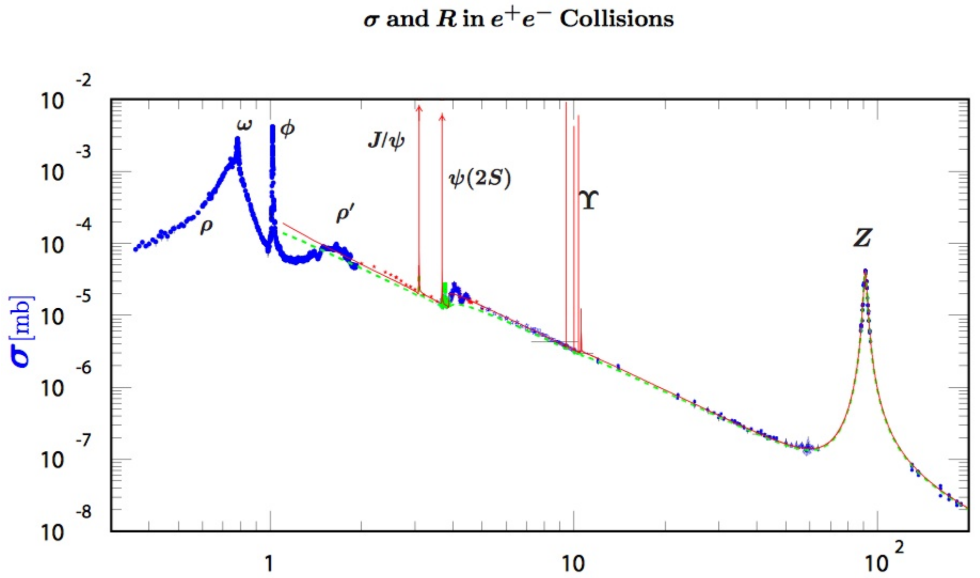

Figure 3.6: the total cross-section $e^+e^- \longrightarrow hadrons$ vs. $\sqrt{s}$ GeV

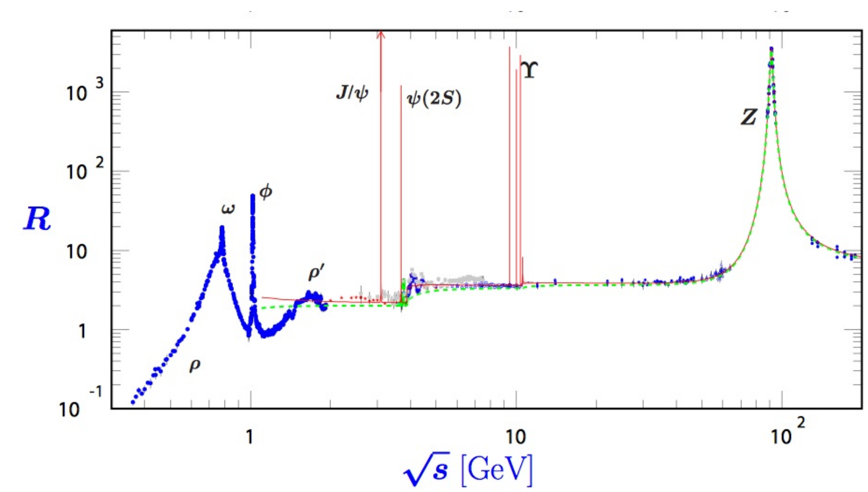

Figure 3.7: the ratio $\sigma(e^+e^- \longrightarrow hadrons, s)/\sigma(e^+e^- \longrightarrow \mu^+\mu^-, s)$

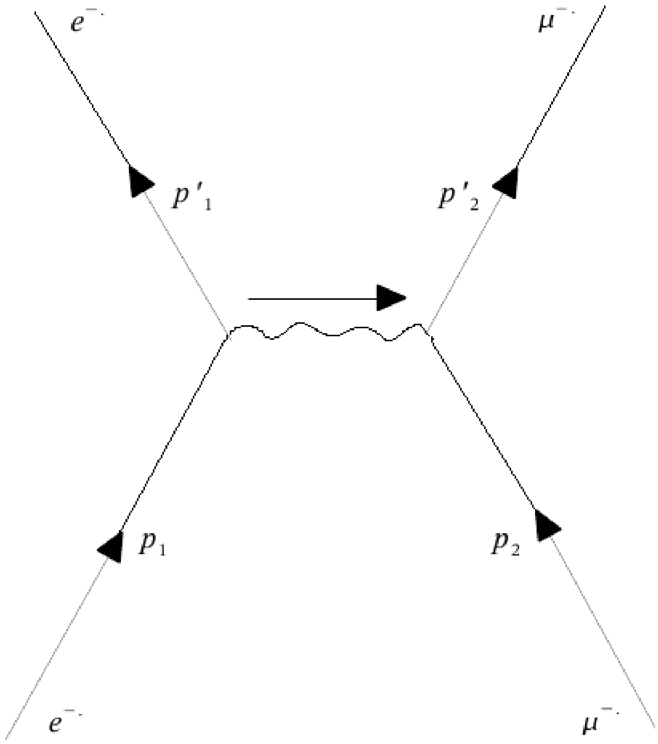

Figure 3.8: the lowest order amplitude for $e^{-}\mu^{-}$ collision

3.4.2 Mott Scattering

Let me now consider another closely related electromagnetic process, *i.e.* the electron versus muon scattering

$$e^{-}\mu^{-} \;\rightarrow\; e^{-}\mu^{-}$$

Again, the Feynman rules give at once the lowest order $O\,(e^2\,)$ amplitude, see figure: namely,

$$\begin{aligned} & \mathcal{M}_{\,r\,r'ss'}\,(p_1,p_1';p_2,p_2') \\ = \;\; & \bar{u}_{\,s'}\,(p_2')\,\gamma_\mu\,u_{\,s}\,(p_2)\;\frac{e^2}{(p_1'-p_1\,)^{\,2}}\;\bar{u}_{\,r'}\,(p_1')\,\gamma^\mu\,u_{\,r}\,(p_1) \end{aligned} \qquad (3.123)$$

Taking the square modulus as well as the average over the incoming particle spin and the sum over the final particle spin we find

$$\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}\;|\,\mathcal{M}_{\,r\,r'ss'}\,(p_1,p_1';p_2,p_2')\,|^{\,2} = \left(\frac{e^2}{t}\right)^2 g_{\,\mu\nu}\,g_{\,\rho\sigma}\;\times \qquad (3.124)$$

$$\tfrac{1}{4}\,\mathrm{tr}\;\left[\,(\,\not p_1' + m_e\,)\,\gamma^\mu\,(\,\not p_1 + m_e\,)\,\gamma^\rho\,\right]\;\mathrm{tr}\;\left[\,(\,\not p_2' + m_\mu\,)\,\gamma^\nu\,(\,\not p_2 + m_\mu\,)\,\gamma^\sigma\,\right]$$

in which I have employed the momentum transfer Mandelstam's variable $t = (p_1' - p_1)^2$. It is worthwhile to gather that (3.124) coincides with the previous expression (3.114) under the replacements

$$p \to p_1 \qquad p' \to -p_1' \qquad q \to p_2' \qquad q' \to -p_2 \qquad s \to t$$

so that, setting once again $m_e \approx 0$,

$$\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'} |\mathcal{M}|^2 = \frac{8e^4}{t^2} \times$$
$$\left[(p_1\cdot p_2)(p_1'\cdot p_2') + (p_1\cdot p_2')(p_1'\cdot p_2) - (p_1\cdot p_1')m_\mu^2\right] \quad (3.125)$$

This trick, which allows to build up the amplitude of the process

$$e^-\mu^- \to e^-\mu^-$$

from the knowledge of the amplitude of the related one

$$e^+e^- \to \mu^+\mu^-$$

is a first example of use of a general rule named crossing symmetry. In general, in fact, the $S-$matrix element for any process involving a particle of energy momentum p in the initial state is equal to the $S-$matrix element for an otherwise identical process, but for the exchange of the antiparticle with 4-momentum $-p$ in the final state.

Conversely, the kinematics in the center of momentum frame will be rather different. Actually we have

$$\begin{aligned}
&\text{incoming electron}: \quad \mathbf{p}_1 = \mathbf{p} \qquad E_1 \approx |\mathbf{p}| = p\\
&\text{incoming muon}: \quad \mathbf{p}_2 = -\mathbf{p} \qquad E_2 = \sqrt{\mathbf{p}^2 + m_\mu^2} = E\\
&\text{outgoing electron}: \quad \mathbf{p}_1' = \mathbf{p}' \qquad E_1' \approx |\mathbf{p}'|\\
&\text{outgoing muon}: \quad \mathbf{p}_2' = -\mathbf{p}' \qquad E_2' = \sqrt{\mathbf{p}'^2 + m_\mu^2}\\
&E_{\mathrm{CM}} = E_1 + E_2 \approx p + E \approx E_1' + E_2' \Leftrightarrow |\mathbf{p}'| \approx p
\end{aligned}$$

and thereby

$$\begin{aligned}
&p_1\cdot p_2 = p_1'\cdot p_2' = p(p+E) \qquad p_1\cdot p_2' = p_1'\cdot p_2 = p(p\cos\theta + E)\\
&p_1\cdot p_1' = p^2(1-\cos\theta) \qquad t = (p_1 - p_1')^2 \approx -2p^2(1-\cos\theta)
\end{aligned}$$

in such a manner that we can write

$$\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'} |\mathcal{M}|^2 = \frac{e^4}{2p^2\sin^4(\theta/2)} \times$$
$$\left[(p+E)^2 + (E + p\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right] \quad (3.126)$$

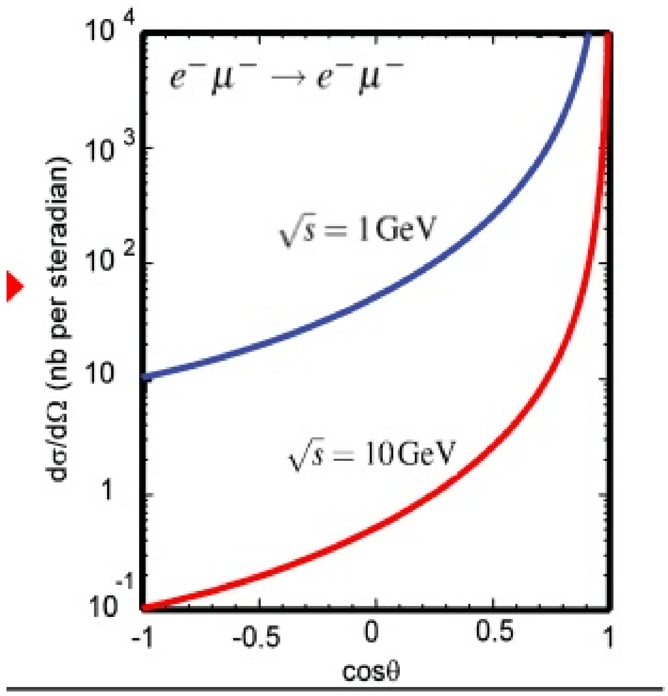

Figure 3.9: the Mott differential cross-section

Now we can use the basic formula (3.98) with $|\,\mathbf{p}'\,| \approx |\,\mathbf{p}\,| = p$ which yields

$$\begin{aligned}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} &= \frac{1}{64\pi^2\,(p+E\,)^2}\cdot\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \\ &\approx \frac{\alpha^2}{2\,E_{\mathrm{CM}}^2\cdot 4\,\beta^2\,\sin^4(\theta/2)} \\ &\times \left[(1+\beta\,)^2+(1+\beta\,\cos\theta\,)^2-2\left(\frac{m_\mu}{E}\right)^2\sin^2\frac{\theta}{2}\right]\end{aligned}$$

where $\beta = v/c \approx p/E$. In the ultra-relativistic limit $E \approx p$ we find

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \overset{\beta\to 1}{\sim} \frac{\alpha^2}{2p^2\cdot 4\,\sin^4(\theta/2)}\,\Big(1+\cos^4(\theta/2)\Big)$$

Consider now the very same process in the incident muon rest frame and retaining the electron mass $m_e \ll m_\mu$ but treating the muon mass as very large. Then, if we disregard the muon recoil, the kinematics reads

$$\left.\begin{array}{lcc} \text{incoming electron :} & \mathbf{p}_1 = \mathbf{p} & E_1 = \sqrt{\mathbf{p}^2 + m_e^2} = E \\ \text{incoming muon :} & \mathbf{p}_2 = 0 & E_2 = m_\mu \\ \text{outgoing muon :} & \mathbf{p}_2' \approx 0 & E_2' \approx m_\mu \\ \text{outgoing electron :} & \mathbf{p}_1' = \mathbf{p}' & E_1' \approx E \ll m_\mu \end{array}\right\} \qquad (3.127)$$

with $|\,\mathbf{p}\,| = p \approx |\,\mathbf{p}'\,|$, whence

$$p_1 \cdot p_2 = E\, m_\mu \approx p_1' \cdot p_2' \approx p_1 \cdot p_2' \approx p_1' \cdot p_2$$
$$p_1 \cdot p_1' = E^2 - p^2 \cos\theta \qquad t = (p_1 - p_1')^2 \approx -\,2p^2\,(1 - \cos\theta\,)$$

in such a manner that now we have

$$\sum_{r,r'} \sum_{s,s'} \tfrac{1}{4}\,|\,\mathcal{M}\,|^2 = \frac{8\,e^4}{t^2} \times \Big[\,(p_1 \cdot p_2\,)(\,p_1' \cdot p_2'\,) + (\,p_1 \cdot p_2'\,)(\,p_1' \cdot p_2\,) - (\,p_1 \cdot p_1'\,)\,m_\mu^2 - (\,p_2 \cdot p_2'\,)\,m_e^2 + 2\,m_\mu^2 m_e^2\,\Big] \qquad (3.128)$$

$$\sum_{r,r'} \sum_{s,s'} \tfrac{1}{4}\,|\,\mathcal{M}\,|^2 = \frac{e^4}{2p^4\,\sin^4(\theta/2)} \times \Big[\,2\,E^2\,m_\mu^2 - m_\mu^2\,(\,E^2 - p^2\,\cos\theta\,) + m_\mu^2 m_e^2\,\Big] \qquad (3.129)$$

If one of the two incident particles is sufficiently heavy, like the muon in the present example, so that its state does not change after the collision, then its role in the process is equivalent to a fixed target for which recoil can be disregarded. Turning back to the main basic formula (3.71) for the differential cross section, in the present case of a 2-particle final state with one very heavy particle we can use the kinematics (3.127) where θ is now the scattering angle of the light particle in the heavy particle rest frame, so that

$$I \equiv \sqrt{(p_1 \cdot p_2)^2 - m^2 M^2} = |\,\mathbf{p}\,|\,M$$

$$\prod_{k=1}^{2} \int \frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\,2\omega(\,\mathbf{p}_k'\,)}\;(2\pi)^4\,\delta\,(p_1' + p_2' - p_1 - p_2)$$
$$= \int \mathrm{d}\Omega\,(\phi,\theta) \int_0^\infty \frac{\mathrm{d}p\,p^2}{16\pi^2 M E\,(p)}\;\delta\left(E_{p'}' - E_p\right)$$
$$= \int \mathrm{d}\Omega\,(\phi,\theta) \int_0^\infty \frac{\mathrm{d}E\,p\,(\,E\,)}{16\pi^2 M}\;\delta\,(E' - E)$$
$$= \frac{|\,\mathbf{p}'\,|}{16\,\pi^2\,M} \int \mathrm{d}\Omega\,(\phi,\theta) \approx \frac{|\,\mathbf{p}\,|}{16\,\pi^2\,M} \int \mathrm{d}\Omega\,(\phi,\theta) \qquad (3.130)$$

Thus, according to the main formula (3.71) and the above fixed target (FT) kinematics, as well as the related final 2-particle phase space integration, we eventually come to the remarkably simple expression

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{FT}} = \frac{|\,\mathcal{M}\,(s,t,u)\,|^{\,2}}{64\,\pi^{2}\,M^{\,2}} \tag{3.131}$$

Inserting the spin averaged and summed amplitude (3.129) and setting $M = m_\mu$ yields

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \frac{\alpha^{2}}{4\,|\,\mathbf{p}\,|^{2}\,\beta^{\,2}\,\sin^{4}(\theta/2)}\,\left(1-\beta^{\,2}\,\sin^{2}\frac{\theta}{2}\right) \tag{3.132}$$

where $\beta \equiv |\,\mathbf{p}\,|\,/\,E$, which is the celebrated Mott formula for the Coulomb scattering of relativistic electrons.

Sir Neville Francis Mott (Leeds, 30.09.1905 - Milton Keynes, 8.8.1996)
The Scattering of Fast Electrons by Atomic Nuclei
Proc. Roy. Soc. A **124**, 425-442 (1929)

In the non-relativistic limit and for a fixed target of atomic number Z we readily recover the Rutherford formula. Actually, for

$$\beta = \frac{v}{c} \qquad \mathbf{p} \approx m\,\mathbf{v} \qquad E \approx m\,c^{2}$$

we get the leading term

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Rutherford}} \overset{\beta\to 0}{\sim} \frac{Z^{2}\,\alpha^{2}\,(\hbar c)^{2}}{4\,m^{2}\,v^{4}\,\sin^{4}(\theta/2)} \tag{3.133}$$

Sir Ernest Rutherford, First Baron of Nelson
(Brightwater, 30.08.1871 - Cambridge, 19.10.1937)
The Scattering of α and β Particles
by Matter and the Structure of the Atom
Philosophical Magazine, **21**, 669-688 (1911)

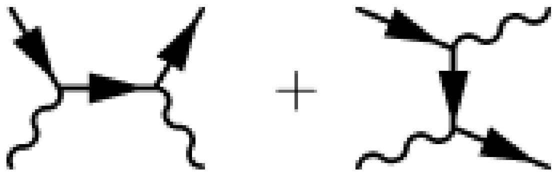

Figure 3.10: the lowest order $e^{-}\gamma$ collision

3.4.3 Compton Scattering

The electron-photon scattering is described, to the lowest order, by the two tree-level diagrams which differ by the 2-photon exchange, see figure. Making use of the rules of correspondence we construct the matrix element

$$\begin{aligned} i\,\mathcal{M}_{rr'}^{AA'}(\mathbf{k},\mathbf{k}') &= -\,ie^2\,\varepsilon_\mu^{A'}(k')\,\bar u_{r'}(\mathbf{p}')\,M^{\mu\nu}\,\varepsilon_\nu^{A}(k)\,u_r \\ &= \varepsilon_\mu^{A'}(k')\,\bar u_{r'}(\mathbf{p}')\,(ie\,\gamma^\mu)\,S(p+k)\,(ie\,\gamma^\nu)\,u_r\,\varepsilon_\nu^{A}(k) \\ &+ [\text{photon exchange} \quad \{k\,\mu\,A\} \leftrightarrows \{(-\,k')\,\nu\,A'\}] \end{aligned} \tag{3.134}$$

$$M^{\mu\nu} = \frac{\gamma^\mu(\not p + \not k + m_e)\gamma^\nu}{(p+k)^2 - m_e^2} + \frac{\gamma^\nu(\not p - \not k' + m_e)\gamma^\mu}{(p-k')^2 - m_e^2} \tag{3.135}$$

where the initial electron four momentum is $p^\lambda = (m_e, 0, 0, 0)$. Since $p^2 = m_e^2$ and $k^2 = 0$ we can simplify the above denominators as

$$(p+k)^2 - m_e^2 = 2\,p^\rho\,k_\rho = 2\,k\,m_e$$

$$(p-k')^2 - m_e^2 = -\,2\,p^\rho\,k'_\rho = -\,2\,k'\,m_e$$

where $k = |\mathbf{k}|$, $k' = |\mathbf{k}'|$, while the numerators can be rearranged using a little Dirac algebra

$$\begin{aligned} (\not p + m_e)\,\gamma^\nu\,u_s(p) &= (2p^\nu - \gamma^\nu\,\not p + \gamma^\nu\,m_e)\,u_s(p) \\ &= 2p^\nu\,u_s(p) - \gamma^\nu(\not p - m_e)\,u_s(p) \\ &= 2p^\nu\,u_s(p) \end{aligned} \tag{3.136}$$

By doing the same for both numerators we get

$$M^{\mu\nu} = \frac{\gamma^\mu\,\not k\,\gamma^\nu + 2\gamma^\mu\,p^\nu}{2\,k\,m_e} + \frac{\gamma^\nu\,\not k'\,\gamma^\mu - 2\,\gamma^\nu\,p^\mu}{2\,k'\,m_e}$$

in such a manner that by putting altogether we can write

$$i\,\mathcal{M}_{rr'}^{AA'}(\mathbf{k},\mathbf{k}') = \frac{-\,i\,e^2}{2\,k\,k'\,m_e}\;\bar{u}_{r'}(\mathbf{k}-\mathbf{k}')\,M^{AA'}\,u_r$$

where

$$\begin{aligned} M^{AA'} &= k'\,\varepsilon_\mu^{A'}(k')\,[\,\gamma^\mu \not{k}\,\gamma^\nu + 2\gamma^\mu\,p^\nu\,]\,\varepsilon_\nu^{A}(k) \\ &+ k\,\varepsilon_\nu^{A}(k)\,[\,\gamma^\nu \not{k}'\,\gamma^\mu - 2\,\gamma^\nu\,p^\mu\,]\,\varepsilon_\mu^{A'}(k') \\ &= k'\not{\varepsilon}^{A'}(k')\not{k}\not{\varepsilon}^{A}(k) + k\not{\varepsilon}^{A}(k)\not{k}'\not{\varepsilon}^{A'}(k') \end{aligned} \tag{3.137}$$

in which we have taken properly into account that

$$\mathbf{p} = 0 \qquad \varepsilon_0^A(k) = 0 \qquad \varepsilon_0^{A'}(k') = 0 \qquad (\,A, A' = 1, 2\,)$$

In calculating the probability of the process we have to average over the initial electron spin index r and sum over the final electron spin index r', so that we obtain

$$\begin{aligned} \tfrac{1}{2}\sum_{r\,,\,r'}\left|\,\mathcal{M}_{rr'}^{AA'}(\mathbf{k},\mathbf{k}')\,\right|^2 &= \frac{e^4}{8\,(k\,k'\,m_e)^2} \\ \times \sum_{r\,,\,r'}\bar{u}_{r'}(\mathbf{k}-\mathbf{k}')\,M^{AA'}\,u_r &\left(\bar{u}_{r'}(\mathbf{k}-\mathbf{k}')\,M^{AA'}\,u_r\right)^* \\ = \frac{e^4}{8\,(k\,k'\,m_e)^2}&\sum_{r\,,\,r'}\bar{u}_{r'}(\mathbf{p}')\,M^{AA'}\,u_r\,\bar{u}_r\,\bar{M}^{AA'}\,u_{r'}(\mathbf{p}') \\ = \frac{e^4}{8\,(k\,k'\,m_e)^2}&\sum_{r'}u_{r'}(\mathbf{p}')\,\bar{u}_{r'}(\mathbf{p}')\,M^{AA'}\sum_{r}u_r\,\bar{u}_r\,\bar{M}^{AA'} \\ = \frac{e^4}{8\,(k\,k'\,m_e)^2}&\,\mathrm{tr}\left[\,(\not{p}' + m_e)\,M^{AA'}\,(\not{p} + m_e)\,\bar{M}^{AA'}\,\right] \\ = \frac{e^4}{8\,(k\,k'\,m_e)^2}&\,\mathrm{tr}\;Q^{AA'} \end{aligned} \tag{3.138}$$

where

$$\begin{aligned} M^{AA'} &= k'\not{\varepsilon}^{A'}(k')\not{k}\not{\varepsilon}^{A}(k) + k\not{\varepsilon}^{A}(k)\not{k}'\not{\varepsilon}^{A'}(k') \\ \bar{M}^{AA'} &= k'\not{\varepsilon}^{A}(k)\not{k}\not{\varepsilon}^{A'}(k') + k\not{\varepsilon}^{A'}(k')\not{k}'\not{\varepsilon}^{A}(k) \\ Q^{AA'} &= (\not{p}' + m_e)\,M^{AA'}\,(\not{p} + m_e)\,\bar{M}^{AA'} \end{aligned}$$

The matrix Q may now be suitably presented as the sum of three matrices

$$Q = Q_1 + Q_2 + Q_3$$

where, by omitting photon polarization indexes for the sake of simplicity,

$$Q_1 = (\not p + m_e)\, M\, (\not p + m_e)\, \bar{M} \tag{3.139}$$
$$Q_2 = (\not k - \not k')\, M\, \not p\, \bar{M} \tag{3.140}$$
$$Q_3 = m_e\, (\not k - \not k')\, M\, \bar{M} \tag{3.141}$$

The matrix Q_3 contains the product of an odd number of Dirac matrices and therefore

$$\mathrm{tr}\, Q_3 = 0$$

and for the same reason we have

$$\mathrm{tr}\, Q_1 = \mathrm{tr}\, [\, \not p\, M\, \not p\, \bar{M}\,] + m_e^2\, \mathrm{tr}\, [\, M\, \bar{M}\,]$$

For the calculation of traces we can make use of the transverse-like conditions for the electromagnetic field

$$k_*^\mu\, \varepsilon_\mu^A(k) = k_*'^\mu\, \varepsilon_\mu^{A'}(k') = k^\mu\, \varepsilon_\mu^A(k) = k'^\mu\, \varepsilon_\mu^{A'}(k') = 0$$

where $k_*^\mu = (k, -\mathbf{k})$ is the dual light-like four vector, with $A, A' = 1, 2$, and the relations

$$p^\mu\, \varepsilon_\mu^A(k) = p^\mu\, \varepsilon_\mu^{A'}(k') = 0$$
$$p_\mu\, k^\mu = k\, m_e \qquad p^\mu\, k'_\mu = m_e\, k' \qquad k_\mu\, k^\mu = k'^\mu\, k'_\mu = 0$$
$$g^{\mu\nu}\, \varepsilon_\mu^A(k)\, \varepsilon_\nu^A(k) = g^{\mu\nu}\, \varepsilon_\mu^{A'}(k')\, \varepsilon_\nu^{A'}(k') = -1$$

In order to calculate $\mathrm{tr}\, Q_1$ it is sufficient to anti-commute the first two terms in Q_1, which yields

$$\begin{aligned} \not p\, M &= k'\, \not p\, \not\varepsilon'\, \not k\, \not\varepsilon + k\, \not p\, \not\varepsilon\, \not k'\, \not\varepsilon' \\ &= -\, k'\, \not\varepsilon'\, \not p\, \not k\, \not\varepsilon - k\, \not\varepsilon\, \not p\, \not k'\, \not\varepsilon' \\ &= (-\, 2 m_e\, k)\, k'\, \not\varepsilon'\, \not\varepsilon + k'\, \not\varepsilon'\, \not k\, \not p\, \not\varepsilon \\ &+ (-\, 2 m_e\, k')\, k\, \not\varepsilon\, \not\varepsilon' + k\, \not\varepsilon\, \not k'\, \not p\, \not\varepsilon' \\ &= -\, M\, \not p - 4kk' m_e\, (\varepsilon \cdot \varepsilon')\, \mathbb{I} \end{aligned} \tag{3.142}$$

and therefore

$$\begin{aligned} \mathrm{tr}\, Q_1 &= \mathrm{tr}\, [\, \not p\, M\, \not p\, \bar{M}\,] + m_e^2\, \mathrm{tr}\, [\, M\, \bar{M}\,] \\ &= -\, \mathrm{tr}\, [\, M\, \not p\, \not p\, \bar{M}\,] + m_e^2\, \mathrm{tr}\, [\, M\, \bar{M}\,] \\ &- 4kk' m_e\, (\varepsilon \cdot \varepsilon')\, \mathrm{tr}\, [\, \not p\, \bar{M}\,] \\ &= -\, 4kk' m_e\, (\varepsilon \cdot \varepsilon')\, \mathrm{tr}\, [\, k'\, \not p\, \not\varepsilon\, \not k\, \not\varepsilon' + k\, \not p\, \not\varepsilon'\, \not k'\, \not\varepsilon\,] \\ &= 32 m_e^2 k^2 k'^2\, (\varepsilon \cdot \varepsilon')^2 = 32 m_e^2 k^2 k'^2\, \cos^2\theta \end{aligned} \tag{3.143}$$

where the shorthand notation $(a \cdot b) = a^{\mu}\, b_{\mu}$ has been used, while θ denotes the angle between the incident and scattered photon momenta. Moreover, for the calculation of $\operatorname{tr} Q_2$ it is convenient to write

$$\operatorname{tr} Q_2 \;=\; \operatorname{tr}\,[\,(\not k - \not k'\,)\, M \not p\, \bar M\,] = \operatorname{tr}\,[\,\bar M\,(\not k - \not k'\,)\, M \not p\,] \tag{3.144}$$

and by carrying out the anti-commutations in such a manner that the identical factors turn out to be next to each other we get

$$\begin{aligned}
& (\not k - \not k'\,)\, M = (\not k - \not k'\,)\,(k' \not\varepsilon' \not k \not\varepsilon + k \not\varepsilon \not k' \not\varepsilon'\,) \\
= \;& \not k (k' \not\varepsilon' \not k \not\varepsilon + k \not\varepsilon \not k' \not\varepsilon'\,) - \not k'\,(k' \not\varepsilon' \not k \not\varepsilon + k \not\varepsilon \not k' \not\varepsilon'\,) \\
= \;& k' \not k \not\varepsilon' \not k \not\varepsilon + k \not k \not\varepsilon \not k' \not\varepsilon' - k' \not k' \not\varepsilon' \not k \not\varepsilon - k \not k' \not\varepsilon \not k' \not\varepsilon' \\
= \;& k'\,\{\not k\,,\, \not\varepsilon'\}\not k \not\varepsilon - k \not\varepsilon \not k \not k' \not\varepsilon' + k' \not\varepsilon' \not k' \not k \not\varepsilon - k\,\{\not k'\,,\, \not\varepsilon\}\not k' \not\varepsilon' \\
= \;& k' \not\varepsilon' \not k' \not k \not\varepsilon - k \not\varepsilon \not k \not k' \not\varepsilon' + 2(k \cdot \varepsilon'\,)\, k' \not k \not\varepsilon - 2(k' \cdot \varepsilon)\, k \not k' \not\varepsilon'
\end{aligned}$$

Hence, if we set

$$a = k' \not\varepsilon' \not k' \not k \not\varepsilon - k \not\varepsilon \not k \not k' \not\varepsilon'$$
$$b = 2(k \cdot \varepsilon'\,)\, k' \not k \not\varepsilon - 2(k' \cdot \varepsilon)\, k \not k' \not\varepsilon'$$

we readily obtain

$$\begin{aligned}
& \operatorname{tr}\,[\,\bar M a \not p\,] \\
= \;& \operatorname{tr}\,[\,(k' \not\varepsilon \not k \not\varepsilon' + k \not\varepsilon' \not k' \not\varepsilon)(k' \not\varepsilon' \not k' \not k \not\varepsilon - k \not\varepsilon \not k \not k' \not\varepsilon'\,) \not p\,] \\
= \;& -\, k'^2 \operatorname{tr}\,[\,\not k \not\varepsilon' \not\varepsilon' \not k' \not k \not\varepsilon \not\varepsilon \not p\,] + \; k^2 \operatorname{tr}\,[\,\not\varepsilon' \not\varepsilon' \not k' \not\varepsilon \not\varepsilon \not k \not k' \not p\,] \\
+ \;& k\, k' \operatorname{tr}\,[\,\not\varepsilon \not k \not\varepsilon' \not k \not\varepsilon \not k' \not\varepsilon' \not p\,] - \; k\, k' \operatorname{tr}\,[\,\not\varepsilon' \not k' \not\varepsilon \not k' \not\varepsilon' \not k \not\varepsilon \not p\,] \\
= \;& -\, k'^2 \operatorname{tr}\,[\,\not k \not k' \not k \not p\,] + \; k^2 \operatorname{tr}\,[\,\not k' \not k \not k' \not p\,] \\
+ \;& 2(k \cdot \varepsilon'\,)\, k\, k' \operatorname{tr}\,[\,\not k \not k' \not\varepsilon' \not p\,] - \; 2(k' \cdot \varepsilon)\, k\, k' \operatorname{tr}\,[\,\not k' \not k \not\varepsilon \not p\,] \\
= \;& 8k\, k'\, m_e\, (k \cdot k'\,)(k - k'\,) \\
+ \;& 8k\, k'\, m_e\, \{\, k\, (k' \cdot \varepsilon)^2 - k'\, (k \cdot \varepsilon'\,)^2\,\}
\end{aligned} \tag{3.145}$$

$$\begin{aligned}
& \operatorname{tr}\,[\,\bar M b \not p\,] \\
= \;& \operatorname{tr}\,[\,(k' \not\varepsilon \not k \not\varepsilon' + k \not\varepsilon' \not k' \not\varepsilon)\{2(k \cdot \varepsilon'\,)\, k' \not k \not\varepsilon - 2(k' \cdot \varepsilon)\, k \not k' \not\varepsilon'\}\not p\,] \\
= \;& 2(k \cdot \varepsilon'\,)\, k'^2 \operatorname{tr}\,[\,\{\not\varepsilon'\,,\, \not k\}\not k \not p\,] 1/2 - 2(k' \cdot \varepsilon)\, kk' \operatorname{tr}\,[\,\not\varepsilon \not k \not k' \not p\,] \\
+ \;& 2(k \cdot \varepsilon'\,)\, kk' \operatorname{tr}\,[\,\not\varepsilon' \not k' \not k \not p\,] - 2(k' \cdot \varepsilon)\, k^2 \operatorname{tr}\,[\,\{\not k'\,,\, \not\varepsilon\}\not k' \not p\,] \\
= \;& 4(k \cdot \varepsilon'\,)^2\, k'^2 \operatorname{tr}\,[\,\not k \not p\,] + 2(k' \cdot \varepsilon)^2\, kk' \operatorname{tr}\,[\,\not k \not p\,] \\
- \;& 4(k' \cdot \varepsilon)^2\, k^2 \operatorname{tr}\,[\,\not k' \not p\,] - 2(k \cdot \varepsilon'\,)^2\, kk' \operatorname{tr}\,[\,\not k' \not p\,] \\
= \;& 16\,(k \cdot \varepsilon'\,)^2\, k'^2\, k\, m_e + 8\,(k' \cdot \varepsilon)^2\, k^2\, k'\, m_e \\
- \;& 16\,(k' \cdot \varepsilon)^2\, k'\, k^2\, m_e + 8\,(k \cdot \varepsilon'\,)^2\, k\, k'^2\, m_e \\
= \;& -\, 8\, k\, k'\, m_e\, \{\, k\, (k' \cdot \varepsilon)^2 - k'\, (k \cdot \varepsilon'\,)^2\,\}
\end{aligned} \tag{3.146}$$

By adding all the above trace calculations we get

$$\begin{aligned}\operatorname{tr} Q &= 32\, m_e^2 k^2 k'^2 \cos^2\theta + 8\, k\, k'\, m_e\, (k \cdot k')(k - k') \\ &= 32 m_e^2 k^2 k'^2\, (\varepsilon \cdot \varepsilon')^2 + 8 k\, k'\, m_e\, (k \cdot k')(k - k')\end{aligned} \tag{3.147}$$

Next, by writing the energy momentum conservation in the form

$$\begin{cases} k + m_e = k' + \sqrt{\mathbf{p}'^2 + m_e^2} \\ \mathbf{k} = \mathbf{k}' + \mathbf{p}' \end{cases} \tag{3.148}$$

and eliminating $\mathbf{p}'$ we come to the famous Compton kinematic relation

$$k^\mu\, k'_\mu \equiv k\, k'\, (1 - \cos\theta) = m_e\, (k - k') \tag{3.149}$$

or, in physical units,

$$c\, (\nu - \nu') = 2\pi\, \lambda_e\; \nu\nu'\, (1 - \cos\theta) \tag{3.150}$$

where

$$\lambda_e = \frac{\hbar}{m_e\, c} = 3.861592678(26) \times 10^{-13}\ \mathrm{m} \tag{3.151}$$

is the celebrated Compton wave length of the electron.

Arthur Holly Compton (Wooster, 10.09.1892 - Berkeley, 15.03.1962)
A Quantum Theory of the Scattering of X-rays by Light Elements
The Physical Review **21**, 483 - 502 (1923)

Notice that after scattering the photon frequency is always shifted towards the infrared. The Compton relation can also be suggestively rewritten as a photon wave length shift: namely,

$$\Delta\,\lambda = \lambda' - \lambda = 2\pi\, \lambda_e\, (1 - \cos\theta)$$

Solving with respect to the scattered photon frequency we get

$$k' \;=\; \frac{k}{1 + (1 - \cos\theta)\, \dfrac{k}{m_e}}$$

Hence we eventually obtain the final result

$$\begin{aligned}\operatorname{tr} Q^{AA'} &= 32\, m_e^2 k^2 k'^2 g^{\mu\nu}\, \varepsilon_\mu^A(k)\, \varepsilon_\nu^{A'}(k')\; g^{\rho\sigma}\, \varepsilon_\rho^A(k)\, \varepsilon_\sigma^{A'}(k') \\ &+\; 8\, m_e^2 k^2 k'^2 \left(\frac{k}{k'} + \frac{k'}{k} - 2 \right) \\ &=\; 8\, m_e^2 k^2 k'^2 \left\{ 4 \left(\varepsilon^A \cdot \varepsilon^{A'} \right)^2 \; - \; 2 + \frac{k}{k'} + \frac{k'}{k} \right\}\end{aligned}$$

in accordance with N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience Publisher, New York, 1959, p.279. If one is not interested in detecting photon polarization, then it is convenient to average over the incident and sum over the scattered photon polarization that yields

$$\begin{aligned} & g^{\mu\nu}\, g^{\rho\sigma} \sum_{A=1,2} \tfrac{1}{2}\, \varepsilon_{\mu}^{A}(k)\, \varepsilon_{\rho}^{A}(k) \sum_{A'=1,2} \varepsilon_{\nu}^{A'}(k')\, \varepsilon_{\sigma}^{A'}(k') \\ = & \; \tfrac{1}{2}\, g^{\mu\nu}\, g^{\rho\sigma}\, \Pi^{\perp}_{\mu\rho}(k)\, \Pi^{\perp}_{\nu\sigma}(k') \end{aligned} \tag{3.152}$$

where

$$\Pi^{\perp}_{\mu\rho}(k) = g_{\mu\rho} - \frac{k_{\mu}\, k^{*}_{\rho} + k_{\rho}\, k^{*}_{\mu}}{k \cdot k^{*}} \qquad (\, k^{*}_{\mu} = k^{\mu}\,)$$

Thus, from the relations

$$(\, k \cdot k^{*}\,) = 2\mathbf{k}^{2} \qquad (\, k' \cdot k'_{*}\,) = 2\mathbf{k}'^{2} \tag{3.153}$$

we find

$$\begin{aligned} \tfrac{1}{2}\, \Pi^{\perp}_{\mu\rho}(k)\, \Pi^{\mu\rho}_{\perp}(k') &= \frac{1}{2}\left(\frac{k_{\mu}\, k^{*}_{\rho} + k_{\rho}\, k^{*}_{\mu}}{k \cdot k^{*}}\right)\left(\frac{k'^{\mu}\, k'^{\rho}_{*} + k'^{\rho}\, k'^{\mu}_{*}}{k' \cdot k'_{*}}\right) \\ &= \frac{(\, k \cdot k'\,)(\, k^{*} \cdot k'_{*}\,) + (\, k \cdot k'_{*}\,)(\, k^{*} \cdot k'\,)}{(\, k \cdot k^{*})(\, k' \cdot k'_{*}\,)} \end{aligned} \tag{3.154}$$

and since we have

$$(\, k \cdot k'\,) = kk'(1 - \cos\theta) = (\, k^{*} \cdot k'_{*}\,)$$

$$(\, k \cdot k'_{*}\,) = kk'(1 + \cos\theta) = (\, k^{*} \cdot k'\,)$$

we come to the result

$$\tfrac{1}{2}\, \Pi^{\perp}_{\mu\rho}(k)\, \Pi^{\mu\rho}_{\perp}(k') = \tfrac{1}{2}\,(1 + \cos^{2}\theta) \tag{3.155}$$

and consequently

$$\tfrac{1}{2} \sum_{A,\, A'=1,2} \operatorname{tr} Q^{AA'} = 16\, m_e^{2} k^{2} k'^{2} \left(\frac{k}{k'} + \frac{k'}{k} - \sin^{2}\theta\right) \tag{3.156}$$

It turns out that the non-polarized differential cross-sections will be given by the general Golden Formula

$$\mathrm{d}\sigma = \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^{3}\, 2k'} \int \frac{\mathrm{d}\mathbf{p}'}{(2\pi)^{3}\, 2E'}\, (2\pi)^{4}\, \delta^{(4)}(k + p - k' - p')$$

$$
\begin{aligned}
& \times \quad \tfrac{1}{4}\,(\,p^{\lambda}k_{\lambda}\,)^{-1}\ \cdot\ \tfrac{1}{2}\sum_{A\,,\,A'=1,2}\tfrac{1}{2}\sum_{r,r'=1,2}\left|\,\mathcal{M}_{rr'}^{AA'}(\mathbf{k},\mathbf{k}'\,)\,\right|^{2} \\
& = \quad \int_{0}^{\infty}\frac{k'^{2}\,\mathrm{d}k'}{(2\pi)^{3}\,2k'}\int\frac{\mathrm{d}\boldsymbol{\Omega}}{2E'(\,k'\,)}\ (2\pi)\,\delta\,(k'+E'(\,k'\,)-m_{e}-k) \\
& \times \quad \frac{1}{4km_{e}}\ \cdot\ \frac{e^{4}}{8(kk'm_{e})^{2}}\ \cdot\ 16\,m_{e}^{2}k^{2}k'^{2}\left(\frac{k}{k'}+\frac{k'}{k}-\sin^{2}\theta\right)
\end{aligned}
$$

in which

$$
E'(\,k'\,)\equiv\sqrt{k'^{2}+k^{2}-2kk'\,\cos\theta+m_{e}^{2}}
$$

$$
\mathrm{d}\boldsymbol{\Omega}=\mathrm{d}\phi\,\mathrm{d}\cos\theta \qquad (\,0\le\theta\le\pi\,,\ 0\le\phi\le2\pi\,)
$$

From the theory of the tempered distributions we get the well known relation

$$
\begin{aligned}
& \int_{0}^{\infty}\frac{k'\,\mathrm{d}k'}{E'(\,k'\,)}\ \delta\,(k'+E'(\,k'\,)-m_{e}-k)\ f(\,k'\,) \\
= \quad & \left[\frac{k'\,f(\,k'\,)}{|\,E'(\,k'\,)+\,k'-k\cos\theta\,|}\right]_{k'=\widetilde{k}'} \qquad [\,\forall\,f\in\mathcal{S}(\mathbb{R})\,]
\end{aligned}
$$

where

$$
\widetilde{k}'+E'(\widetilde{k}'\,)=k+m_{e}\ \Leftrightarrow\ \widetilde{k}'\ \equiv\ \frac{k\,m_{e}}{m_{e}+k\,(1-\cos\theta)}
$$

in such a manner that

$$
\left[\frac{k'}{|\,E'(\,k'\,)+\,k'-k\cos\theta\,|}\right]_{k'=\widetilde{k}'}=\frac{k'}{m_{e}+k\,(1-\cos\theta)}=\frac{\widetilde{k}'^{2}}{k\,m_{e}}
$$

It follows that we finally obtain the non-polarized differential cross-section

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)_{\mathrm{Klein-Nishina}} & = \frac{e^{4}}{16\pi m_{e}^{2}c^{4}}\left(\frac{k'}{k}\right)^{2}\left(\frac{k}{k'}+\frac{k'}{k}-\sin^{2}\theta\right) \\
& = \frac{\pi\hbar^{2}\alpha^{2}}{m_{e}^{2}c^{2}}\left(\frac{k'}{k}\right)^{2}\left(\frac{k}{k'}+\frac{k'}{k}-\sin^{2}\theta\right) \qquad (3.157) \\
\frac{k'}{k} & \equiv \frac{1}{1+\hbar k\,(1-\cos\theta)/m_{e}c}
\end{aligned}
$$

which is the celebrated Klein-Nishina formula, first obtained in 1928 at the Niels Bohr Institute of Coopenhagen

Oskar Klein (Stockholm, 15.09.1894 - Stockholm, 5.02.1977)
Yoshio Nishina (Hamanaka, 6.12.1890 - Tokyo, 10.01.1951)
Über die Streung von Strahlung durch freie Elektronen

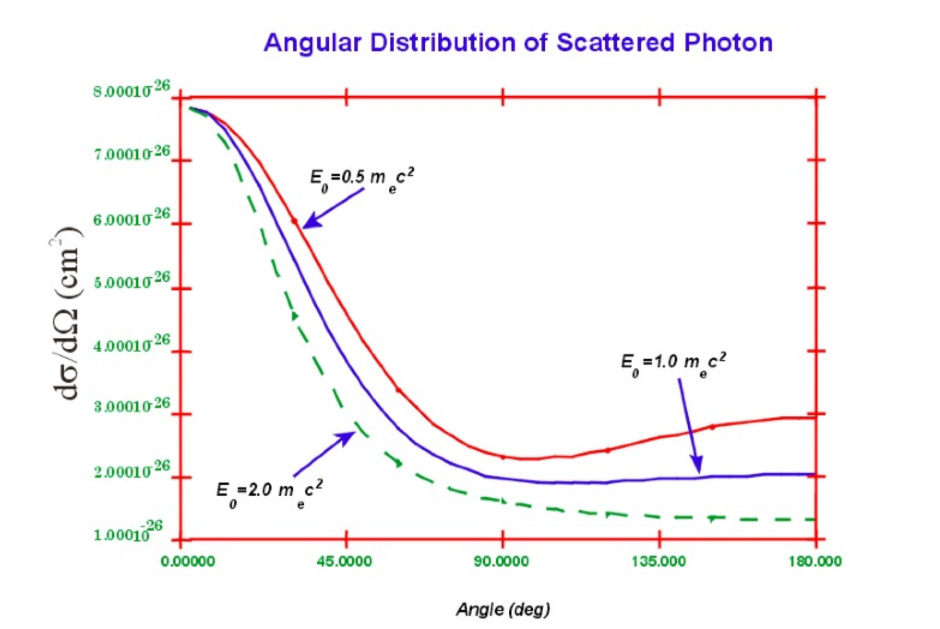

Figure 3.11: Klein-Nishina differential cross-section vs. angle

nach der neuen relativistischen Quantendynamik von Dirac
Zeitschrift für Physik **52**, 853 - 864 (1929)

Y. Nishina, *Die Polarisation der Comptonstreung nach der Diracschen Theorie des Elektrons*
Z. Physik **52**, 869 - 877 (1929)

Notice that in the low frequency limit $k \to 0$ which entails $k'/k \to 1$ the non-relativistic Thomson formula for the differential cross-section of light by free electrons: namely,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)_{\text{Thomson}} = \pi\alpha^2\lambda_e^2(\,1+\cos^2\theta\,) \tag{3.158}$$

and after angular integration

$$\sigma_{\mathrm{T}} = \tfrac{8}{3}\,\pi\alpha^2\lambda_e^2 = \tfrac{8}{3}\,\pi\,r_e^2 = 0.665\,245\,873(13)\text{ barn} \tag{3.159}$$

where $r_e = \alpha\,\lambda_e = 2.817\,940\,325(28) \times 10^{-13}$ cm is the classical electron radius, while 1 barn $= 10^{-28}$ m^2. The Thomson cross section of nearly half a barn is used as the paradigmatic reference scale for all cross sections of the high energy sub-nuclear particle Physics. Note that the photons of the visible part of the electromagnetic spectrum have energies of the eV order.

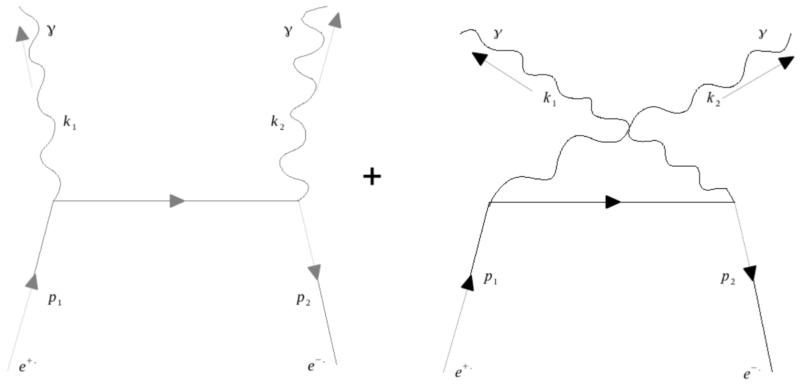

Figure 3.12: the direct and exchange diagrams for e^-e^+ annihilation into a $\gamma\gamma$ pair to the lowest order

3.4.4 Annihilation into $\gamma\,\gamma$ of $e^-\,e^+$ Pairs

The two photon annihilation is described, to the lowest order, by the two tree-level diagrams which differ by the two–photon exchange, see figure. In the center of momentum frame of the e^+e^- pair we have

$$\begin{array}{llll} \text{electron :} & \mathbf{p}_1 = \mathbf{p} & E_1 = \sqrt{p^2+m^2} & |\,\mathbf{p}\,| = p \\ \text{positron :} & \mathbf{p}_2 = -\,\mathbf{p} & E_2 = E_1 = E & \\ \text{first photon :} & \mathbf{k}_1 = \mathbf{k}\,, & k_1^0 = |\,\mathbf{k}\,| & \\ \text{second photon :} & \mathbf{k}_2 = -\,\mathbf{k}\,, & k_2^0 = k_1^0 = |\,\mathbf{k}\,| \;\equiv\; k & \end{array}$$

Moreover, since we will finally sum over both photon polarization states, we associate to the two final photon the real and physical linear polarization vectors

$$\varepsilon_\mu^{\,A}(k) \quad (\,A = 1, 2\,) \qquad\qquad \varepsilon_\nu^{\,B}(k) \quad (\,B = 1, 2\,)$$

that satisfy

$$-\,g^{\mu\nu}\,\varepsilon_\mu^{\,A}(\,k\,)\,\varepsilon_\nu^{\,B}(\,k\,) = \delta^{AB} \qquad\qquad A, B \;=\; 1, 2$$

The Mandelstam variables are

$$s = (p_1+p_2)^2 = (k_1+k_2)^2 = 2m^2 + 2p_1\cdot p_2 = 4E^{\,2} = 2k_1\cdot k_2 = 4k^{\,2}$$

that evidently yields $E = k$ together with

$$\begin{aligned} t &= (p_1-k_1)^2 = (p_2-k_2)^2 = m^2 - 2p_1\cdot k_1 = m^2 - 2p_2\cdot k_2 \\ u &= (p_1-k_2)^2 = (p_2-k_1)^2 = m^2 - 2p_1\cdot k_2 = m^2 - 2p_2\cdot k_1 \end{aligned}$$

with $s+t+u=2m^2$. Making use of the rules of correspondence we construct the matrix elements

$$\begin{aligned} i\,\mathcal{M} &= -\,ie^2\,\varepsilon_\nu\,\varepsilon_\mu\,\bar{v}_r(-\mathbf{p})\,M^{\,\mu\nu}\,u_{\,s}(\mathbf{p}) \\ &= \bar{v}_r(\mathbf{p}_2)\,(-ie\,\gamma^{\,\nu})\,\varepsilon_\nu^{\,B}(\mathbf{k}_2)\,S(p_1-k_1)\,(-ie\,\gamma^{\,\mu})\,\varepsilon_\mu^{\,A}(\mathbf{k}_1)\,u_{\,s}(\mathbf{p}_1) \\ &+ \;[\,\text{photon exchange}\quad \mathbf{k}_1 \leftrightarrow \mathbf{k}_2 \qquad A \leftrightarrow B \qquad \mu \leftrightarrow \nu\,] \\ &= -\,2ie^2\,\bar{v}_r\,(-\,\mathbf{p})\,Q\,u_{\,s}(\mathbf{p})\,[\,(t-m^2)\,(u-m^2)\,]^{-1} \end{aligned}$$

where we have set

$$Q=(p_1\cdot k_2)\,\not\varepsilon_2\,(\not p_1-\not k_1+m)\,\not\varepsilon_1+(p_1\cdot k_1)\,\not\varepsilon_1\,(\not p_1-\not k_2+m)\,\not\varepsilon_2$$

Notice that we have

$$\begin{aligned} Q\,u_s\,(\,\mathbf{p}\,) &= (p_1\cdot k_2)\,\not\varepsilon_2\,(\not p_1-\not k_1+m)\,\not\varepsilon_1\,u_s\,(\,\mathbf{p}_1\,) \\ &+ (p_1\cdot k_1)\,\not\varepsilon_1\,(\not p_1-\not k_2+m)\,\not\varepsilon_2\,u_s\,(\,\mathbf{p}_1\,) \\ &= (p_1\cdot k_2)\,\varepsilon_2\,(2p_1\cdot\varepsilon_1-\not k_1\not\varepsilon_1)\,u_s\,(\,\mathbf{p}_1\,) \\ &+ (p_1\cdot k_1)\,\varepsilon_1\,(2p_1\cdot\varepsilon_2-\not k_2\not\varepsilon_2)\,u_s\,(\,\mathbf{p}_1\,) \end{aligned}$$

in such a way that we can write

$$\begin{aligned} Q &\doteq (p_1\cdot k_2)\,\not\varepsilon_2\,[\,2(p_1\cdot\varepsilon_1)\;-\;\not k_1\,\not\varepsilon_1\,] \\ &+ (p_1\cdot k_1)\,\not\varepsilon_1\,[\,2(p_1\cdot\varepsilon_2)\;-\;\not k_2\,\not\varepsilon_2\,] \end{aligned}$$

where $\doteq$ means equality up to evanescent terms when acting upon $u\,(\,\mathbf{p}\,)$. If we are looking for annihilation of non-polarized e^+e^- pairs then we have to average with respect to the polarization of the incident e^-e^+ pair so that we can write

$$[\,\bar{v}_r\,(-\,\mathbf{p}\,)\,Q\,u_s\,(\,\mathbf{p}\,)\,]^*=\bar{u}_s\,(\,\mathbf{p}\,)\;\bar{Q}\;\,v_r\,(-\,\mathbf{p}\,)$$

so that

$$\begin{aligned} \bar{Q} &\doteq (p_1\cdot k_2)\,[\,2p_1\cdot\varepsilon_1\;-\;\not\varepsilon_1\,\not k_1\,]\,\not\varepsilon_2 \\ &+ (p_1\cdot k_1)\,[\,2p_1\cdot\varepsilon_2\;-\;\not\varepsilon_2\,\not k_2\,]\,\not\varepsilon_1 \end{aligned}$$

As a consequence, after summation over spinor polarization and making use of the property of the cyclic property of the trace operation, in accordance with the Golden Formula (3.98) for the quasi-elastic scattering in the center of momentum frame, we can definitely write

$$\begin{aligned} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} &= \left(\frac{\alpha}{4k}\right)^2\cdot\frac{\mathrm{tr}\,[\,(\not p_2-m)\,Q\,(\not p_1+m)\;\bar{Q}\,]}{[\,(\,t-m^2\,)(u-m^2\,)\,]^2}\cdot\frac{k}{p} \\ &= \alpha^2\,\frac{\mathrm{tr}\,[\,(\not p_2-m)\,Q\,(\not p_1+m)\;\bar{Q}\,]}{16\,pk\,(\,t-m^2\,)^2\,(\,u-m^2\,)^2} \\ &\equiv \alpha^2\,\mathrm{tr}\,\Upsilon/[\,16\,pk\,(t-m^2)^2\,(u-m^2)^2\,] \end{aligned}$$

where tr $[\cdots]$ refers to trace over spinor indexes. Making use of the law of the conservation of tetra-momentum

$$p_2 = k_1 + k_2 - p_1$$

we shall write Υ in the form of the sum

$$\Upsilon = \Upsilon_1 + \Upsilon_2 + \Upsilon_3$$

where

$$\begin{aligned}
\Upsilon_1 &= (\not k_1 + \not k_2)\, Q\, \not p_1\ \bar{Q} \\
\Upsilon_2 &= -\,(\not p_1 + m)\, Q\, (\not p_1 + m)\ \bar{Q} \\
\Upsilon_3 &= m\,(\not k_1 + \not k_2)\, Q\ \bar{Q}
\end{aligned}$$

It is immediate to check by direct inspection that Υ_3 is a sum of products of an odd number of Dirac matrices, whence $\mathrm{tr}\,\Upsilon_3 = 0$. Moreover, after setting $k_1 + k_2 = K$, we have

$$\begin{aligned}
\mathrm{tr}\,\Upsilon_1 &= \mathrm{tr}\,[\,(\not k_1 + \not k_2)\, Q\, \not p_1\ \bar{Q}\,] \\
&= \mathrm{tr}\Big\{(p_1\cdot k_2)\,\not K \not\varepsilon_2[\,2(p_1\cdot\varepsilon_1) - \not k_1 \not\varepsilon_1\,] \\
&+\ (p_1\cdot k_1)\,\not K \not\varepsilon_1[\,2(p_1\cdot\varepsilon_2) - \not k_2 \not\varepsilon_2\,]\Big\} \\
&\times\ \Big\{(p_1\cdot k_2)\,\not p_1[\,2(p_1\cdot\varepsilon_1) - \not\varepsilon_1 \not k_1\,]\not\varepsilon_2 \\
&+\ (p_1\cdot k_1)\,\not p_1\,[\,2(p_1\cdot\varepsilon_2) - \not\varepsilon_2 \not k_2\,]\not\varepsilon_1\Big\} \\
&= \mathrm{tr}\Big\{(p_1\cdot k_2)\,\not K \not\varepsilon_2[\,2(p_1\cdot\varepsilon_1) - \not k_1 \not\varepsilon_1\,]\Big\} \\
&\times\ \Big\{(p_1\cdot k_2)\,\not p\,[\,2(p_1\cdot\varepsilon_1) - \not\varepsilon_1 \not k_1\,]\not\varepsilon_2\Big\} \\
&+\ \mathrm{tr}\Big\{(p_1\cdot k_2)\,\not K \not\varepsilon_2\,[\,2(\,p_1\cdot\varepsilon_1\,) - \not k_1 \not\varepsilon_1\,]\Big\} \\
&\times\ \Big\{(p_1\cdot k_1)\,\not p\,[\,2(p_1\cdot\varepsilon_2) - \not\varepsilon_2 \not k_2\,]\not\varepsilon_1\Big\} \\
&+\qquad \Big\{1\ \leftrightarrow\ 2\Big\}
\end{aligned}$$

This expression contains the sum of two groups of sixteen terms which are related by the exchange operation $\{1\ \leftrightarrow\ 2\}$. In turn, each group is done of traces of products of four, six and eight Dirac matrices. The general method of calculating traces consists of successive displacements of identical matrix-four-vector. The calculation of $\mathrm{tr}\,\Upsilon_1$ and $\mathrm{tr}\,\Upsilon_2$ is then straightforward and elementary, although tedious. For this calculation we shall use

$$\begin{aligned}
\mathrm{tr}\,(\gamma^\mu\gamma^\nu) &= g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} \;=\; 4\,g^{\mu\nu} \\
\tfrac{1}{4}\,\mathrm{tr}\,\left(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\right) &= g^{\kappa\lambda}\,g^{\mu\nu} \;-\; g^{\kappa\mu}\,g^{\lambda\nu} \;+\; g^{\kappa\nu}\,g^{\lambda\mu}
\end{aligned}$$

as well as the forthcoming relations, which are a direct consequence of the corresponding definitions: namely,

$$\begin{aligned}
& k_1^2 = k_2^2 = 0\ ,\ \ k_1\cdot k_2 = 2k^2 \\
& k_1\cdot\varepsilon_1 = k_2\cdot\varepsilon_2 = k_1\cdot\varepsilon_2 = k_2\cdot\varepsilon_1 = 0 \\
& k_1\cdot p_1 = k\,\omega_p - \mathbf{k}_1\cdot\mathbf{p}_1 = k\,(\,k - p\,\cos\theta\,) \\
& k_2\cdot p_1 = k\,\omega_p - \mathbf{k}_2\cdot\mathbf{p}_1 = k\,(\,k + p\,\cos\theta\,)
\end{aligned}$$

where θ is the angle between the vectors $\mathbf{k}_1$ and $\mathbf{p}_1$. The result is

$$\begin{aligned}
\mathrm{tr}\,\Upsilon_1 &= 32\,k^2\,(p_1\cdot k_1)\,(p_1\cdot k_2) \\
&\times\ [\,2(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_1)(p_1\cdot\varepsilon_2) + k^2 + (p_1\cdot\varepsilon_1)^2 + (p_1\cdot\varepsilon_2)^2\,] \\
\mathrm{tr}\,\Upsilon_2 &= -\,32\,k^2\,(p_1\cdot k_1)\,(p_1\cdot k_2) \\
&\times\ [\,2(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_1)(p_1\cdot\varepsilon_2) + (p_1\cdot\varepsilon_1)^2 + (p_1\cdot\varepsilon_2)^2\,] \\
&-\ 32\,[\,2k^2\,(p_1\cdot\varepsilon_1)(p_1\cdot\varepsilon_2) + (\varepsilon_1\cdot\varepsilon_2)(p_1\cdot k_1)(p_1\cdot k_2)\,]^2 \\
\mathrm{tr}\,\Upsilon &= 32\,(p_1\cdot k_1)^2\,(p_1\cdot k_2)^2\,\left\{\frac{k^4}{(p_1\cdot k_1)(p_1\cdot k_2)}\;-\;(\varepsilon_1\cdot\varepsilon_2)^2\;-\right. \\
&-\ \left.\frac{4k^4\,(p_1\cdot\varepsilon_1)^2(p_1\cdot\varepsilon_2)^2}{(p_1\cdot k_1)^2(p_1\cdot k_2)^2} - \frac{4k^2\,(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_1)(p_1\cdot\varepsilon_2)}{(p_1\cdot k_1)(p_1\cdot k_2)}\right\}
\end{aligned}$$

The very last expression contains a shorthand notation, though the various addenda do actually depend upon two physical polarization indexes, just like in the Compton scattering. For example we have

$$(\varepsilon_1\cdot\varepsilon_2)^2 = g^{\mu\nu}\varepsilon^A_\mu(k)\varepsilon^B_\nu(k)\,g^{\rho\sigma}\varepsilon^A_\rho(k)\varepsilon^B_\sigma(k)$$

so that the sum over all the physical polarization indexes yields

$$g^{\mu\nu}\,g^{\rho\sigma}\sum_{A=1,2}\varepsilon^A_\mu(k)\,\varepsilon^A_\rho(k)\sum_{B=1,2}\varepsilon^B_\nu(k)\,\varepsilon^B_\sigma(k) = g^{\mu\nu}\,g^{\rho\sigma}\,\Pi^\perp_{\mu\rho}(k)\,\Pi^\perp_{\nu\sigma}(k)$$

where projector on the physical polarization two dimensional space is

$$\Pi^\perp_{\mu\rho}(k) = g_{\mu\rho} - \frac{k_\mu k^*_\rho + k_\rho k^*_\mu}{k\cdot k^*}\qquad\qquad (\,k^*_\mu = k^\mu\,)$$

Now, since we obtain

$$\Pi_{\mu\rho}^{\perp}(k)\,\Pi_{\perp}^{\mu\rho}(k) = \mathrm{tr}\,\Pi_{\perp} = 2$$

we definitely get

$$\sum_{A=1,2}\ \sum_{B=1,2}(\,\varepsilon_1^A\cdot\varepsilon_2^B\,)^2 = 2$$

Quite analogously we obtain

$$\begin{aligned}
&\sum_{A=1,2}\left(p_1\cdot\varepsilon_1^A\right)^2 = \sum_{A=1,2}\left(p_1\cdot\varepsilon_2^A\right)^2 \\
=\ & g_{\mu\nu}\,g_{\rho\sigma}\,p_1^\mu\,p_1^\rho\sum_{A=1,2}\varepsilon_A^\nu(k)\,\varepsilon_A^\sigma(k) = p_1^\mu\,p_1^\rho\,\Pi_{\mu\rho}^{\perp}(k) \\
=\ & m^2 - \frac{(\,p_1\cdot k)(\,p_1\cdot k_*)}{\mathbf{k}^2} = m^2-(E-p\cos\theta)(E+p\cos\theta) \\
=\ & -\,p^{\,2}\sin^2\theta
\end{aligned}$$

Finally we get

$$\begin{aligned}
&\sum_{A=1,2}\ \sum_{B=1,2}(\,\varepsilon_1^A\cdot\varepsilon_2^B\,)\,(\,p_1\cdot\varepsilon_1^A\,)\,(\,p_1\cdot\varepsilon_2^B\,) \\
=\ & p_1^\mu\,p_1^\rho\,\Pi_{\mu\nu}^{\perp}(k)\,\Pi_{\rho\sigma}^{\perp}(k)\,g^{\,\nu\sigma} = p_1^\mu\,p_1^\rho\,\Pi_{\mu\rho}^{\perp}(k) = -\,p^{\,2}\sin^2\theta
\end{aligned}$$

Putting altogether we obtain

$$\begin{aligned}
\sum_{A=1,2}\sum_{B=1,2}\mathrm{tr}\,\Upsilon_{AB} \ &=\ 4\,(2p_1\cdot k_1)^2\,(2p_1\cdot k_2)^2\ \times \\
&\times\ \left\{\frac{k^{\,2}+p^{\,2}\,(1+\sin^2\theta)}{k^{\,2}-p^{\,2}\cos^2\theta}\ -\ \frac{2\,p^{\,4}\sin^4\theta}{(\,k^{\,2}-p^{\,2}\cos^2\theta\,)^2}\right\}
\end{aligned}$$

In the center of momentum frame we have

$$2p_1\cdot k_1 = m^2-t \qquad 2p_1\cdot k_2 = m^2-u \qquad s = 4\,k^{\,2}$$

so that, by substituting the above expression for $\mathrm{tr}\,\Upsilon$ we obtain the mass independent formula for the differential cross-section for the annihilation of a particle-antiparticle pair first obtained by Paul Adrian Maurice Dirac (1930)

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{\alpha^2}{4kp}\left\{\frac{k^{\,2}+p^{\,2}\,(\,1+\sin^2\theta\,)}{k^{\,2}-p^{\,2}\cos^2\theta} - \frac{2\,p^{\,4}\sin^4\theta}{(\,k^{\,2}-p^{\,2}\cos^2\theta\,)^2}\right\}$$

according to [7] § **23** eq. (23.12) p. 282 and [6] § **88** eq. (88,13) p. 431. Let us calculate the lowest order total cross-section for the process of e^-e^+

pairs production out of quantum electromagnetic radiation in spinor QED. Actually we have

$$\sigma(\, e^- e^+ \,\leftrightarrow\, 2\gamma\,) = 2\pi \int_0^1 \mathrm{d}\cos\theta \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\rm CM} = \frac{\pi\alpha^2}{pk}\left(\frac{1}{2} + p^2 I_1 - J_2\right)$$

where

$$I_1 \equiv \int_0^1 \frac{\mathrm{d}x}{k^2 - p^2x^2} = \frac{1}{2pk}\,\ln\frac{1+\chi}{1-\chi} = \frac{1}{pk}\,\mathrm{Arcth}\,\chi > 0$$

$$I_2 \equiv \int_0^1 \frac{\mathrm{d}x}{(k^2 - p^2x^2)^2} = -\,\frac{1}{2k}\cdot\frac{\partial I_1}{\partial k} = \frac{1}{2pk^3}\left[\frac{\chi}{1-\chi^2} + \mathrm{Arcth}\,\chi\right]$$

with $\chi \equiv p/k$ $(0 \le \chi < 1)$ whereas

$$\begin{aligned} J_2 &= \int_0^1 \mathrm{d}x\,\left[\frac{p^2(1-x^2)}{k^2-p^2x^2}\right]^2 = \int_0^1 \mathrm{d}x\,\left[1 - \frac{k^2-p^2}{k^2-p^2x^2}\right]^2 \\ &= 1 - 2k^2(1-\chi^2)I_1 + k^4(1-\chi^2)^2 I_2 \\ &= 1 + \frac{1}{2}(1-\chi^2)\left(1 - \frac{3+\chi^2}{\chi}\,\mathrm{Arcth}\,\chi\right) \end{aligned}$$

so that we definitely obtain

$$\begin{aligned} \sigma_{\,2\gamma\to e^-e^+} &= \frac{\pi\alpha^2}{pk}\left[\chi\mathrm{Arcth}\chi - \frac{1}{2} - \frac{1}{2}(1-\chi^2)\left(1 - \frac{3+\chi^2}{\chi}\,\mathrm{Arcth}\chi\right)\right] \\ &= \frac{\pi\alpha^2}{pk}\left[-\,\frac{1}{2} + \frac{1}{2}\left(2+\chi^2\right) + \cdots\right] \simeq \frac{\pi\alpha^2}{2pk} \qquad (3.160) \end{aligned}$$

which becomes very large close to the threshold $\hbar k = m_e c \Leftrightarrow p = 0\,$.

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3.5 Appendix A

Here we shall first report the so called *heuristic memento derivation* of the differential cross section formula – V.B. Berestetskij, E.M. Lifšits and L.P. Pitaevskij, *Teoria quantistica relativistica*, Editori Riuniti, Roma, 1978, § **65** eq. (65,18) p. 302. From the basic formula

$$S_{f\leftarrow\iota} = \langle\, f \,|\, S \,|\, \iota\,\rangle = \delta_{\iota f} + (2\pi)^4\, i\, \delta\,(P_f' - P_\iota)\, \mathcal{M}(\,p_\iota \mapsto p_f')$$

for the scattering process $1 + 2 + \ldots + M \quad\longmapsto\quad 1' + 2' + \ldots + N'$, we immediately obtain that the transition probability over all space-time is then formally given by

$$dw_{f\leftarrow\iota} = [\,(2\pi)^4\, \delta\,(P_f' - P_\iota)\,]^{\,2}\, |\, \mathcal{M}(\,p_\iota \mapsto p_f')\,|^{\,2}\ \mathrm{d}\mathbf{P}_\iota\, \mathrm{d}\mathbf{P}_f' \tag{3.161}$$

where

$$\mathrm{d}\mathbf{P}_\iota \;=\; \prod_{j=1}^{M} [\,(2\pi)^3\, 2\omega(\mathbf{p}_j)\,]^{-1}\, \mathrm{d}\mathbf{p}_j \quad \mathrm{d}\mathbf{P}_f' \;=\; \prod_{k=1}^{N} [\,(2\pi)^3\, 2\omega(\mathbf{p}_k')\,]^{-1}\, \mathrm{d}\mathbf{p}_k'$$

The square of the δ–distribution is understood in the sense that $(2\pi)^4\, \delta^{\,(4)}(0)$ is nothing but the space-time total volume. This can be readily seen in terms of the formal identities

$$\begin{aligned} 2\pi\, \delta(p) \;&=\; \lim_{L\to\infty} \int_{-L}^{L} \mathrm{e}^{\,ipx}\, \mathrm{d}x \\ &=\; \lim_{L\to\infty}\, \frac{2}{p}\, \sin(pL) \\ \lim_{p\to 0} 2\pi\delta(p) \;&=\; \lim_{L\to\infty}\; 2L \end{aligned}$$

Notice that in natural units $\hbar = c = 1$ we have the following canonical engineering dimensions:

$$[\,|\,\mathbf{k}\,\rangle\,] = \mathrm{cm}\,, \qquad [\,\mathcal{M}(\,p_\iota \mapsto p'_f)\,] = \mathrm{cm}^{\,M+N-4}$$
$$[\,|\,\mathcal{M}(\,p_\iota \mapsto p'_f)\,|^2\,\delta\,(P'_f - P_\iota)\,] = \mathrm{cm}^{2M+2N-4}$$

It follows that the transition probability per element of space-time is

$$\mathrm{d}W_{f\leftarrow\iota} = (2\pi)^4\,\delta\,(P'_f - P_\iota)\,|\,\mathcal{M}(\,p_\iota \mapsto p'_f)\,|^2\;\mathrm{d}\mathbf{P}_\iota\,\mathrm{d}\mathbf{P}'_f \tag{3.162}$$

In scattering experiments one is usually interested in the differential cross section of two incident particles into many. For two incident particles we have

$$\begin{aligned}\mathrm{d}W_{f\leftarrow\iota} &= \frac{1}{16\pi^2}\,\delta\,(P'_f - p_1 - p_2)\,|\,\mathcal{M}(\,p_1,p_2 \mapsto p'_f)\,|^2\,\frac{\Delta\mathbf{p}_1\Delta\mathbf{p}_2}{\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)}\;\mathrm{d}\mathbf{P}'_f \\ &= \tfrac{1}{4}\,(2\pi)^4\,\delta\,(P'_f - p_1 - p_2)\,\frac{|\,\mathcal{M}(\,p_1,p_2 \mapsto p'_f)\,|^2}{\omega(\mathbf{p}_1)\,\omega(\mathbf{p}_2)\,\Delta V_1\,\Delta V_2}\;\mathrm{d}\mathbf{P}'_f\end{aligned}$$

where $\Delta\mathbf{p}_1\,,\ \Delta\mathbf{p}_2$ are very small regions in momentum space centered around $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively. Then, taking eq. (3.77) into account, we eventually find the differential cross-section in the form

$$\begin{aligned}&\mathrm{d}\sigma\,(1+2\ \longmapsto\ 1'+2'+\ldots+N') \\ &\equiv\ \frac{(2\pi)^3}{\Delta\mathbf{p}_1}\cdot\frac{(2\pi)^3}{\Delta\mathbf{p}_2}\cdot\frac{\mathrm{d}W_{f\leftarrow\iota}}{v_{\mathrm{rel}}} = \frac{1}{\beta_{\mathrm{rel}}}\;\Delta V_1\,\Delta V_2\,\mathrm{d}W_{f\leftarrow\iota} \\ &=\ \tfrac{1}{4}\,[\,(p_1\cdot p_2)^2 - m_1^2m_2^2\,]^{-1/2}\;(2\pi)^4\,\delta\,(P'_f - P_\iota) \\ &\times\ |\,\mathcal{M}(\,p_1,p_2 \mapsto p'_f)\,|^2\prod_{k=1}^{N}[\,(2\pi)^3\,2\omega(\mathbf{p}'_k)\,]^{-1}\,\mathrm{d}\mathbf{p}'_k\end{aligned} \tag{3.163}$$

in perfect agreement with eq. (3.71).

Chapter 4

Radiative Corrections

4.1 Evaluation of Feynman Integrals

The Feynman rules lead to loop integrals which are admittedly poorly defined divergent expressions. The divergences we have to face with are caused by the bad large wave number behavior of the loop integrating functions: these are the ultraviolet divergences of quantum field theories. In the case of field theory models involving mass-less particles, *e.g.* photons, another kind of low 4−momentum non-integrable singularities indeed appear, the so named infrared divergences, which will not be treated in the present context.

The simplest examples arise immediately in the real scalar self-interacting field theory and in the Yukawa spinor-meson field theory. Specifically, from the lowest order expressions (2.27) and (2.28) for the 2−point and 4−point connected Green's functions in momentum space, after truncation of the external free propagators, we can pick out the divergent parts

$$\Sigma_1(0) \simeq \frac{\lambda}{2}\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{i}{\ell^2 - m^2 + i\varepsilon}$$

$$\prod_{\jmath=1}^{4}(\,k_\jmath^2 - m^2\,)\,\widetilde{G}_c^{(4)}(k_1,k_2,k_3,k_4) + i\lambda \simeq \tfrac{1}{2}\,(-\,i\,\lambda\,)^2 \sum_{(\imath\,\jmath)} (2\pi)^4$$
$$\times \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\,\frac{i}{\ell_1^2 - m^2 + i\varepsilon}\int \frac{\mathrm{d}^4\ell_2}{(2\pi)^4}\,\frac{i}{\ell_2^2 - m^2 + i\varepsilon}\,\delta\,(\,\ell_1 + \ell_2 - k_\imath - k_\jmath\,)$$

where the sum $(\,\imath\,\jmath\,)$ runs over the three pairs $(12),(13),(14)$, which turns out to be related, as we shall see further on, to the first radiative correction to the self-interaction coupling λ.

Finally, the first two divergent terms arising in the perturbative expansion of the fermion determinant (2.36) in the Yukawa field theory formally read

$$\Gamma_2(\emptyset) \stackrel{\text{def}}{=} -iy\int\frac{\mathrm{d}^4p}{(2\pi)^4}\;\mathrm{tr}\,\frac{i}{\not p - M + i\varepsilon} \qquad (\text{spinor tadpole}) \qquad (4.1)$$

$$\Sigma_2(\not k) \stackrel{\text{def}}{=} i\,y^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\;\mathrm{tr}\,\frac{i}{\not p - M + i\varepsilon}\cdot\frac{i}{\not p + \not k - M + i\varepsilon} \qquad (4.2)$$

which corresponds to the so called π^0 meson (neutral pion) self-energy.

In order to give a precise mathematical meaning to the above listed ill defined integral expressions, we have to introduce from the outset some kind of REGULARIZATION PROCEDURE, the aim of which is to build up absolutely convergent loop integrals. Thus, I will briefly review here below the most commonly used ultraviolet regulators, by applying the latter ones to the above written paradigmatic simple divergent loop integrals.

4.1.1 Cut−Off Regularization

This is the most intuitive and physically motivated ultraviolet regulator, that I have already employed in the discussion of the vacuum energy and the cosmological constant – see the First Semester Course. It is based upon the rather plausible belief that the validity of the principles of modern quantum field theory, as well as the classical theory of gravitation based upon Einstein general relativity, can not be pushed beyond a very high energy scale such as the Planck scale

$$\begin{aligned} M_P &= \sqrt{\hbar c/G_N} = 1.220\,90(9)\times 10^{19}\ \mathrm{GeV/c^2} \\ &= 2.176\,45(16)\times 10^{-11}\ \mathrm{g} \approx 20\ \mathrm{pg} \\ G_N &= 6.674\,2(10)\times 10^{-8}\ \mathrm{cm^3\ g^{-1}\ s^{-2}} \\ &= 6.7087(10)\times 10^{-39}\ \hbar c\ (\mathrm{GeV}/c^2)^{-2} \end{aligned}$$

where G_N is the Newtonian gravitational constant. The matter is that at the Plack scale some new physics is expected to govern the quantum gravity phenomena, a realm which does not seem to be experimentally accessible nowadays [1]. If we trust in general relativity and in quantum field theory up

[1] Nonetheless, at the moment there are indeed big efforts in trying to detect ultra-high energy cosmic rays (UHECR), ultrahigh energy $\gamma-$ray bursts as well as black hole particle physics effects at LHC, due to the possible existence of space-time extra-dimensions.

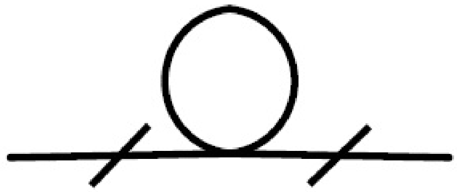


Figure 4.1: the amputated scalar tadpole

to the Planck scale but not beyond, it turns out to be quite natural to cut-off the loop integrations at a very high wave number of the order

$$K \simeq \sqrt{c^3/8\pi\hbar G_N} = (8\pi G_N)^{-1/2}$$

Consider therefore the cut-off regularized scalar tadpole

$$\begin{aligned}
\texttt{reg}\,\Sigma_1(0;K) &= (-i)(k^2-m^2)^2\,\widetilde{G}_c^{(2)}(k;K) \\
&\stackrel{\text{def}}{=} \frac{\lambda}{2}\int\frac{\mathrm{d}\boldsymbol{\ell}}{(2\pi)^4}\,\theta(K^2-\boldsymbol{\ell}^2)\int_{-\infty}^{\infty}\mathrm{d}\ell_0\,\frac{i}{\ell_0^2-\boldsymbol{\ell}^2-m^2+i\varepsilon} \\
&= \frac{\lambda}{4}\int\frac{\mathrm{d}\boldsymbol{\ell}}{(2\pi)^3}\,\theta(K^2-\boldsymbol{\ell}^2)\left(\boldsymbol{\ell}^2+m^2\right)^{-1/2} \\
&= \frac{\lambda}{8\pi^2}\int_0^K \mathrm{d}\ell\,\ell^2\,\left(\ell^2+m^2\right)^{-1/2} \\
&= \frac{\lambda}{m}\cdot\frac{\mathrm{d}}{\mathrm{d}m}\int_0^K\frac{\mathrm{d}\ell}{8\pi^2}\,\ell^2\sqrt{\ell^2+m^2} = \frac{\lambda}{m}\cdot\frac{\mathrm{d}\langle\rho\rangle}{\mathrm{d}m} \qquad (4.3)
\end{aligned}$$

where the quantity $\langle\rho\rangle$ is nothing but the vacuum energy density that I have already introduced in the first part of these notes and which is related to the so called zero−point energy of the quantized scalar field. From [14] eq. **2.272**3. p. 105 we obtain

$$\begin{aligned}
\texttt{reg}\,\Sigma_1(0;K) &= \frac{\lambda K^2}{16\pi^2}\left\{\sqrt{1+m^2/K^2}-\frac{m^2}{K^2}\right. \\
&\times\left.\left[\ln\frac{K}{m}+\ln\left(1+\sqrt{1+m^2/K^2}\right)\right]\right\} \qquad (4.4)
\end{aligned}$$

and from the explicit expression of the vacuum energy density

$$\langle\, \rho \,\rangle = \frac{K^4}{16\pi^2} + \frac{K^2 m^2}{16\pi^2} - \frac{m^4}{32\pi^2}\left[\ln\frac{K}{m} - \frac{1}{4} + \ln 2 + O\left(\frac{m}{K}\right)^2\right]$$

we eventually understand the ultraviolet cut−off regularized scalar tadpole as follows : namely,

$$\texttt{reg}\,\Sigma_1(0\,;\,K\,) \stackrel{\mathrm{def}}{=} \frac{\lambda}{16\pi^2}\left\{K^2 - m^2\left[\ln\frac{K}{m} - \frac{1}{2} + \ln 2 + O\left(\frac{m}{K}\right)^2\right]\right\} \tag{4.5}$$

which means a quadratic divergence and a logarithmic divergence at the Planck scale. It follows that if we remove the zero-point vacuum energy through normal ordering then the divergent scalar tadpole disappears.

The spinor tadpole can be treated in the very same way: namely,

$$\begin{aligned}
\texttt{reg}\,\Gamma_2(\emptyset) &= \frac{g}{(2\pi)^4}\int \mathrm{d}\mathbf{p}\;\theta\,(\,K^2 - \mathbf{p}^2\,)\int_{-\infty}^{\infty}\frac{\mathrm{tr}\;(\,\not p + M\,)\,\mathrm{d}p_0}{p_0^2 - \mathbf{p}^{\,2} - M^{\,2} + i\varepsilon}\\
&= \frac{4gM}{i\pi^2}\int_0^K \frac{p^2\,\mathrm{d}p}{\sqrt{(p^2 + M^2)}}\\
&= \frac{4gM}{2i\pi^2}\left\{K\,\sqrt{K^2 + M^2} - M^2\,\ln\left[\frac{K}{M} + \sqrt{1 + \frac{K^2}{M^2}}\,\right]\right\}\\
&= \frac{2gM}{i\pi^2}\left\{K^2 - M^2\left[\ln\frac{K}{M} - \frac{1}{2} + \ln 2 + O\left(\frac{M}{K}\right)^2\right]\right\}
\end{aligned} \tag{4.6}$$

which means, as expected, that we have again a quadratic divergence and a logarithmic divergence at the Planck scale.

4.1.2 Pauli−Villars Regularization

This method for the ultraviolet regularization of fermion cycles has been introduced in the quantum field theory by one of the main Father Founders and one of his students

Wolfgang Pauli & Felix Villars
(Swiss Federal Institute of Technology, Zurich, Switzerland)
On the Invariant Regularization in Relativistic Quantum Theory
Review of Modern Physics **21**, 434 - 444 (1949) [Issue 3 − July 1949]

See also : Claude Itzykson and Jean−Bernard Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980, § **7-1-1** p. 319 ; Nicolai Nicolaievic Bogoliubov and D.M. Shirkov,

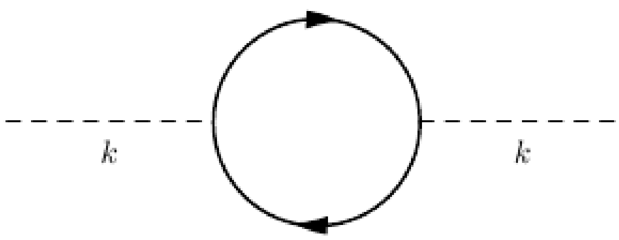


Figure 4.2: the pion self-energy 1-loop diagram

Introduction to the theory of quantized fields, Interscience Publishers, New York, 1959, § 30.2, p. 364 ; Ludwig Dimitrievich Fadde'ev & Andrei Alexe'evich Slavnov, *Gauge fields. Introduction to quantum theory*, Benjamin, Reading (MA), 1980, § **4.3**, p. 131.

To implement it consider the spinor propagator in momentum space

$$S_F(p\,,\,M) \;=\; \frac{i}{\not p - M + i\varepsilon} \;=\; \frac{i\,(\not p + M)}{p^2 - M^2 + i\varepsilon}$$

and let me define the pion self-energy, or the vacuum polarization scalar, for the Yukawa theory with the Pauli−Villars regularization, *viz.*

$$\mathtt{reg}\,\Sigma(\not k\,;\Lambda) \;\stackrel{\mathrm{def}}{=}\; i\,y^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\sum_{s=0}^{S} C_s\;\mathrm{tr}\,S_F(p\,,\,M_s)\,S_F(p+k\,,\,M_s)$$

where $M_0 = M$, $C_0 = 1$ while $\{\,M_s \equiv \lambda_s\,M\,|\,\lambda_s \gg 1\ (\,s = 1,2,\ldots,S\,)\,\}$ is a collection of very large auxiliary masses. The set of constants C_s will be suitably selected, as we shall see in the sequel, in such a manner to obtain a specific and mathematically meaningful form for the ultraviolet divergences that will manifest themselves in the limit $\lambda_s \to \infty$. Further, we have denoted the collection of very large auxiliary masses M_s ($s = 1,2,\ldots,S$) by the symbol Λ . Then we get

$$\mathtt{reg}\,\Sigma(\not k\,;\Lambda) \;=\; -\;iy^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\sum_{s=0}^{S}\frac{C_s\,\mathrm{tr}\,[\,(\not p + M_s)\,(\not p + \not k + M_s)\,]}{(p^2 - M_s^2 + i\varepsilon)\,[\,(p+k)^2 - M_s^2 + i\varepsilon\,]}$$

Taking into account that we have

$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu) \;=\; g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} \;=\; 4g^{\mu\nu} \qquad\qquad \mathrm{tr}\,\gamma^\lambda \;=\; 0$$

we can write

$$\begin{aligned}&\mathtt{reg}\,\Sigma(\not k;\Lambda) \;=\\ -\;&4iy^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\sum_{s=0}^{S}\frac{C_s\,(p^2+p\cdot k+M_s^2)}{(p^2-M_s^2+i\varepsilon)\,[\,(p+k)^2-M_s^2+i\varepsilon\,]}\end{aligned}$$

It is convenient to introduce the Feynman parametric formula

$$\begin{aligned}&\frac{1}{(p^2-M_s^2+i\varepsilon)\,[\,(p+k)^2-M_s^2+i\varepsilon\,]} \;=\\ &\int_0^1\frac{\mathrm{d}x}{\{\,x\,(p^2-M_s^2)+(1-x)\,[\,(p+k)^2-M_s^2\,]+i\varepsilon\,\}^2}\\ &=\;\int_0^1\frac{\mathrm{d}x}{[\,p^2-M_s^2+2p\cdot k\,(1-x)+(1-x)\,k^2+i\varepsilon\,]^2}\end{aligned}$$

and by exchanging the integrals

$$\begin{aligned}&\mathtt{reg}\,\Sigma(\not k;\Lambda) \;=\; -\,4iy^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4p}{(2\pi)^4}\\ &\sum_{s=0}^{S}\frac{C_s\,(p^2+p\cdot k+M_s^2)}{[\,p^2-M_s^2+2p\cdot k\,(1-x)+(1-x)\,k^2+i\varepsilon\,]^2}\end{aligned}$$

Let us shift the integration variable

$$p\;\mapsto\;p'=p+k\,(1-x)$$

which yields

$$\begin{aligned}&\mathtt{reg}\,\Sigma(\not k;\Lambda) = -\,4iy^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4p'}{(2\pi)^4}\\ &\sum_{s=0}^{S}\,C_s\,\frac{M_s^2-x\,(1-x)\,k^2+p'^2-p'\cdot k\,(1-2x)}{[\,p'^2-M_s^2+x\,(1-x)\,k^2+i\varepsilon\,]^2}\end{aligned}$$

The very last term in the numerator is odd and vanishes after integration. Thus we are left with

$$\begin{aligned}&\mathtt{reg}\,\Sigma(\not k;\Lambda) \;=\\ -\;&4iy^2\int_0^1\mathrm{d}x\sum_{s=0}^{S}\,C_s\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{p^2+M_s^2-x\,(1-x)\,k^2}{[\,p^2-M_s^2-(x^2-x)\,k^2+i\varepsilon\,]^2}\end{aligned}$$

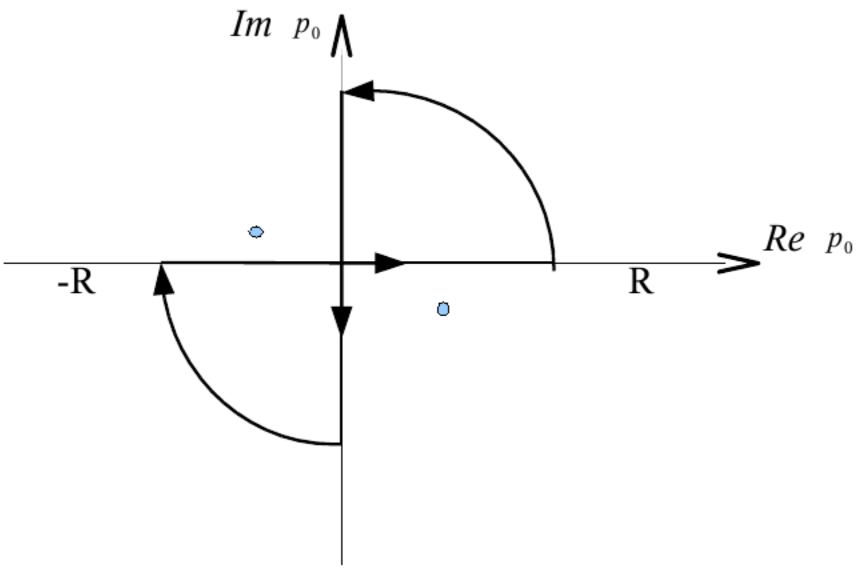


Figure 4.3: the oriented contour γ^+

Notice that, for $k^2 < 4M_s^2$, the integrating function does exhibit two double poles for each $s = 0, 1, 2, \ldots, S$ at

$$p_s^0 = \begin{cases} \sqrt{\mathbf{p}^2 + k^2\, R(x, a_s)} - i\varepsilon \\ -\sqrt{\mathbf{p}^2 + k^2\, R(x, a_s)} + i\varepsilon \end{cases} \qquad \left(a_s \equiv M_s^2 / k^2 \right)$$

because

$$k^2\, R(x, a_s) = k^2 \left(x^2 - x + a_s \right) > 0 \qquad \Leftrightarrow \qquad k^2 < 4M_s^2$$

To go further on let me first wisely perform the Wick rotation, that is, let us consider the closed oriented contour γ^+ in the complex energy plane. Note that, thanks to the causal prescription, the set of pairs of double poles $\{p_s^0 \,|\, s = 0, 1, 2, \ldots, S\}$ of the integrating function lie outside the contour γ^+ for $k^2 < M^2 = \min\{M_s^2 \,|\, s = 0, 1, 2, \ldots, S\}$. Since the contributions due to the two arcs of the large circle of radius R do vanish when $R \to \infty$ we obtain

$$\texttt{reg}\,\Sigma_E(\,k_E\,;\,\Lambda\,) = 4y^2 \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \sum_{s=0}^{S} C_s\, \frac{M_s^2 + x\,(1-x)\,k_E^2 - p_E^2}{[\,p_E^2 + M_s^2 + x\,(1-x)\,k_E^2\,]^2}$$

in which I have set $k^0 = i\,k_4\,,\ p^0 = i\,p_4$ and $p_{E\mu} = (\,\mathbf{p}\,,\,p_4\,)\,,\ p_E^2 = \mathbf{p}^2 + p_4^2\,.$

Let us now consider the Pauli–Villars regularization for the generating integral representation that reads

$$I_n\,(z_E\,,\,\Lambda) \;\equiv\; (-1)^n \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4}\;\sum_{s=0}^{S} C_s\;\frac{\exp\{\,i\,p_E\cdot z_E\,\}}{(\,p_E^2+\Delta_s^2\,)^{\,n}} \tag{4.7}$$

in which I have set for short $\Delta_s^2 \;=\; \lambda_s^2 M^2 + x\,(1-x)\,k_E^2\,.$ It follows that we can write

$$\mathtt{reg}\,\Sigma_E\,(\,k_E\,;\,\Lambda\,) \;=\; 4\,g^2\int_0^1 \mathrm{d}x\,\lim_{z_E\,\to\,0}\big(\,2\,\Delta^2\,I_2 + I_1\,\big)\,\exp\{\,i\,k_E\cdot z_E\,\}$$

where

$$\Delta^2\,I_2 \equiv \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4}\;\sum_{s=0}^{S} C_s\,\Delta_s^2\;\frac{\exp\{\,i\,p_E\cdot z_E\,\}}{(\,p_E^2+\Delta_s^2\,)^{\,2}}$$

Taking the Mellin transform we find

$$\begin{aligned}
I_n\,(z_E\,,\,\Lambda) \;&=\; \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p_E\;\sum_{s=0}^{S} C_s\;\frac{(-1)^n}{\Gamma\,(\,n\,)}\\
&\times\;\int_0^\infty \mathrm{d}t\;t^{\,n-1}\exp\{-t\,p_E^2 - t\,\Delta_s^2 + i\,p_E\cdot z_E\}\\
&=\;\frac{(-1)^n}{\Gamma(n)}\int_0^\infty \mathrm{d}t\;t^{\,n-1}\sum_{s=0}^{S} C_s\;\exp\{-\,t\,\Delta_s^2\}\\
&\times\;\frac{1}{(2\pi)^4}\int \mathrm{d}^4 p_E\;\exp\left\{-\,t\left(p_E - i\,\frac{z_E}{2t}\right)^2 - \frac{z_E^2}{4t}\right\}\\
&=\;\frac{(-1)^{\,n}}{16\pi^2\,\Gamma(n)}\int_0^\infty \mathrm{d}t\;t^{\,n-3}\sum_{s=0}^{S} C_s\;\exp\{\,-\,t\,\Delta_s^2 - z_E^2\,/\,4t\,\}\\
&=\;\frac{2\,(-1)^{\,n}}{16\pi^2\,\Gamma(n)}\sum_{s=0}^{S} C_s\;\left(\frac{2\,\Delta_s}{|\,z_E\,|}\right)^{2-n} K_{\,2-n}(\,\Delta_s\,|\,z_E\,|\,)
\end{aligned}$$

where

$$|\,z_E\,| = \sqrt{\mathbf{z}^2 + z_4^2} = \sqrt{\mathbf{z}^2 - z_0^2} = \sqrt{-\,z^{\,2}} \qquad (\,z^2 < 0\,)$$

For $n = 1$ and $z_s \equiv \Delta_s\,|\,z_E\,|$ we obtain

$$I_1\,(\,z_E\,,\,\Lambda\,) \;=\; -\left(\frac{1}{2\pi z_E}\right)^2\sum_{s=0}^{S} C_s\;z_s\,K_1\,(\,z_s\,)$$

$$
\begin{aligned}
&= \left(\frac{1}{2\pi z_E}\right)^2 \sum_{s=0}^{S} C_s\ z_s\ \frac{\mathrm{d}}{\mathrm{d}z_s}\, K_0(z_s) \\
I_2(z_E\,,\,\Lambda) &= \frac{1}{8\pi^2}\sum_{s=0}^{S} C_s\, K_0(z_s)
\end{aligned}
\tag{4.8}
$$

where K_0 is the modified Bessel function of the third kind, also named Basset−McDonald function, of order zero, the series representation of which is provided by

$$
K_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}\left[\psi(k+1) - \ln\frac{z}{2}\right]
$$

while

$$
K_1(z) = \frac{1}{z} + \sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{2k+1}\frac{1}{k!(k+1)!}\left[\ln\frac{z}{2} - \frac{1}{2}\,\psi(k+1) - \frac{1}{2}\,\psi(k+2)\right]
$$

Now if we fix the auxiliary constants $C_1, C_2, \ldots, C_s$ so that

$$
\sum_{s=0}^{S} C_s = 0 \qquad \sum_{s=0}^{S} C_s\,\lambda_s^2 = 0 \tag{4.9}
$$

then we eventually obtain a regular and finite generating integral $I_n(z_E, \Lambda)$ in the limit $z_E \to 0$, *viz.*,

$$
\begin{aligned}
I_1 + 2\Delta^2 I_2 &= \left(\frac{1}{2\pi z_E}\right)^2 \sum_{s=0}^{S} C_s\,\left[\,z_s^2\,K_0(z_s) \ - \ z_s\,K_1(z_s)\right] \\
&= \left(\frac{1}{2\pi z_E}\right)^2 \sum_{s=0}^{S} C_s\, z_s\left(z_s + \frac{\mathrm{d}}{\mathrm{d}z_s}\right) K_0(z_s) \\
&= -\,\frac{3}{4}\left(\frac{1}{2\pi}\right)^2 \sum_{s=0}^{S} C_s\,\Delta_s^2\,\ln z_s \ + \ O(z_E^2)
\end{aligned}
$$

and finally

$$
\begin{aligned}
&-\lim_{z_E\to 0}\{2\Delta^2\, I_2(z_E\,,\,\Lambda) + I_1(z_E\,,\,\Lambda)\} \ = \\
&\frac{3M^2}{16\pi^2}\sum_{s=0}^{S} C_s\left[\lambda_s^2 + x(1-x)\,\frac{k_E^2}{M^2}\right]\ln\left[\lambda_s^2 + x(1-x)\,\frac{k_E^2}{M^2}\right]
\end{aligned}
\tag{4.10}
$$

with

$$\sum_{s=1}^{S} C_s \;=\; -1 \qquad\qquad \sum_{s=1}^{S} C_s\,\lambda_s^2 \;=\; -1 \tag{4.11}$$

the minimal choice being $S = 2$ that yields

$$C_0 = \lambda_0 = 1 \qquad\qquad C_1 = \frac{\lambda_2^2 - 1}{\lambda_1^2 - \lambda_2^2} \qquad\qquad C_2 = \frac{\lambda_1^2 - 1}{\lambda_2^2 - \lambda_1^2}$$

Turning back to the invariant polarization function we obtain

$$\begin{aligned} \texttt{reg}\,\Sigma_E\,(k_E\,;\,\Lambda\,) \;&=\; -\;\; \frac{3y^2M^2}{4\pi^2}\sum_{s=0}^{S} C_s \int_0^1 \mathrm{d}x\,\left[\,\lambda_s^2 + x\,(1-x)\,\frac{k_E^2}{M^2}\,\right] \\ &\quad\times\;\; \ln\left[\,\lambda_s^2 + x\,(1-x)\,\frac{k_E^2}{M^2}\,\right] \end{aligned}$$

and thereby

$$\begin{aligned} \texttt{reg}\,\Sigma_E\,(k_E\,;\,\Lambda\,) \;&=\; -\;\; \frac{3y^2M^2}{4\pi^2}\int_0^1 \mathrm{d}x\,\left[\,1 + x\,(1-x)\,\frac{k_E^2}{M^2}\,\right] \\ &\quad\times\;\; \ln\left[\,1 + x\,(1-x)\,\frac{k_E^2}{M^2}\,\right] \\ &\quad-\;\; \frac{3y^2M^2}{4\pi^2}\sum_{s=1}^{S} C_s \int_0^1 \mathrm{d}x\,\left[\,\lambda_s^2 + x\,(1-x)\,\frac{k_E^2}{M^2}\,\right] \\ &\quad\times\;\; \ln\left[\,\lambda_s^2 + x\,(1-x)\,\frac{k_E^2}{M^2}\,\right] \end{aligned}$$

which clearly suggests how to segregate the divergent and finite parts in the limit $\lambda_s \to \infty$ ($s = 1, 2, \ldots, S$) of very large nonphysical auxiliary fermion masses. As a matter of fact, we can rewrite the very last term of the right-hand-side of the above expression in the form

$$\begin{aligned} &\int_0^1 \mathrm{d}x \sum_{s=1}^{S} C_s\lambda_s^2\left[\,1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\,\right] \ln\lambda_s^2\left[\,1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\,\right] \\ =\;\; &\sum_{s=1}^{S} C_s\,\lambda_s^2\,\ln\lambda_s^2 \int_0^1 \mathrm{d}x\left[\,1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\,\right] \\ +\;\; &\sum_{s=1}^{S} C_s\lambda_s^2 \int_0^1 \mathrm{d}x\left[\,1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\,\right] \ln\left[\,1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\,\right] \end{aligned}$$

so that

$$\sum_{s=1}^{S} C_s\,\lambda_s^2\,\ln\lambda_s^2 \int_0^1 \mathrm{d}x \left[1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\right]$$

$$= \sum_{s=1}^{S} C_s \ln\lambda_s^2 \left[\lambda_s^2 + \frac{k_E^2}{6M^2}\right]$$

$$\sum_{s=1}^{S} C_s\lambda_s^2 \int_0^1 \mathrm{d}x \left[1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\right] \ln\left[1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\right]$$

$$= \frac{k_E^2}{M^2}\sum_{s=1}^{S} C_s \int_0^1 \mathrm{d}x\, x(1-x)\left[1 + x\,(1-x)\,\frac{k_E^2}{M_s^2}\right] + O\left(\Lambda^{-4}\right)$$

$$= -\frac{k_E^2}{6M^2} + \text{evanescent terms}$$

Then we eventually come to the suggestive result

$$\texttt{reg}\,\Sigma_E\,(k_E\,;\Lambda) \;=\; \frac{y^2}{8\pi^2}\left[k_E^2 - \sum_{s=1}^{S} C_s\,\ln\lambda_s^2\left(k_E^2 + 6M_s^2\right)\right]$$
$$-\;\frac{3y^2M^2}{4\pi^2}\int_0^1 \mathrm{d}x\;\left[1 + x\,(1-x)\,\frac{k_E^2}{M^2}\right]\,\ln\left[1 + x\,(1-x)\,\frac{k_E^2}{M^2}\right]$$

and turning back to Minkowski space

$$\texttt{reg}\,\Sigma(\not k\,;\Lambda) \;=\; \frac{y^2}{8\pi^2}\;\sum_{s=1}^{S}\;C_s\,\ln\lambda_s^2\left(k^2 - 6M_s^2\right)$$
$$-\;\frac{y^2}{8\pi^2}\left[k^2 + (6M^2 - k^2)\,\ln\frac{k^2}{M^2}\right]$$
$$-\;\frac{3y^2}{4\pi^2}\int_0^1 dx\;\left[\,M^2 - x\,(1-x)\,k^2\,\right]\,\ln(x^2 - x + M^2/k^2) \quad (4.12)$$

It is convenient to rewrite the above expression in a more convenient and standard form. To this purpose let me define the quantities

$$a \equiv \frac{M^2}{k^2} \qquad\qquad R \equiv a - x + x^2 \qquad (4.13)$$

$$\bar{I}_0 = \int_0^1 \mathrm{d}x\;\ln\frac{a}{R} \qquad\qquad \bar{I}_2 = \int_0^1 \mathrm{d}x\,x\,(1-x)\,\ln\frac{a}{R} \qquad (4.14)$$

so that the vacuum polarization function for the Yukawa field theory in the Pauli−Villars regularization can be eventually cast in the simple form

$$\texttt{reg}\,\Sigma\,(\not k, M\,;\Lambda) \;=\; \frac{y^2}{8\pi^2}\;\sum_{s=1}^{S}\;C_s\,\ln\lambda_s^2\left(k^2 - 6M_s^2\right)$$

$$- \frac{y^2 k^2}{8\pi^2} - \frac{3y^2}{4\pi^2}\left(M^2\, \bar{I}_0 - k^2\, \bar{I}_2 \right) \tag{4.15}$$

Consider the finite part of the pion self-energy

$$\texttt{reg}\,\widehat{\Sigma}\,(k^2\,;M) \;=\; -\frac{y^2 k^2}{8\pi^2} - \frac{3y^2}{4\pi^2}\int_0^1 \mathrm{d}x\;[\,M^2 - k^2\,x\,(1-x)\,]\;\ln\frac{k^2 R}{M^2}$$

and let me look at the analytic structure of the integrand in the complex $s-$plane with $\Re\mathrm{e}\, s = k^2$. One can readily verify that the argument of the logarithm is always positive definite for $k^2 < 4M^2$ and $0 \le x \le 1$, while it becomes negative definite for

$$\Re\mathrm{e}\, s > 4M^2 \;\vee\; \frac{1}{2} - \frac{\beta}{2} < x < \frac{1}{2} + \frac{\beta}{2} \qquad \beta = \sqrt{1 - \frac{4M^2}{k^2}} \qquad 0 < \beta < 1$$

It follows that for $k^2 < 4M^2$ the vacuum polarization invariant function is real and analytic $\forall\, x \in [\,0,1\,]$, while the logarithm develops a branch point when $R = 0$, viz.,

$$\{k^2 = 4M^2 \;\vee\; x = \tfrac{1}{2}\} \cup \{k^2 > 4M^2 \;\vee\; x = \tfrac{1}{2} \pm \tfrac{\beta}{2}\}$$

Notice that $k^2 = 4M^2$ precisely corresponds to the threshold for a creation of a fermion-antifermion real pair, in such a manner that the complex $s-$plane has a cut just above the threshold, *i.e.* for $\Re\mathrm{e}\, s > 4M^2$. As a consequence, the imaginary part of the the vacuum polarization invariant function can be readily obtained above/below the cut by

$$\ln\left[\frac{k^2 \pm i0}{M^2}\left(x^2 - x + \frac{M^2}{k^2 \pm i0}\right)\right] = \ln\left[\frac{k^2}{M^2}\,(\,-R\,)\right] \mp i\pi \qquad (\,R < 0\,)$$

which yields

$$\begin{aligned} \Im\mathrm{m}\;\widehat{\Sigma}\,(\,k^2 \pm i0\,,M^2\,) \;&=\; \pm\,\frac{3y^2}{4\pi}\int_{(1-\beta)/2}^{(1+\beta)/2} \mathrm{d}x\;[\,M^2 - k^2\,x\,(1-x)\,] \\ &=\; \pm\,\frac{y^2 k^2}{8\pi}\left(1 - \frac{4M^2}{k^2}\right)^{3/2} \end{aligned} \tag{4.16}$$

Moreover, the parametric integrals of the Appendix eventually yield

$$\texttt{reg}\,\widehat{\Sigma}(\,k^2\,;M\,) = M^2\,\frac{3y^2}{4\pi^2}\,\bar{I}_0 \;-\; k^2\,\frac{3y^2}{4\pi^2}\,\left(\tfrac{1}{6} + \bar{I}_2\right) \tag{4.17}$$

The first addendum in the right hand side of the above equality is closely related to the invariant polarization function of Quantum Electrodynamics and can be rewritten for $0 < k^2 < 4M^2$ as

$$\frac{y^2 k^2}{4\pi^2}\left\{-\,\frac{1}{3}+\left(1+\frac{2M^2}{k^2}\right)\right.$$
$$\left.\times\left[\left(\frac{4M^2}{k^2}-1\right)^{1/2}\operatorname{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}-1\right]\right\}$$

Exercise : let me compare the second and third line of the above equality with eq. (7-9) § **7-1-1** p. 323 of the book by Claude Itzykson and Jean−Bernard Zuber, *Quantum Field Theory*, McGraw−Hill, New York, 1980. We find

$$\begin{aligned} -\,k^2\,\frac{3y^2}{4\pi^2}\,\bar{I}_2 &= \frac{y^2k^2}{2\cdot 4\pi^2}\int_0^1\frac{\mathrm{d}x}{R}\;\left(4x^4-8x^3+3x^2\right)\\ &= \frac{y^2}{4\pi}\cdot\frac{k^2}{2\pi}\left\{\frac{44}{6}-4a+\frac{4-16a+8a^2}{2}\int_0^1\frac{\mathrm{d}x}{R}\right.\\ &\left.-12-(4-12a)\int_0^1\frac{\mathrm{d}x}{R}+3+\frac{3-6a}{2}\int_0^1\frac{\mathrm{d}x}{R}\right\}\\ = \frac{y^2}{4\pi}\cdot\frac{k^2}{2\pi}&\left\{-\,\frac{5}{3}-\frac{4M^2}{k^2}+\left[\frac{M^2}{k^2}\left(1+\frac{4M^2}{k^2}\right)-\frac{1}{2}\right]\int_0^1\frac{\mathrm{d}x}{R}\right\}\end{aligned}$$

for $0 < k^2 < 4M^2$ we have

$$\int_0^1\frac{\mathrm{d}x}{R}\;=\;\frac{4}{\sqrt{\Delta}}\operatorname{arcctg}\sqrt{\Delta}\qquad\qquad \Delta=\frac{4M^2}{k^2}-1$$

and thereby

$$\begin{aligned} -\,k^2\,\frac{3y^2}{4\pi^2}\,\bar{I}_2 &= \frac{g^2k^2}{2\cdot 4\pi^2}\int_0^1\frac{\mathrm{d}x}{R}\;\left(4x^4-8x^3+3x^2\right)\;=\\ \frac{y^2}{4\pi}\cdot\frac{k^2}{2\pi}&\left\{-\,\frac{5}{3}-\frac{4M^2}{k^2}+\left[\frac{4M^2}{k^2}\,(2+\Delta)-2\right]\,\frac{1}{\sqrt{\Delta}}\operatorname{arcctg}\sqrt{\Delta}\right\}\\ &= \frac{y^2}{4\pi}\cdot\frac{k^2}{2\pi}\left\{-\,\frac{5}{3}-\frac{4M^2}{k^2}+2\left(1+\frac{2M^2}{k^2}\right)\sqrt{\Delta}\operatorname{arcctg}\sqrt{\Delta}\right\}\\ &= \frac{y^2}{4\pi}\cdot\frac{k^2}{2\pi}\left\{\frac{1}{3}+2\left(1+\frac{2M^2}{k^2}\right)\left[\left(\frac{4M^2}{k^2}-1\right)^{1/2}\right.\right.\\ &\left.\left.\times\;\operatorname{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}-1\right]\right\}\end{aligned}$$

which is in perfect agreement. *Quod Erat Demonstrandum*

It turns out that the very same first addendum in the right hand side of eq. (4.17) can be eventually recast in the form

$$M^2\,\frac{3y^2}{4\pi^2}\,\bar{I}_0=\frac{3y^2M^2}{2\pi^2}\left\{1-\left(\frac{4M^2}{k^2}-1\right)^{1/2}\operatorname{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}\right\}$$

so that putting altogether we finally obtain

$$\texttt{reg}\,\Sigma(k^2, M^2; \Lambda) = \frac{y^2}{8\pi^2} \sum_{s=1}^{S} C_s \ln \lambda_s^2 \left(k^2 - 6M_s^2 \right) \tag{4.18}$$

$$+ \frac{3y^2M^2}{2\pi^2} \left\{ 1 - \left(\frac{4M^2}{k^2} - 1 \right)^{1/2} \operatorname{arcctg} \left(\frac{4M^2}{k^2} - 1 \right)^{1/2} \right\}$$

$$- \frac{y^2k^2}{4\pi^2} \left\{ \frac{1}{3} - \left(1 + \frac{2M^2}{k^2} \right) \left[\left(\frac{4M^2}{k^2} - 1 \right)^{1/2} \operatorname{arcctg} \left(\frac{4M^2}{k^2} - 1 \right)^{1/2} - 1 \right] \right\}$$

$$= \frac{3y^2}{4\pi^2} \sum_{s=1}^{S} C_s \ln \lambda_s^2 \left(\tfrac{1}{6} k^2 - M_s^2 \right) + \frac{y^2}{\pi^2} \left(M^2 - \tfrac{1}{3} k^2 \right)$$

$$- \frac{y^2k^2}{4\pi^2} \left(\frac{4M^2}{k^2} - 1 \right)^{3/2} \operatorname{arcctg} \left(\frac{4M^2}{k^2} - 1 \right)^{1/2} \tag{4.19}$$

4.1.3 Dimensional Regularization

The technique of dimensional regularization has been invented by

Gerardus 't Hooft and Martinus Justinus Godefriedus Veltman
Regularization and renormalization of gauge fields
Nuclear Physics B **44**, 189-213 (1972)
C.G. Bollini and J.J. Giambiagi
Lowest order divergent graphs in v-dimensional space
Physics Letters B **40**, 566-568 (1972)
J. F. Ashmore
A method of gauge invariant regularization
Lettere al Nuovo Cimento **4**, 289-290 (1972)
G. M. Cicuta and E. Montaldi
Analytic renormalization via continuous space dimension
Lettere al Nuovo Cimento **4**, 329-332 (1972)

The basic idea behind this tool is very simple : by lowering the number of dimensions over which one integrates, it might happen that the divergences trivially disappear. Then we can give a precise meaning to some divergent loop integral trough the method of analytic continuation in the number of space-time dimensions D that could be eventually turned into a complex number $2\omega \in \mathbb{C}$. In so doing, the divergences appear as poles in the complex $\omega-$plane. Let me do a simple calculation to understand how this technique is at work.

Consider a $D-$dimensional Minkowski space $\mathbb{R}^{1,\,D-1}$ with $D-1$ space dimensions with negative signature and one temporal dimension of positive signature. Then we can perform the Wick rotation to calculate any Feynman integral and produce an absolutely convergent integral in a $2\omega-$dimensional Euclidean space with ω integer and sufficiently small. A paradigmatic case is actually provided by

$$I = \mu^{4-2\omega}\int\frac{\mathrm{d}^{2\omega}k_E}{(2\pi)^{2\omega}}\ (\,k_E^2+\Delta\,)^{-2}$$

with $\Delta > 0$, which is absolutely convergent for $\omega < 2$. The arbitrary mass scale μ has been introduced with a suitable power, in such manner to deal with a dimensionless quantity I. The spherical polar coordinates of $k_{E\mu}$ are $k, \phi, \theta_1, \theta_2, \ldots, \theta_{2\omega-2}$ and we have

$$\begin{cases} k_1 = k\cos\theta_1 \\ k_2 = k\sin\theta_1\cos\theta_2 \\ k_3 = k\sin\theta_1\sin\theta_2\cos\theta_3 \\ \cdots\cdots\cdots \\ k_{2\omega-1} = k\sin\theta_1\sin\theta_2\cdots\sin\theta_{2\omega-2}\cos\phi \\ k_{2\omega} = k\sin\theta_1\sin\theta_2\cdots\sin\theta_{2\omega-2}\sin\phi \end{cases}$$

with $0 \le \theta_\imath \le \pi$ for $\imath = 1, 2, \ldots, 2\omega-2$ and $0 \le \phi \le 2\pi$ while $k = |k_E| \ge 0$. It turns out that

$$\frac{\partial\,(k_1, k_2, \cdots, k_{2\omega})}{\partial\,(k, \phi, \theta_1, \cdots, \theta_{2\omega-2})} = k^{2\omega-1}(\sin\theta_1)^{2\omega-2}(\sin\theta_2)^{2\omega-3}\cdots(\sin\theta_{2\omega-2})$$

Hence we immediately obtain

$$\begin{aligned} I \;&=\; \frac{\mu^{4-2\omega}}{(2\pi)^{2\omega}}\int_0^\infty\frac{\mathrm{d}k\ k^{2\omega-1}}{(k^2+\Delta)^2} \\ &\times\;\; (2\pi)\prod_{\jmath=1}^{2\omega-2}\int_0^\pi \mathrm{d}\theta_\jmath\ (\sin\theta_\jmath)^{2\omega-\jmath-1} \end{aligned}$$

Now we have

$$\begin{aligned} &\int_0^\pi \mathrm{d}\theta_\jmath\ (\sin\theta_\jmath)^{2\omega-j-1} = 2\int_0^1 \mathrm{d}t_\jmath\ (1-t_\jmath^2)^{\omega-1-\jmath/2} \\ =\;\; &\int_0^1 \mathrm{d}y\ y^{-1/2}(1-y)^{\omega-1-\jmath/2} = B(1/2, \omega-\jmath/2) = \frac{\Gamma(\omega-j/2)\sqrt{\pi}}{\Gamma(\omega-j/2+1/2)} \end{aligned}$$

where

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

is the Euler Beta function, so that

$$
\begin{aligned}
& (2\pi) \prod_{\jmath=1}^{2\omega-2} \int_0^{\pi} \mathrm{d}\theta_\jmath \, (\sin\theta_\jmath)^{2\omega-\jmath-1} \\
&= \frac{2\pi^\omega \, \Gamma(1)\Gamma(3/2)\Gamma(2)\cdots\Gamma(\omega-1/2)}{\Gamma(3/2)\Gamma(2)\cdots\Gamma(\omega-1/2)\Gamma(\omega)} \;=\; \frac{2\pi^\omega}{\Gamma(\omega)} \qquad (4.20)
\end{aligned}
$$

and thereby

$$
\begin{aligned}
I &= \frac{2\pi^\omega \mu^{4-2\omega}}{\Gamma(\omega)\,(2\pi)^{2\omega}} \int_0^\infty k\,\mathrm{d}k \, \frac{(k^2)^{\omega-1}}{(k^2+\Delta)^2} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty \frac{\mathrm{d}q \, q^{\omega-1}}{(q+\Delta)^2} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^\omega \Gamma(\omega)} \left(-\frac{\mathrm{d}}{\mathrm{d}\Delta}\right) \int_0^\infty \frac{\mathrm{d}q \, q^{\omega-1}}{q+\Delta} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^\omega \Gamma(\omega)} \left(-\frac{\mathrm{d}}{\mathrm{d}\Delta}\right) \int_0^\infty \mathrm{d}q \, q^{\omega-1} \int_0^\infty \mathrm{d}t \, \exp\{-tq - t\Delta\} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^\omega \Gamma(\omega)} \left(-\frac{\mathrm{d}}{\mathrm{d}\Delta}\right) \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-t\Delta} \int_0^\infty \mathrm{d}q \, q^{\omega-1} \, \mathrm{e}^{-tq} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^\omega} \int_0^\infty \mathrm{d}t \, t^{1-\omega} \, \mathrm{e}^{-t\Delta} = \frac{\Gamma(2-\omega)}{(4\pi)^\omega} \left(\frac{\mu^2}{\Delta}\right)^{2-\omega} \qquad (4.21)
\end{aligned}
$$

which is legitimate in the strip $\Re\mathrm{e}\,\omega < 2$ of the complex $\omega-$plane. Expanding around $2-\omega \equiv \epsilon\,,\; 0<\epsilon<1$, we find

$$
\begin{aligned}
\Gamma(-n+\epsilon) &= \frac{(-1)^n}{n!} \left\{ \frac{1}{\epsilon} + \psi(n+1) \right. \\
&+ \left. \frac{\epsilon}{2} \left[\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1) \right] \right\} \\
&+ O\left(\epsilon^2\right) \qquad n = 0, 1, 2, \ldots \qquad (4.22)
\end{aligned}
$$

where

$$
\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln\Gamma(z)
$$

$$
\psi(n+1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \mathbf{C}
$$

$$
\psi'(n+1) = \frac{\pi^2}{6} + \sum_{\jmath=1}^{n} \jmath^{-2} \qquad [\,\psi'(1) = \pi^2/6\,]
$$

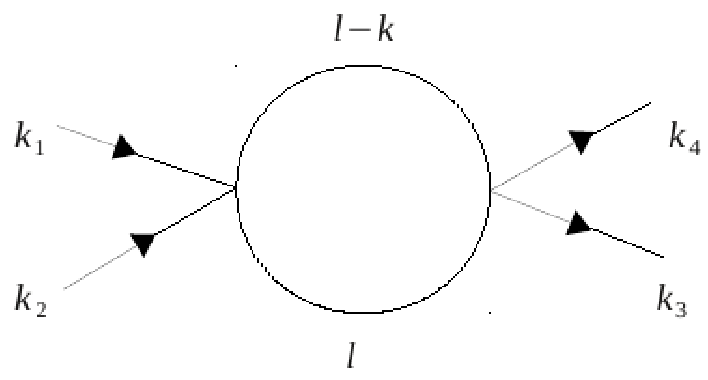


Figure 4.4: the Sweet Candy 1-loop diagram

C being the Euler−Mascheroni constant

$$\psi(1) = -\,\mathbf{C} = -\,0.5772\ldots$$

Hence we finally come to the expansion

$$\begin{aligned} I &= \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} - \mathbf{C}\ + \cdots\right)\left(1 + \epsilon\,\ln\frac{4\pi\mu^2}{\Delta}\ + \cdots\right) \\ &= \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{4\pi\mu^2}{\Delta}\right) + O(\epsilon) \end{aligned} \tag{4.23}$$

which shows that the divergence can be segregated as a simple pole in the so named $\epsilon-$expansion.

4.1.4 $\lambda\phi_4^4$ Elastic Scattering Amplitude

Consider now the so called *Sweet Candy diagram* which corresponds to any of the three diagrams contributing to the 4−point connected Green's functions in momentum space

$$J_{12}(k) \;\equiv\; \frac{1}{2i}\,(-\,i\lambda)^2 \int \frac{\mathrm{d}\ell}{(2\pi)^4}\;\frac{i}{\ell^2 - m^2 + i\varepsilon}\cdot\frac{i}{(\ell-k)^2 - m^2 + i\varepsilon} \tag{4.24}$$

where $k = k_1 + k_2$ and after setting for the sake of brevity

$$\int_\ell \;\stackrel{\text{def}}{=}\; \mu^{2\epsilon}\int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \qquad [\,\epsilon = 2 - \omega\,]$$

we come to the dimensional regularized Feynman integral

$$J_s(k) \;=\; \frac{\lambda^2}{2i}\int_\ell \frac{1}{(\,\ell^2-m^2+i\varepsilon\,)[\,(\,\ell-k\,)^2-m^2+i\varepsilon\,]} \tag{4.25}$$

the index s standing for the Mandelstam variable $s = (k_1 + k_2)^2$. Using Feynman parametric formula we get

$$J_s(k) \;=\; \frac{\lambda^2}{2i}\int_0^1 \mathrm{d}x \int_\ell \frac{1}{[\,\ell^2-m^2-2\ell\cdot k\,(1-x)+k^2\,(1-x)+i\varepsilon\,]^2}$$

and after a shift of the integration variable

$$\ell' = \ell - (1-x)\,k$$

we end up with

$$J_s(k) \;=\; \frac{\lambda^2}{2i}\int_0^1 \mathrm{d}x \int_\ell \frac{1}{[\,\ell^2-m^2+x(1-x)k^2+i\varepsilon\,]^2} \tag{4.26}$$

The denominator in the integrating function can be rewritten as

$$\left[\,\ell^2-k^2\left(x^2-x+\frac{m^2}{k^2}\right)+i\varepsilon\,\right]^2 \equiv \left[\,\ell^2-k^2\,R(x,a)+i\varepsilon\,\right]^2$$

where $a \equiv m^2/k^2$ and $0 \le x \le 1$. Since we have

$$\frac{\mathrm{d}R}{\mathrm{d}x} = 2x-1 \qquad \frac{\mathrm{d}^2R}{\mathrm{d}x^2} = 2 \qquad R(0,a) = R(1,a) = a$$

we find that the 1-parameter family of parabolas $y = R(x,a)$ exhibits the vertexes at $x = \frac{1}{2}$ with

$$R(\tfrac{1}{2},a) \;=\; a - \tfrac{1}{4} \equiv \frac{\Delta}{4} \quad \begin{cases} \ge 0 & \text{for} \quad 0 < k^2 \le 4m^2 \\ \le 0 & \text{for} \quad k^2 < 0 \,\vee\, k^2 \ge 4m^2 \end{cases}$$

in such a manner that we eventually find

$$k^2 R(x,a) > 0 \qquad \text{for} \qquad 0 \le x \le 1 \,\vee\, k^2 < 4m^2$$

i.e. below the threshold for two real scalar particles production, so that the integrand (4.26) has two double poles at

$$\ell_0 = \begin{cases} \sqrt{\boldsymbol{\ell}^2 + k^2R(x,a)} - i0^+ \\ -\sqrt{\boldsymbol{\ell}^2 + k^2R(x,a)} + i0^+ \end{cases}$$

Now we can safely perform the Wick rotation for $k^2 < 4m^2$ with $\ell_0 = i\ell_4$ and get

$$J_s(k) \;=\; \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x \int_{\ell_E} \frac{1}{[\,\ell_E^2 + m^2 + x(1-x)k_E^2\,]^2}$$

with $k_0 = ik_4$ and $k_E^2 = \mathbf{k}^2 + k_4^2 > 0$. Taking the Mellin transform

$$\begin{aligned}
J_s(k) &= \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x \int_0^\infty t\,\mathrm{d}t\;\exp\{-\,t\,[\,m^2 + x(1-x)k_E^2\,]\}\int_{\ell_E}\exp\{-\,t\,\ell_E^2\} \\
&= \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x \int_0^\infty t^{1-\omega}\,\mathrm{d}t\;\exp\{-\,t\,[\,m^2 + x(1-x)k_E^2\,]\}\,\frac{\mu^{2\epsilon}}{(4\pi)^\omega} \\
&= \frac{\lambda^2}{32\pi^2}\,\Gamma\,(\,\epsilon\,)\int_0^1 \mathrm{d}x\;\left[\frac{4\pi\mu^2}{m^2 + x\,(1-x)\,k_E^2}\right]^\epsilon \\
&= \frac{\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon} - \mathbf{C} + \int_0^1 \mathrm{d}x\;\ln\frac{4\pi\mu^2}{m^2 - x\,(1-x)\,k^2}\right\} + \;\text{irrelevant} \\
&\doteq \frac{\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon} + \psi(2) - \ln\frac{m^2}{4\pi\mu^2} - \int_0^1 \mathrm{d}x\left(1 + \ln\frac{k^2 R}{m^2}\right)\right\}
\end{aligned}$$

Notice that in the very last integral the argument of the logarithm is such that

$$\frac{k^2 R(x,a)}{m^2}\qquad \begin{cases} > 0 & \text{for}\quad k^2 < 4m^2\ \vee\ 0 \le\ x \le\ 1 \\ = 0 \quad \text{for} & \{x_\pm = \frac12 \pm \frac12\sqrt{-\Delta}\,|\,\Delta < 0\} \vee \{k^2 = 0\} \\ < 0 & \text{for}\quad x_- <\ x\ < x_+\ \vee\ k^2 > 4m^2 \end{cases}$$

Now, for example, when $0 < k^2 < 4m^2$ and $\Delta > 0$ we obtain

$$\begin{aligned}
J_s(k) &\doteq \frac{\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{4\pi\mu^2}{m^2} + \int_0^1 \frac{\mathrm{d}x}{R}\;(\,2x^2 - x\,)\right\} \\
&= \frac{\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{4\pi\mu^2}{m^2} + 2\left(1 - \sqrt{\Delta}\,\mathrm{arcctg}\sqrt{\Delta}\right)\right\} \\
&= \frac{\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon} + \psi(2) - \ln\frac{m^2}{4\pi\mu^2}\right. \\
&+\;\; \left. 1 - 2\left(\frac{4m^2}{k^2} - 1\right)^{\frac12}\;\mathrm{arcctg}\sqrt{\frac{4m^2}{k^2} - 1}\,\right\} \qquad (4.27)
\end{aligned}$$

We have to remember that in the evaluation of the 4−point connected Green's function for the self-interacting real scalar field theory, there will be three such contributions with

$$k_{12} = k_1 + k_2 \qquad\quad k_{23} = k_2 + k_3 \qquad\quad k_{31} = k_3 + k_1$$

corresponding to the s , u and t channels respectively, owing to

$$s = (k_1 + k_2)^2 \qquad t = (k_2 + k_3)^2 \qquad u = (k_1 + k_3)^2$$

all four momenta being incoming.

Unitarity Relation and the Optical Theorem.
From the definition (3.57) we can translate the requirement of unitarity for the scattering operator into the suggestive relations

$$S\,S^\dagger = S^\dagger S = \mathbb{I} = (\,\mathbb{I} + i\,T\,)\left(\,\mathbb{I} - i\,T^{\,\dagger}\,\right)$$

that yields the so called unitarity relation for the transition or reaction matrix

$$-\,i\,(\,T - T^{\,\dagger}\,) = T\,T^{\,\dagger} \tag{4.28}$$

which must be satisfied order by order in perturbation theory. Hence, going back to the LSZ reduction formulas (3.18) for *e.g.* the elastic scattering of two neutral scalar particles we obtain

$$\begin{aligned} \langle\, k_3\, k_4 \,|\, S \,|\, k_1\, k_2 \,\rangle &= (2\pi)^4\,\delta(\,k_1 + k_2 + k_3 + k_4\,) \\ &\times\ \tfrac{1}{2} \prod_{\jmath=1}^{4} \lim_{k_\jmath^{\,2} \to m^{\,2}} \left(,m^{\,2} - k_\jmath^{\,2}\,\right) \times\ \widetilde{G}_c^{(4)}\,(k_1, k_2\,;\,-\,k_3, -\,k_4) \end{aligned}$$

that yields for *e.g.* the $s-$channel with $(k_1 + k_2)^2 = s = (k_3 + k_4)^2$

$$\begin{aligned} \langle\, k_3\, k_4 \,|\, T \,|\, k_1\, k_2 \,\rangle &= (2\pi)^4\,\delta(\,k_1 + k_2 + k_3 + k_4\,)\,\mathcal{M}(k_1 k_2 \to\ k_3 k_4) \\ i\,\mathcal{M}(s) &\equiv \tfrac{1}{2} \prod_{\jmath=1}^{4} \lim_{k_\jmath^{\,2} \to m^{\,2}} \left(\,m^{\,2} - k_\jmath^{\,2}\,\right) \times\ \widetilde{G}_c^{(4)}\,(k_1, k_2\,;\,-\,k_3, -\,k_4) \\ &= -\,\tfrac{1}{2}\,i\,\{\lambda - J(s)\} + O(\lambda^3) \end{aligned}$$

where, using dimensional regularization, we understand

$$J(s) = \frac{\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{4\pi\mu^2}{m^{\,2}} - \int_0^1 \mathrm{d}x\ \ln\frac{sR}{m^{\,2}}\right\} + O(\epsilon)$$

To the lowest order we obviously have $\mathcal{M}(s) = -\frac{1}{2}\lambda$ so that the unitarity relation (4.28) is trivially satisfied. Turning to the second order in λ on the one hand we can write

$$\begin{aligned} (-\,i)\langle\, k_3\, k_4 \,|\, T - T^{\,\dagger} \,|\, k_1\, k_2 \,\rangle &= \frac{i}{2}(2\pi)^4\,\delta(\,k_1 + k_2 + k_3 + k_4\,)\,[\,J(s + i0) - J^*(s - i0)\,] \\ J(s + i0) - J^*(s - i0) &= 2i\,\Im\mathrm{m}\,J_s(k) = \frac{\lambda^2}{16\pi^2}\,(-\,i\pi)(x_+ - x_-) \\ &= (-\,i)\,\frac{\lambda^2}{16\pi}\,\sqrt{1 - \frac{4m^{\,2}}{s}} \end{aligned}$$

where $x_\pm$ are the solutions of $sR(x, a) = 0$ with $s \neq 0$, so that

$$(-\,i)\langle\, k_3\, k_4 \,|\, T - T^{\,\dagger} \,|\, k_1\, k_2 \,\rangle = \frac{\lambda^2}{32\pi}\,\sqrt{1 - \frac{4m^{\,2}}{s}}$$

On the other hand, taking equations (3.97) and (3.102) properly into account, we readily find up to the same approximation

$$
\begin{aligned}
\langle\, k_3\, k_4 \,|\, T\, T^{\dagger} \,|\, k_1\, k_2 \,\rangle &= (2\pi)^4\, \delta(\, k_1 + k_2 + k_3 + k_4 \,) \\
&\times \int Dp \int Dp'\, \langle\, k_3\, k_4 \,|\, T \,|\, p\, p' \,\rangle \langle\, p'\, p \,|\, T^{\dagger} \,|\, k_1\, k_2 \,\rangle \\
&= (2\pi)^4\, \delta(\, k_1 + k_2 + k_3 + k_4 \,) \\
&\times \int Dp \int Dp'\, (2\pi)^4\, \delta(k_1 + k_2 + p + p'\,)\ |\, \mathcal{M}(k_1 k_2 \to\ pp'\,)\,|^2
\end{aligned}
$$

where

$$
\begin{aligned}
&\int Dp \int Dp'\, \langle\, k_3\, k_4 \,|\, T \,|\, p\, p' \,\rangle \langle\, p'\, p \,|\, T^{\dagger} \,|\, k_1\, k_2 \,\rangle \\
&= \left(-\tfrac{1}{2}\,\lambda\right)^2 \int Dp \int Dp'\, (2\pi)^4\, \delta(k_3 + k_4 + p + p'\,) \\
&= \tfrac{1}{4}\,\lambda^2\, F(s, m^2) \textstyle\int \mathrm{d}\Omega / 32\pi^2 s \\
&= \frac{\lambda^2}{32\pi} \sqrt{1 - \frac{4m^2}{s}}
\end{aligned}
$$

which vindicates the unitarity relation $O(\lambda^2)$ as expected.

For the important special case of the forward scattering, in which $k_1 + k_3 = 0 = k_2 + k_4$, we obtain a simpler formula. Taking the supplementary kinematical flux factor (3.99) into account, with $\mathbf{k} = \mathbf{k}_1 = -\mathbf{k}_3$, we can write the relationships

$$
I_{\mathrm{fw}} = |\,\mathbf{k}\,| \sqrt{s} = 2\,|\,\mathbf{k}\,|\, E_{\mathrm{CM}} \ = \ \tfrac{1}{2} F(s, m)
$$

$$
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{fw}} = \frac{|\,\mathcal{M}(s)\,|^2}{64\pi^2 s} \approx \frac{\lambda^2}{4} \cdot \frac{1}{64\pi^2 s} + O(\lambda^4)
$$

so that we come to the standard form of the so called **Optical Theorem**

$$
\Im\mathrm{m}\, \mathcal{M}(k_1 k_2 \to\ k_1 k_2) = F(s, m)\, \sigma(\, k_1 k_2 \to\ \text{anything}\,)
$$

This important equality connects the forward scattering amplitude with the total cross section for production of all the final states. Since the imaginary part of the forward scattering amplitude describes the absorption of the forward going wave as the beam crosses the target, it is intuitive that this quantity has to be proportional to the probability of the diffusion. This is nothing but the requirement of conservation of the probability *i.e.* the unitarity of the scattering operator. Finally, it is worthwhile to notice that to the lowest order in perturbation theory and in the non-relativistic limit, taking eq.s (3.51) and (3.52) suitably into account, we can write the non-relativistic potential the corresponds to the $\lambda\phi_4^4$ interaction: namely,

$$
\begin{aligned}
& f(\theta) = \mathrm{T}_{\mathbf{k}} = \frac{\hbar}{8\pi mc}\, \mathcal{M}(\mathbf{k}_1 \mathbf{k}_3 \to \mathbf{k}_2 \mathbf{k}_4) \simeq \frac{\hbar}{8\pi mc}\ \left(-\tfrac{1}{2}\,\lambda\right) \\
& \widetilde{V}(\mathbf{k}_3 - \mathbf{k}_1) = -\,\frac{2\pi\hbar^2}{m} \cdot \left(-\tfrac{1}{2}\,\lambda\right) \cdot \frac{\hbar}{8\pi mc} \qquad \Longleftrightarrow \qquad V(\mathbf{r}) = \frac{\lambda\hbar^3}{8m^2 c}\, \delta(\mathbf{r})
\end{aligned}
$$

that represents the so called repulsive contact interaction.

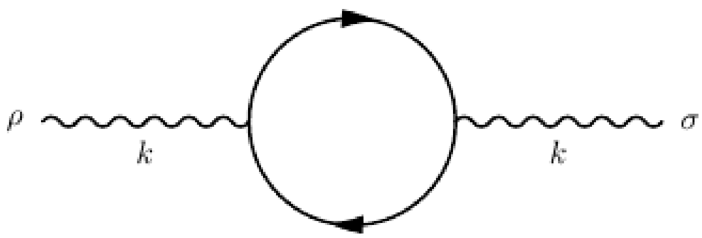


Figure 4.5: the 1-loop vacuum polarization diagram in QED

4.2 Vacuum Polarization

The calculation of the 1-loop diagram called vacuum polarization or photon self-energy is truly very instructive, since it allows to appreciate one of the most surprising consequences and predictions of perturbative quantum field theory: namely, the charge screening effect, just provided by the emission and absorption of virtual particle-antiparticle pairs, leading to a modification of the classical Coulomb interaction between point-like charges in the non-relativistic limit. It turns out that charge screening by radiative corrections is an observable effect that leads to the spectacular energy levels hyper-fine splitting of hydrogen-like atoms. On the other side, in the high energy-momentum limit, the vacuum polarization effect drives to the effective fine structure coupling, which appears to increase with energy in a way which is experimentally verified in the High Energy Particle Physics.

4.2.1 The Self-Energy Tensor in QED

Consider the photon self-energy diagram in quantum electrodynamics that gives rise to the vacuum polarization tensor which is defined to be

$$i\mathtt{reg}\,\Pi^{\rho\sigma}(k,M,\mu) = (-1)(ie)^2 \int_p \mathrm{tr}\,\gamma^{\rho} S_F(p,M)\gamma^{\sigma} S_F(p+k,M)$$

where

$$\int_p \stackrel{\mathrm{def}}{=} \mu^{2\epsilon} \int \frac{\mathrm{d}^{2\omega} p}{(2\pi)^{2\omega}} \qquad [\,\epsilon = 2-\omega\,]$$

Taking the traces

$$\begin{aligned} \mathrm{tr}\,\gamma^{\mu}\gamma^{\nu} &= g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} \;=\; 2^{\omega}\,g^{\mu\nu} \\ \mathrm{tr}\,\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu} &= 2^{\omega}\left(g^{\kappa\lambda}g^{\mu\nu} \;-\; g^{\kappa\mu}g^{\lambda\nu} \;+\; g^{\kappa\nu}g^{\lambda\mu}\right) \end{aligned}$$

we readily come to the expression

$$\begin{aligned}\texttt{reg}\,\Pi^{\mu\nu}(k,M,\mu) &= 2^{\omega}\,ie^2\int_p \frac{2p^\mu p^\nu + p^\mu k^\nu + p^\nu k^\mu}{(\,p^2-M^2+i\varepsilon\,)\,[\,(\,p+k\,)^2-M^2+i\varepsilon\,]}\\ &- 2^{\omega}\,ie^2\int_p \frac{g^{\mu\nu}\,[\,p\cdot(p+k)-M^2\,]}{(\,p^2-M^2+i\varepsilon\,)\,[\,(\,p+k\,)^2-M^2+i\varepsilon\,]}\end{aligned}$$

To calculate the integrals it is convenient to employ the Feynman parameter in such a manner that we can write

$$\begin{aligned}&\int_p \frac{2p^\mu p^\nu + p^\mu k^\nu + p^\nu k^\mu + (M^2-p^2-p\cdot k)\,g^{\mu\nu}}{(\,p^2-M^2+i\varepsilon\,)\,[\,(\,p+k\,)^2-M^2+i\varepsilon\,]}\\ &= \int_0^1 \mathrm{d}x\int_p \frac{2p^\mu p^\nu + p^\mu k^\nu + p^\nu k^\mu + g^{\mu\nu}[\,M^2-p\cdot(p+k)\,]}{[\,p^2-M^2+i\varepsilon+2p\cdot k\,(1-x)+k^2\,(1-x)\,]^2}\end{aligned}$$

A translation of the integration variable $\ell^\mu \;=\; p^\mu + x\,k^\mu$ yields

$$\int_0^1 \mathrm{d}x\int_\ell \frac{2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)k^\mu k^\nu + g^{\mu\nu}[\,M^2+x(1-x)k^2\,]}{[\,\ell^2-M^2+x(1-x)k^2+i\varepsilon\,]^2}$$

up to terms linear in ℓ which are irrelevant for we evidently get

$$\int_\ell \ell^\mu\,[\,\ell^2-k^2\,R(x\,,\,a)+i\varepsilon\,]^z \equiv 0 \qquad (\,\forall\, z\in\mathbb{C}\,)$$

owing to symmetric integration, where $R(x\,,\,a)\equiv x^2-x+a\,,\;a=M^2/k^2\,.$ Moreover, the obvious replacement holds true, owing again to symmetric integration: namely,

$$\int_p p^\mu p^\nu\,f(p^2) = \int_p g^{\mu\nu}\,\frac{p^2}{2\omega}\,f(p^2)$$

Then we can write

$$\texttt{reg}\,\Pi^{\mu\nu} = 2^{\omega}\,ie^2\Big\{g^{\mu\nu}[\,J_2+J_1\,]-k^\mu k^\nu\,J_0\Big\}$$

where

$$\begin{aligned}J_2(k,M,\mu) &= \left(\frac{1}{\omega}-1\right)\int_0^1 \mathrm{d}x\int_\ell \frac{\ell^2}{[\,\ell^2-k^2\,R(x\,,\,a)+i\varepsilon\,]^2}\\ J_0(k,M,\mu) &= \int_0^1 \mathrm{d}x\int_\ell \frac{2x(1-x)}{[\,\ell^2-k^2\,R(x\,,\,a)+i\varepsilon\,]^2}\\ J_1(k,M,\mu) &= \int_0^1 \mathrm{d}x\int_\ell \frac{M^2+x(1-x)k^2}{[\,\ell^2-k^2\,R(x\,,\,a)+i\varepsilon\,]^2}\end{aligned}$$

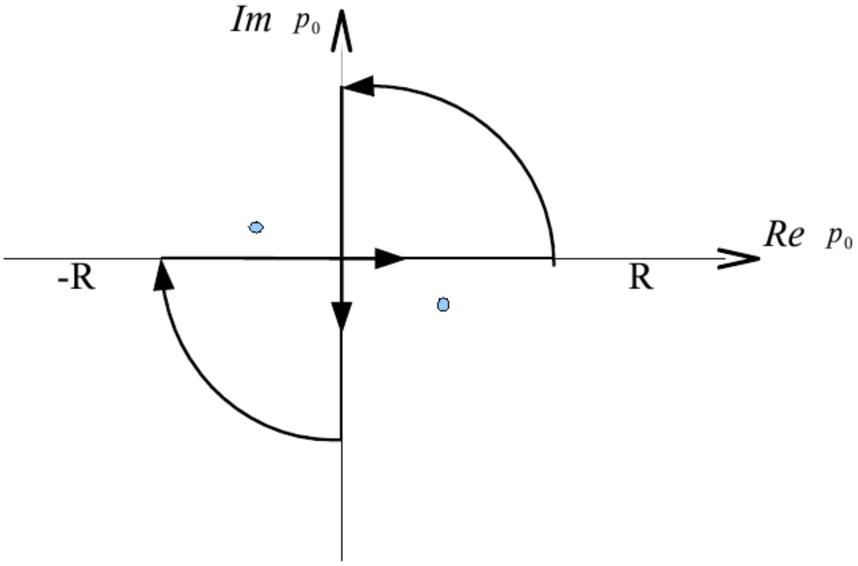


Figure 4.6: the oriented contour γ^+

As we have already seen in the previous paragraph, for $k^2 < 4M^2$ we can evaluate the above integrals by performing the Wick rotation and referring to the oriented contour γ^+ in the complex energy plane. Since the contributions of the two large arcs do vanish when $R \to \infty$, for $\ell_0 = i\,\ell_4\,, k_0 = ik_4\,,$ after setting $\ell_{E\mu} = (\ell_1, \ell_2, \ldots, \ell_{2\omega-1}, \ell_4)$ and $k_{E\mu} = (k_1, k_2, k_3, k_4)$ we get

$$
\begin{aligned}
J_2(k,M,\mu) &= -\,i\left(\frac{1}{\omega}-1\right)\int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}t\,t\,e^{-t[\,M^2+x(1-x)k_E^2\,]}\int_{\ell_E}\ell_E^2\,e^{-t\ell_E^2}\\
&= -\,i\,(4\pi)^{-\omega}\,(1-\omega)\,\mu^{4-2\omega}\int_0^1 \mathrm{d}x\int_0^\infty \mathrm{d}t\,t^{-\,\omega}\,e^{-t[\,M^2+x(1-x)k_E^2\,]}\\
&= \frac{-\,i}{16\pi^2}\,\left(4\pi\mu^2\right)^\epsilon\,\Gamma(\epsilon)\int_0^1 \mathrm{d}x\,[\,M^2+x(1-x)k_E^2\,]^{\,1-\epsilon}\\
&= \frac{-\,i}{16\pi^2}\,\Gamma(\epsilon)\int_0^1 \mathrm{d}x\,[\,M^2-x(1-x)k^2\,]\left(\frac{4\pi\mu^2}{k^2R}\right)^\epsilon\\
J_1(k,M,\mu) &= \frac{i}{16\pi^2}\,\Gamma(\epsilon)\int_0^1 \mathrm{d}x\,[\,M^2+x(1-x)k^2\,]\left(\frac{4\pi\mu^2}{R\,k^2}\right)^\epsilon\\
J_0(k,M,\mu) &= \frac{i}{16\pi^2}\,\Gamma(\epsilon)\int_0^1 \mathrm{d}x\,2x(1-x)\left(\frac{4\pi\mu^2}{R\,k^2}\right)^\epsilon
\end{aligned}
$$

Then we definitely obtain

$$\texttt{reg}\,\Pi^{\mu\nu}(k,M,\mu) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\,\texttt{reg}\,\Pi(k^2,M^2) \tag{4.29}$$

in which the invariant vacuum polarization or photon self-energy function in $\mathrm{D} = 2\omega$ space-time dimensions is provided by the parametric integral

$$\begin{aligned}\texttt{reg}\,\Pi(k^2,M^2) &= \\ (-2)\,\frac{\alpha}{\pi}\,\Gamma(\epsilon)\int_0^1 \mathrm{d}x\,x(1-x)\,&\left[\frac{2\pi\mu^2}{M^2 - x(1-x)k^2}\right]^\epsilon\end{aligned} \tag{4.30}$$

It is very important to gather the fact that dimensional regularization provides an expression for the photon self-energy tensor which satisfies the Ward identity (see in the sequel)

$$k_\mu\,\texttt{reg}\,\Pi^{\mu\nu}(k,M,\mu) = 0$$

which is nothing but the expression of gauge invariance in QED.

The Laurent expansion of the nvariant vacuum polarization function around $\omega = 2$ yields

$$\begin{aligned}\texttt{reg}\,\Pi(k^2,M^2) &= (-2)\,\frac{\alpha}{\pi}\int_0^1 \mathrm{d}x\,x(1-x) \\ &\times \left\{\frac{1}{\epsilon} - \mathbf{C} - \ln\left[\frac{M^2 - x(1-x)k^2}{2\pi\mu^2}\right] + \mathrm{O}(\epsilon)\right\} \\ &\doteq \frac{-\,\alpha}{3\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{2\pi\mu^2}{M^2}\right\} + \widehat{\Pi}(k^2,M^2)\end{aligned} \tag{4.31}$$

where the finite part of the invariant vacuum polarization function reads

$$\begin{aligned}\widehat{\Pi}(k^2,M^2) &\equiv \Pi(k^2,M^2) - \Pi(0,M^2) \\ &= \frac{-\,2\alpha}{\pi}\int_0^1 \mathrm{d}x\,x\,(1-x)\,\ln\frac{M^2}{Rk^2}\end{aligned} \tag{4.32}$$

`Consistency check`. From ref. [14] eq. **2.172** p. 81, eq. **2.174**1. p. 82 with

$$a = \frac{M^2}{k^2} > 0 \qquad b = -\,1 \qquad c = 1 \qquad \Delta = \frac{4M^2}{k^2} - 1 > 0$$

for $0 < k^2 < 4M^2$ we get

$$\begin{aligned}\widehat{\Pi}(k^2,M^2) &= \frac{\alpha}{3\pi}\int_0^1 \frac{\mathrm{d}x}{R}\left(4x^4 - 8x^3 + 3x^2\right) = -\,\frac{2\alpha}{\pi}\,\bar{I}_2 \\ &= \frac{-\,\alpha}{3\pi}\left\{\frac{5}{3} + \frac{4M^2}{k^2} - \left[\frac{M^2}{k^2}\left(1 + \frac{4M^2}{k^2}\right) - \frac{1}{2}\right]\int_0^1 \frac{\mathrm{d}x}{R}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{3\pi}\left\{\frac{1}{3}+2\left(1+\frac{2M^2}{k^2}\right)\left[\left(\frac{4M^2}{k^2}-1\right)^{1/2}\right.\right. \\
&\times \left.\left.\operatorname{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}-1\right]\right\} \qquad (4.33)
\end{aligned}
$$

in agreement with [16] § **7-1-1** eq. (7-9) p. 323. Notice that in the neighborhood of the photon mass shell $k^2 = 0$, that means in the vicinity of the light-cone, we have the behavior

$$
\frac{-\alpha}{3\pi}\int_0^1 \frac{\mathrm{d}x}{R}\left(4x^4-8x^3+3x^2\right) \sim \frac{\alpha k^2}{15\pi M^2} \qquad (k^2 \to 0)
$$

□

Now, in order to unravel the analytic structure of the invariant polarization function it is convenient to come back to the integral representation

$$
\widehat{\Pi}(k^2, M^2) = \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\,\ln\frac{k^2 R}{M^2}
$$

After setting $k^2 = \Re\mathrm{e}\, s$, it appears that the integrating function is real and analytic in $s \in \mathbb{C}$ for $k^2 < 4M^2 \vee 0 \le x \le 1$, while it exhibits a branch point when the argument of the logarithm does vanish, which drives to a cut in the complex $s-$plane when the argument of the logarithm is negative: namely,

$$
R(x,a) = x^2 - x + \frac{M^2}{k^2} \le 0
$$

$$
\Updownarrow
$$

$$
\Re\mathrm{e}\, s = k^2 \ \ge\ 4M^2 \qquad \vee \qquad \frac{1}{2}-\frac{\beta}{2} \le x \le \frac{1}{2}+\frac{\beta}{2}
$$

$$
\beta \equiv \sqrt{1-\frac{4M^2}{k^2}} \qquad [\,0 \le \beta < 1\,]
$$

As a consequence, the imaginary part of the the polarization function can be readily obtained above/below the cut by

$$
\ln\left(x^2 - x + \frac{M^2}{k^2 \pm i0}\right) = \ln(-R) \mp i\pi \qquad (\,R < 0\,)
$$

which yields

$$
\begin{aligned}
\Im\mathrm{m}\,\widehat{\Pi}(k^2 \pm i0, M^2) &= \mp\, 2\alpha \int_{(1-\beta)/2}^{(1+\beta)/2} \mathrm{d}x\, x(1-x) \\
&= \mp\,\frac{\alpha}{3}\left(1+\frac{2M^2}{k^2}\right)\sqrt{1-\frac{4M^2}{k^2}} \qquad (4.34)
\end{aligned}
$$

in accordance [2] with [22] § **7.5** eq. (7.92) p. 253 and [16] § **7-1-1** eq. (7-11) p. 323. It turns out that the discontinuity across the cut

$$\texttt{reg}\,\Pi(k^2+i0,M^2)-\texttt{reg}\,\Pi(k^2-i0,M^2)=2i\,\Im\mathrm{m}\,\widehat{\Pi}(k^2+i0,M^2)$$

does not depend upon regularization, *i.e.* it is finite. Furthermore, it is not by an accident albeit just because of the unitarity relation (4.28) that the discontinuity across the cut has exactly the same energy dependence, up to the substitution $k^2 \leftrightarrow E_{\mathrm{CM}}^2 = 4E^2$, of the cross-section (3.120) for the production of a fermion anti-fermion pair, the parameter β being precisely the fermion velocity in the center of momentum frame.

Exercise: it is quite interesting to calculate the one loop photon self-energy in QED using the UV cut-off regularisation. To this concern one has to suitably introduce a very large cut-off for the photon wave number, of the order *e.g.* of the inverse of the Planck length $K=l_P^{-1}$, where $cM_P=\hbar l_P^{-1}\simeq 10^{19}$ GeV$/c$. The starting point is again

$$\texttt{reg}\,\Pi^{\rho\sigma}(k,M,\mu)=i\,(ie)^2\int_p \mathrm{tr}\,\gamma^{\rho}S_F(p,M)\gamma^{\sigma}S_F(p+k,M)$$

where $p^{\mu}=(E,\vec{p})$ with

$$\int_p \equiv i\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi i}\int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\theta(K^2-\vec{p}^{\,2})$$

so that manifest Lorentz covariance is evidently lost in the intermediate steps of the calculations. By taking trace over gamma matrices we get

$$\begin{aligned}\texttt{reg}\,\Pi^{\mu\nu}(k,M;K) &= 4ie^2\int_p\frac{2p^{\mu}p^{\nu}+p^{\mu}k^{\nu}+p^{\nu}k^{\mu}}{(p^2-M^2+i\varepsilon)\,[\,(p+k)^2-M^2+i\varepsilon\,]}\\ &- 4ie^2\int_p\frac{g^{\mu\nu}\,[\,p\cdot(p+k)-M^2\,]}{(p^2-M^2+i\varepsilon)\,[\,(p+k)^2-M^2+i\varepsilon\,]}\end{aligned}$$

To calculate the integrals it is convenient to set

$$\texttt{reg}\,\Pi^{\mu\nu}(k,M;K)=A(k,M;K)\,k^2g^{\mu\nu}+B(k,M;K)\,k^{\mu}k^{\nu}$$

where the Lorentz invariant and dimensionless form factors A and B can be obtained by the Lorentz contractions

$$\begin{aligned}k_{\mu}\,k_{\nu}\,(k^2)^{-2}\,\texttt{reg}\,\Pi^{\mu\nu}(k,M;K) &= A(k,M;K)+B(k,M;K)\\ &= \frac{4ie^2}{(k^2)^2}\int_p\frac{2p\cdot k\,(k^2+p\cdot k)-k^2\,p\cdot k-k^2(\,p^2-M^2)}{(\,p^2-M^2+i\varepsilon\,)\,[\,(p+k)^2-M^2+i\varepsilon\,]}\\ &= \frac{4ie^2}{(k^2)^2}\int_p\frac{p\cdot k\,(k^2+2p\cdot k)}{(p^2-M^2+i\varepsilon)\,[\,(p+k)^2-M^2+i\varepsilon\,]}\\ &- \frac{4ie^2}{k^2}\int_p\frac{1}{(p+k)^2-M^2+i\varepsilon}\end{aligned}$$

[2]Notice however that the two textbooks use opposite signs, *i.e.* $\bar{\omega}(k^2,m,\Lambda)=-\,\Pi_2(q^2)$.

$$
\begin{aligned}
&= \frac{4ie^2}{(k^2)^2}\int_p \frac{p\cdot k\,[\,(k+p)^2-p^{\,2}\,]}{(\,p^2-M^2+i\varepsilon\,)\,[\,(\,p+k\,)^2-M^2+i\varepsilon\,]}\\
&- \frac{4ie^2}{k^2}\int_p \frac{1}{(\,p+k\,)^2-M^2+i\varepsilon}\\
&= \frac{4ie^2}{(k^2)^2}\int_p \left[\frac{p\cdot k}{p^2-M^2+i\varepsilon}-\frac{(k+p)\cdot k}{(\,p+k\,)^2-M^2+i\varepsilon}\right]
\end{aligned}
$$

Now, it turns out that the first addendum in the above expression vanishes owing to symmetric integration, *viz.,*

$$
\int_{-\infty}^{\infty}\frac{E\,\mathrm{d}E}{E^2-\vec{p}^{\,2}-M^2+i\varepsilon}=0
$$

$$
\int \mathrm{d}\vec{p}\;\theta(M_P^2-\vec{p}^{\,2})\,\frac{\vec{p}}{E^2-\vec{p}^{\,2}-M^2+i\varepsilon}=0
$$

Moreover, if we translate the integration variable in the second addendum of the above expression according to $\vec{p}=\vec{q}-\vec{k}\,, E=\mathcal{E}-k_0$ then we obtain

$$
\begin{aligned}
&\int \mathrm{d}\vec{q}\;\theta(K^2-\vec{q}^{\,2}-\vec{k}^{\,2}+2\vec{q}\cdot\vec{k}\,)\int_{-\infty}^{\infty}\mathrm{d}\mathcal{E}\;\frac{\mathcal{E}\,k_0-\vec{q}\cdot\vec{k}}{\mathcal{E}^{\,2}-\vec{q}^{\,2}-M^2+i\varepsilon}\\
= \;\; &-\vec{k}\cdot\int \vec{q}\;\mathrm{d}\vec{q}\;\theta(K^2-\vec{q}^{\,2}-\vec{k}^{\,2}+2\vec{q}\cdot\vec{k}\,)\int_{-\infty}^{\infty}\frac{\mathrm{d}\mathcal{E}}{\mathcal{E}^{\,2}-\vec{q}^{\,2}-M^2+i\varepsilon}\equiv 0
\end{aligned}
$$

the latter identity being evidently true, either because the Heaviside distribution vanishes or by symmetric integration when the step function equals one.

As a consequence $A+B=0$, which means that gauge invariance of the vacuum polarization tensor holds true even in 1-loop QED with UV cut-off regularisation, *viz.,*

$$
\texttt{reg}\,\Pi^{\,\mu\nu}(k,M;K)=A(k,M;K)(\,k^{\,2}g^{\,\mu\nu}-\,k^{\,\mu}k^{\,\nu}\,)
$$

so that we can definitely write

$$
\begin{aligned}
A(k,M;K) &= g_{\mu\nu}(\,3k^2)^{-1}\,\texttt{reg}\,\Pi^{\,\mu\nu}(k,M;K)\\
&= \frac{4ie^2}{3k^2}\int_p \frac{4M^2-2p\cdot k-2p^{\,2}}{(\,p^2-M^2+i\varepsilon\,)\,[\,(\,p+k\,)^2-M^2+i\varepsilon\,]}
\end{aligned}
$$

From the customary Feynman parametrisation formula we obtain

$$
A(k,M;K) = \frac{4ie^2}{3k^2}\int_0^1\mathrm{d}x\int_p \frac{4M^2-2p\cdot k-2p^{\,2}}{[\,(\,p+xk\,)^2+x(1-x)k^2-M^2+i\varepsilon\,]^2}
$$

in such a manner that, after setting $p=q-xk$ with $q=(\,E,\vec{q}\,)$ and disregarding terms which vanish owing to symmetric integration, we readily get

$$
\begin{aligned}
4M^2-2p\cdot k-2p^{\,2} &= 4M^2-2k\cdot(q-xk)-2(q^{\,2}-2xk\cdot q+x^2k^2)\\
&\backsimeq 4M^2+2x(1-x)k^2-2q^{\,2}
\end{aligned}
$$

so that

$$
A(k,M;K) = -\frac{8e^2}{3k^2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}\vec{q}}{(2\pi)^3}\;\theta(K^2-\vec{q}^{\,2}-x^2\vec{k}^{\,2}+2x\vec{q}\cdot\vec{k}\,)
$$

$$
\begin{aligned}
\times & \quad \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi i}\, \frac{M^2 + 2x(1-x)k^2 - [\,E^{\,2} - \vec{q}^{\,2} - k^2 R(x,a)\,]}{[\,E^{\,2} - \vec{q}^{\,2} + x(1-x)k^2 - M^2 + i\varepsilon\,]^2} \\
\doteq & \quad \frac{8e^2}{3k^2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\vec{q}}{(2\pi)^3}\, \theta(4K^2 - \vec{q}^{\,2}) \Big\{ I_1 - [\,M^2 + 2x(1-x)k^2\,] I_2 \Big\}
\end{aligned}
$$

where $R(x,a) \equiv x^2 - x + a$ ($a = M^2/k^2$) , whereas

$$
I_1 \;\equiv\; \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{E^{\,2} - \vec{q}^{\,2} - k^2 R(x,a) + i\varepsilon} \qquad\qquad I_2 = \frac{1}{2M} \cdot \frac{\partial I_1}{\partial M}
$$

while, in order to find a general UV upper bound which could be homogeneous for all $0 \le x \le 1$ and for any photon wave number within the UV cut-off $|\,\vec{k}\,| \le K$, we can always conservatively and safely set $|\,\vec{q}\,| \le 2K$. From the basic elementary Cauchy integral

$$
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 - \eta^2 + i0} = -\,\frac{1}{2\eta} \qquad\qquad (\,\eta > 0\,)
$$

Hence, for $k^2 < 0$ we find

$$
\begin{aligned}
I_1 &= -\,\tfrac{1}{2} \left[\, \vec{q}^{\,2} - x(1-x)k^2 + M^2 \,\right]^{-1/2} \\
I_2 &= \tfrac{1}{4} \left[\, \vec{q}^{\,2} - x(1-x)k^2 + M^2 \,\right]^{-3/2}
\end{aligned}
$$

and consequently

$$
\begin{aligned}
& A(k, M; K) = \frac{-\,e^2}{3\pi^2 k^2} \int_0^1 \mathrm{d}x \int_0^{2K} \mathrm{d}q\, q^{\,2} \\
\times & \quad \left\{ \frac{M^2 + 2x(1-x)k^2}{[\,q^{\,2} - x(1-x)k^2 + M^2\,]^{3/2}} + \frac{2}{[\,q^{\,2} - x(1-x)k^2 + M^2\,]^{1/2}} \right\} \\
\equiv & \quad -\,\frac{4\alpha}{3\pi} \left[\, A_3(k, M; K) + 2A_1(k, M; K) \,\right]
\end{aligned}
$$

Consider $u = \sqrt{\upsilon + y^2}$ where $\upsilon = [\,M^2 - x(1-x)k^2\,]/4K^2 > 0$, then from [14] eq.(**2.272**4.) p. 86 we get the elementary result

$$
\int_0^1 \mathrm{d}y\, \frac{y^2}{u^3} = \Big[-\,\frac{y}{u} + \ln(y + u) \Big]_0^1 = \frac{-1}{\sqrt{\upsilon + 1}} + \ln\big(1 + \sqrt{\upsilon + 1}\,\big) - \ln \sqrt{\upsilon}
$$

For a very large cut-off $K \to \infty$ we obtain

$$
\begin{aligned}
A_3(k, M; K) \;\; &= \;\; \int_0^1 \mathrm{d}x\, [\,a + 2x(1-x)\,] \\
&\times \;\; \left\{ -1 + 2\ln 2 + \frac{1}{2} \ln \frac{4K^2}{k^2 R(x,a)} \right\} + \text{evanescent} \\
&\doteq \;\; \left(2\ln 2 - 1 + \frac{1}{2} \ln \frac{4K^2}{M^2} \right) \left(a + \frac{1}{3} \right) \\
&- \;\; \frac{1}{2} \int_0^1 \mathrm{d}x\, [\,a + 2x(1-x)\,] \ln \frac{k^2 R}{M^2}
\end{aligned}
$$

Moreover we have from eq. (**2.272**3.) of Ref. [14]

$$\begin{aligned} 2\int_0^1 \mathrm{d}y\,\frac{y^2}{u} &= \left[\,\frac{y}{u} - \upsilon\,\ln(y+u)\right]_0^1 = \frac{1}{\sqrt{1+\upsilon}} - \upsilon\,\ln\left(1+\sqrt{\upsilon+1}\,\right) + \frac{\upsilon}{2}\,\ln\upsilon \\ &= 1 - \frac{\upsilon}{2} - \upsilon\ln 2 + \frac{\upsilon}{2}\ln\upsilon + O(\upsilon^{\,2}) \end{aligned}$$

that yields

$$\begin{aligned} 2A_1(k,M;K) &= \frac{4K^2}{k^2}\int_0^1 \mathrm{d}x\left\{1 - \left(\frac{1}{2}+\ln 2\right)\frac{M^2 - x(1-x)k^2}{4K^2}\right. \\ &+ \left.\frac{1}{2}\cdot\frac{M^2 - x(1-x)k^2}{4K^2}\ln\frac{M^2 - x(1-x)k^2}{4K^2}\right\} + \text{evanescent} \\ &= \frac{4K^2}{k^2} - \frac{1}{2}\int_0^1 \mathrm{d}x\left[1 + \ln 4 + \ln\frac{4K^2}{k^2 R(x,a)}\right] R(x,a) + \text{evanescent} \\ &\mathrel{\mathring{=}} \frac{4K^2}{k^2} - \frac{1}{2}\left(1 + 2\ln 2 + \ln\frac{4K^2}{M^2}\right)\left(a - \frac{1}{6}\right) \\ &+ \frac{1}{2}\int_0^1 \mathrm{d}x[\,a - x(1-x)\,]\ln\frac{k^2 R}{M^2} \end{aligned}$$

Summing up we eventually find out

$$\begin{aligned} \mathtt{reg}\,\Pi_{\mu\nu}(k,M;K)|_{\mathrm{UV}} &= (\,g_{\mu\nu}\,k^2 - k_\mu k_\nu\,)\,\mathtt{reg}\,\Pi(k,M;K)|_{\mathrm{UV}} \\ A(k,M;K) &\equiv -\frac{4\alpha}{3\pi}\,[\,A_3(k,M;K) + 2A_1(k,M;K)\,] \\ &= \frac{-\,\alpha}{3\pi}\left[\frac{(4K)^2}{k^2} + \ln\frac{K^2}{M^2} + 4\ln 2\left(\frac{4}{3} - \frac{M^2}{k^2}\right) - 1 - \frac{6M^2}{k^2}\right] \\ &+ \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\,x(1-x)\ln\left[1 - x(1-x)\frac{k^2}{M^2}\right] \\ &\equiv \mathtt{reg}\,\Pi(k,M;K)|_{\mathrm{UV}} \end{aligned}$$

It is interesting and instructive to make a comparison between the results of the dimensional and UV cut-off regularisations. We have

$$\begin{aligned} \mathtt{reg}\,\Pi(k,M;\mu\epsilon)|_{\mathrm{DR}} &= \frac{-\,\alpha}{3\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{2\pi\mu^2}{M^2}\right\} \\ &+ \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\,x(1-x)\ln\left[1 - x(1-x)\frac{k^2}{M^2}\right] \end{aligned}$$

It turns out that

$\varnothing$	DR	UV
divergent part	$\{-\,\alpha/3\pi\epsilon\}$	$\{(-\,\alpha/3\pi)\left[(4K)^2/k^2 + 2\ln(K/M)\right]\}$
finite arbitrary	$\{-\mathbf{C} + \ln 2\pi\mu^2/M^2\}$	$\{4\ln 2(4/3 - M^2/k^2) - 1 - 6M^2/k^2\}$
finite physical	$\widehat{\Pi}(k,M)$	$\widehat{\Pi}(k,M)$

$$\widehat{\Pi}(k,M) \equiv \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\,x(1-x)\ln\left[1 - x(1-x)\frac{k^2}{M^2}\right]$$

the physical part being distinguished by the behavior $\lim_{k^2\to 0}\widehat{\Pi}(k,M) = 0$ and by the crucial feature that it develops an imaginary part for $k^2 > 4M^2$, *i.e.* the threshold for the producion of a fermion-antifermion pair.

Exercise: let us calculate the 1-loop neutral meson π^0-self-energy of Yukawa theory with dimensional regularization and compare the result with Pauli$-$Villars regularization of eq. (4.19). Accordingly we start from the expression

$$\begin{aligned}
\mathtt{reg}\,\Sigma(k\,;\mu,\omega) &\equiv iy^{\,2}\,\mu^{\,4-2\omega}\,(2\pi)^{-\,2\omega}\int \mathrm{d}^{\,2\omega}p\;\mathrm{tr}\,[\,S_F(p)\,S_F(p+k)\,] \\
&= -\,2^{\,\omega}iy^{\,2}\int_p \frac{p^2+M^2+p\cdot k}{(p^2-M^2+i\varepsilon)[\,(p+k)^2-M^2+i\varepsilon\,]} \\
&= -\,2^{\,\omega}iy^{\,2}\int_p \frac{2M^2+p\cdot k}{(p^2-M^2+i\varepsilon)[\,(p+k)^2-M^2+i\varepsilon\,]} \\
&- \;\;2^{\,\omega}iy^{\,2}\int_p \frac{1}{(p+k)^2-M^2+i\varepsilon}
\end{aligned}$$

where the Clifford algebra is understood in 2ω space-time dimensions. By making use of the list of integrals in the Appendix we readily get

$$\begin{aligned}
\mathtt{reg}\,\Sigma(k\,;\mu,\omega) &= -\,4iy^{\,2}\left\{g^{\,\mu\nu}\,I_{\,\mu\nu}(1,1)-k^{\,\mu}\,I_{\mu}(1,1)+M^2\,I(1,1)\right\} \\
&\doteq \frac{3y^2M^2}{4\pi^2}\left(\frac{1}{\epsilon}-\mathbf{C}+I_0+\frac{2}{3}\right) \\
&- \frac{y^2k^2}{8\pi^2}\left(\frac{1}{\epsilon}-\mathbf{C}+6I_2+\frac{2}{3}\right) \\
&= \frac{3y^2}{4\pi^2}\,\left(M^2-\tfrac{1}{6}\,k^2\right)\left(\frac{1}{\epsilon}-\mathbf{C}+\ln\frac{4\pi\mu^2}{M^2}+\frac{2}{3}\right) \\
&- \frac{3y^2}{4\pi^2}\,(\bar{I}_2\,k^2-\bar{I}_0\,M^2) \\
&= \frac{3y^2}{4\pi^2}\,\left(M^2-\tfrac{1}{6}\,k^2\right)\left(\frac{1}{\epsilon}-\mathbf{C}+\ln\frac{4\pi\mu^2}{M^2}\right)-\frac{7y^{\,2}k^{\,2}}{24\pi^{\,2}} \\
&+ \frac{3y^{\,2}M^{\,2}}{2\pi^{\,2}}-\frac{y^{\,2}k^{\,2}}{4\pi^{\,2}}\left(\frac{4M^2}{k^2}-1\right)^{3/2}\mathrm{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}
\end{aligned}$$

It is important to realize that the sign of the divergent part, as well as the whole non-polynomial part, *i.e.* the very last term

$$-\frac{y^{\,2}k^{\,2}}{4\pi^{\,2}}\left(\frac{4M^2}{k^2}-1\right)^{3/2}\mathrm{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}$$

exactly coincide with the corresponding quantities (4.19) which has been obtained in the Pauli$-$Villars regularization. In other words, it turns out that the arbitrariness in the finite part of the above 1$-$loop regularized quantity does merely concern the polynomial part in momentum space, that is the local part in configuration space. This feature will represent, as we shall see further on, the key point of the renormalization procedure.

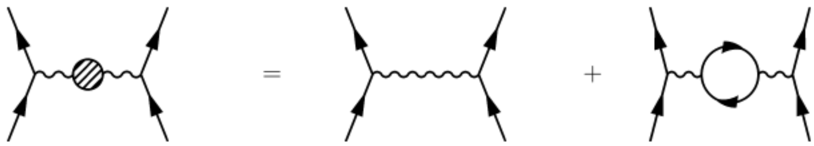


Figure 4.7: the 1-loop correction to the Coulomb potential

4.2.2 Vacuum Polarization Effects

Next let us examine how the finite part of the invariant polarization function $\widehat{\Pi}(k^2, M^2)$ does modify the electromagnetic interaction. Actually, it turns out that in the non-relativistic limit it makes sense to compute the potential $V(r)$, that will contain the modifications to the classical Coulomb potential caused by the Heisenberg Uncertainty Principle and the Theory of Relativity: the emission and absorption of virtual pairs, that is the vacuum polarization effect. Let me recall that for two incoming and two outgoing distinguishable particles of equal mass M but unlike charges $-e$ and $-Ze$ respectively, the leading order contribution to the scattering amplitude is given by eq. (3.55)

$$\bar{u}_{r'}(p')\,(ie\gamma^{\mu})\,u_r(p)\,\frac{-i\,g_{\mu\nu}}{(p-p')^2}\,\bar{u}_{s'}(q')\,(i\,Ze\gamma^{\nu})\,u_s(q)$$

Now, after inclusion of the lowest order radiative correction of Figure 4.7 we readily get

$$\bar{u}_{r'}(p')\,(ie\gamma^{\mu})\,u_r(p)\,\widetilde{G}^{(2)}_{\mu\nu}(p-p')\,\bar{u}_{s'}(q')\,(i\,Ze\gamma^{\nu})\,u_s(q) \qquad (4.35)$$

$$\widetilde{G}^{(2)}_{\mu\nu}(k) = \frac{-i}{k^2}\left[g_{\mu\nu} + \mathtt{reg}\,\Pi(k^2, M^2)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\right] \qquad (4.36)$$

As already emphasized – see equation (3.56) – the very last term in the RHS of the above equality does not contribute for $k = p - p'$, owing to the gauge invariance and the Ward identities, so that we can safely write

$$\widetilde{G}^{(2)}_{\mu\nu}(k) = \frac{g_{\mu\nu}}{ik^2}\,[\,1 + \mathtt{reg}\,\Pi(k^2, M^2)\,] \approx \frac{-\,ig_{\mu\nu}}{k^2\,[\,1 - \mathtt{reg}\,\Pi(k^2, M^2)\,]}$$

. Thus, once again, in the non-relativistic limit we can write

$$k^2 = (p-p')^2 \approx -\,\mathbf{k}^2 \qquad \bar{u}_{r'}(p')\,\gamma^0\,u_r(p) \approx 2M\,\delta_{rr'}$$

where M is the particle mass in such a manner that we can write

$$\frac{-\,iZe^2}{|\,\mathbf{p}-\mathbf{p}'\,|^2}\,2M\,\delta_{rr'}\,2M\,\delta_{ss'} = 4\pi i\,T_{\mathbf{p},\mathbf{p}'}\,2M\,\delta_{rr'}\,\delta_{ss'}$$

and consequently

$$T_{\mathbf{p}\,,\mathbf{p}'} = f(\theta) = \frac{-\,2\,M\,Z\alpha}{|\,\mathbf{p}-\mathbf{p}'\,|^2}$$

which corresponds to the repulsive Coulomb potential

$$V(r) = \frac{Ze^2}{4\pi\,r} = Z\,\frac{\alpha}{r}$$

so that

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \;|\,f(\theta)\,|^2\; = \;\frac{Z^2\,\alpha^2}{4\,M^2\,v^4\,\sin^4(\theta/2)} \qquad (\,\mathbf{p}=M\,\mathbf{v}\,)$$

which is nothing but the celebrated Rutherford classical cross-section.

Now, to the aim of taking into account the radiative corrections in the non-relativistic limit, I can write in analogy

$$\begin{aligned} \widehat{V}(r) \;&=\frac{Ze^2}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\;\frac{\mathrm{e}^{i\,\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2\,[\,1-\mathtt{reg}\,\Pi\,(-\,\mathbf{k}^2,M^2\,)\,]} \\ \simeq\;\; & \lim_{\mu\to 0}\;\frac{-\,i\,Ze^2}{4\pi^2\,r}\int_{-\infty}^{\infty}\mathrm{d}k\;\frac{k\,\mathrm{e}^{ikr}}{k^2+\mu^2}\;\left[\,1+\mathtt{reg}\,\Pi\,(-\,k^2\,,M^2)\,\right] \end{aligned} \tag{4.37}$$

where I have introduced the small photon mass μ as an infrared regulator for the Coulomb potential. To calculate this integral – see the exercise here below – we consider the complex $k-$plane and a big half-circle in the upper half-plane centered at the origin, with diameter on the real axis and very large ray $R\,\to\,\infty$. Notice however the the upper half-plane has a cut starting from $\Im\mathrm{m}\,k = 2M$ to infinity, for the invariant polarization function has a branch point at $-\,k^2\,=\,(ik)^2\,=\,4M^2$, as I have discussed before. Furthermore there is a simple pole at $k\,=\,i\mu$, leading to the Coulomb potential after removal of the infrared regulator μ . Since the real part of the finite quantity $\mathtt{reg}\,\Pi\,(-\,k^2\,,M^2)$ takes the same value on both sides of the cut, it follows that **the modifications to the Coulomb potential are solely due to the imaginary part of the invariant polarization function**, *i.e.* to its discontinuity. Consequently, it is crucial to gather that the vacuum polarization effect does not depend neither upon the divergent part of the photon self-energy tensor, nor upon the regulator mass or length scale - *e.g.* μ in the dimensional regularization - as it does for a true and real physical effect. Hence we readily obtain from eq. (4.34)

$$\begin{aligned} \delta\,\widehat{V}(r) \;&=\; \frac{Ze^2}{2\pi^2\,r}\int_{2M}^{\infty}\mathrm{d}k\;\frac{\mathrm{e}^{-\,k\,r}}{k}\;\Im\mathrm{m}\,\widehat{\Pi}\,(k^2-i\,0,M^2) \\ &=\; \frac{2Z\alpha^2}{3\pi\,r}\int_{2M}^{\infty}\mathrm{d}k\;\frac{\mathrm{e}^{-\,k\,r}}{k}\;\left(1+\frac{2M^2}{k^2}\right)\sqrt{1-\frac{4M^2}{k^2}} \end{aligned} \tag{4.38}$$

and changing the integration variable according to $k = 2M(1+\xi)$ we find

$$\delta\,\widehat{V}(r) \;=\; \frac{2Z\alpha^2}{3\pi\,r}\,\mathrm{e}^{-2Mr}\int_0^\infty \mathrm{d}\xi\;\mathrm{e}^{-2Mr\,\xi}\,f(\xi)$$

$$f(\xi) \;=\; \frac{1}{1+\xi}\left[1+\frac{1}{2(1+\xi)^2}\right]\sqrt{1-\left(\frac{1}{1+\xi}\right)^2}$$

At large distances $r \gg 1/M$ this integral is dominated by the region where $\xi \simeq 0$, so that we can safely approximate the the integrating function $f(\xi)$ with its leading value around $\xi = 0$: namely,

$$f(\xi) \simeq \tfrac{3}{2}\sqrt{1-(1-\xi)^2} \simeq \tfrac{3}{2}\sqrt{2\xi} \qquad (\,\xi \simeq 0\,)$$

in such a manner that we can eventually write

$$\begin{aligned}
\delta\,\widehat{V}(r) &\approx \frac{Z\alpha^2}{\pi\,r}\,e^{-2Mr}\int_0^\infty \mathrm{d}\xi\,e^{-2Mr\,\xi}\,\sqrt{2\xi} \\
&= \frac{Z\alpha^2}{2\pi\,r}\,e^{-2Mr}(Mr)^{-3/2}\,\Gamma(\tfrac{3}{2}) \\
&\approx \frac{Z\alpha^2}{4\,r\sqrt{\pi}}\,(Mr)^{-3/2}\,e^{-2Mr} \qquad (\,Mr \gg 1\,) \qquad (4.39)
\end{aligned}$$

Thus we see that the range of the correction term is of the order of the Compton wavelength $\hbar/Mc$ of the particles. The radiative correction to the Coulomb potential is named the Serber–Uehling potential

Robert Serber
Linear Modifications in the Maxwell Field Equations
Physical Review **48** (1935) 49 - 54 [Issue 1 – July 1935]
Edwin A. Uehling
Polarization Effects in the Positron Theory
Physical Review **48** (1935) 55 - 63 [Issue 1 – July 1935]

We can interpret the result as being due to charge screening effect. When the two point-like charges, for instance two electrons, are at the distance of the electron Compton wavelength $\lambda_e = \hbar/m_e c = 3.861\,592\,678(26) \times 10^{-13}$ m $\simeq$ 386 fm, then the continuous emission and absorption of virtual $e^- e^+$ pairs, owing to the energy-time uncertainty principle, is such that the vacuum acts as a dielectric medium, in which the apparent finite charge e is less than the divergent *bare charge* e_0 or, in other words,

$$\alpha_{\mathrm{eff}}(r,M) \sim \frac{\alpha}{1-\dfrac{\alpha}{4\sqrt{\pi}}\,(Mr)^{-3/2}\,\mathrm{e}^{-2Mr}} \qquad r \gg \frac{\hbar}{Mc} \qquad (4.40)$$

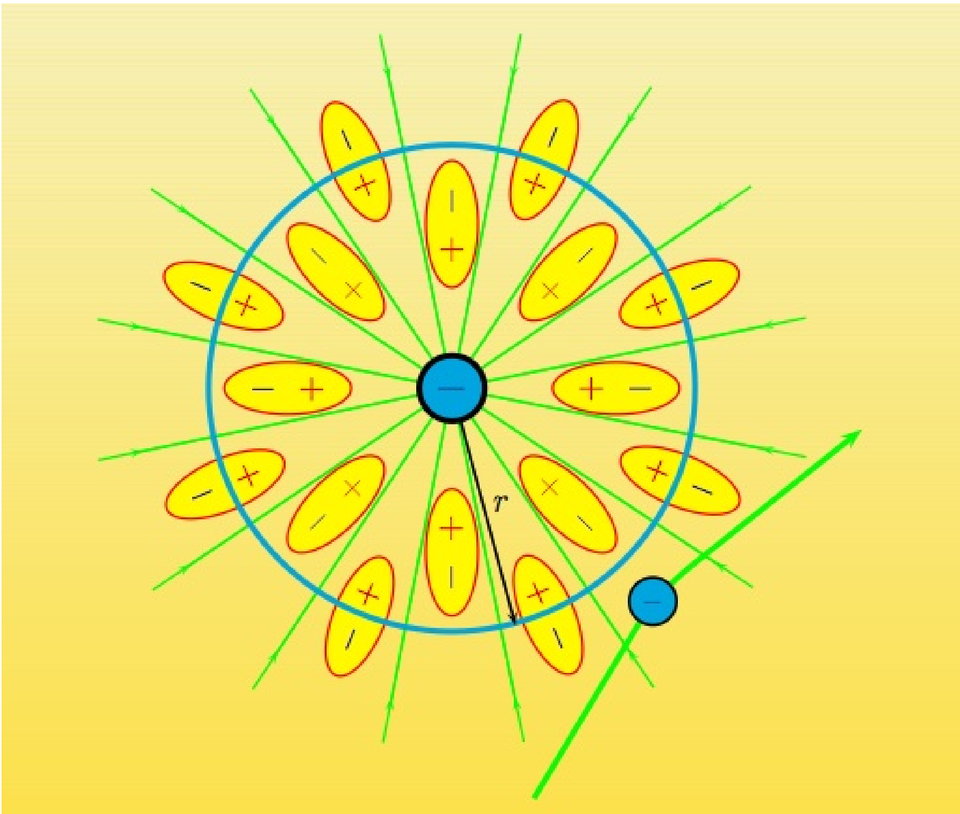


Figure 4.8: the charge screening effect in QED

At shorter and shorter distances we begin to penetrate the polarization cloud and see the bare charge, which is bigger and bigger as far as we penetrate closer and closer. This is known as the vacuum polarization effect, which can be unraveled experimentally in the famous Lamb shift discovered by Lamb and Retherford in 1947 in hydrogen

Willis E. Lamb, Jr. and Robert C. Retherford (1947)
Fine Structure of the Hydrogen Atom by a Microwave Method
The Physical Review **72**, 241-243.

The radiative correction to the Coulomb potential breaks the degeneracy between the $2S_{1/2}$ and $2P_{1/2}$ states of the relativistic hydrogen atom, as it is described by the Dirac equation. The current experimental value is

$$\frac{\Delta E}{h} = \Delta\nu_{\rm exp} = 1,057.845(9)\ \text{MHz}$$

to be compared with the theoretical value

$$\Delta\nu_{\rm th} = 1,057.86\ \text{MHz}$$

Actually the pair of very accurate experimental findings, *i.e.* the discovery of the $2S_{1/2}$ and $2P_{1/2}$ level shift in hydrogen along with the anomaly in the magnetic moment of the electron discovered by Foley and Kush in the very same year, did enormously trigger the rapid development of the Quantum Field Theory. Conversely, in the very small distances limit $\mathbf{k}^2 = -\,k^2 \gg M^2$ we can safely approximate

$$\begin{aligned}
\widehat{\Pi}(k^2, M^2) &= \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\ x(1-x)\ \ln\left[\frac{M^2 - x(1-x)k^2}{M^2}\right] \\
&\approx \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\ x\,(1-x)\ \left\{\ln\left(-\frac{k^2}{M^2}\right) + \ln[\,x\,(1-x)\,]\right\} \\
&= \frac{\alpha}{3\pi}\left\{\ln\left(\frac{-\,k^2}{\mathrm{e}^{5/3}\,M^2}\right) \ +\ O(M^2/k^2)\right\} \qquad (4.41)
\end{aligned}$$

As a consequence the effective electric coupling in the limit of very short distances becomes approximately

$$\begin{aligned}
\alpha_{\rm eff}(k, M) &\approx \alpha\left[1 + \frac{\alpha}{3\pi}\ \ln\left(\frac{-\,k^2}{\mathrm{e}^{5/3}\,M^2}\right)\right] \\
&\approx \alpha\left[1 - \frac{\alpha}{3\pi}\ \ln\left(\frac{-\,k^2}{\mathrm{e}^{5/3}\,M^2}\right)\right]^{-1} \qquad (4.42)
\end{aligned}$$

Of course, the above approximate short−distance behavior of the effective charge can be trusted as long as

$$\frac{\alpha}{3\pi}\,\ln\left(\frac{-\,k^2}{\mathrm{e}^{5/3}\,M^2}\right)\ <\ 1 \tag{4.43}$$

and leads to a singularity for

$$|\,\mathbf{k}\,| = \frac{Mc}{\hbar}\ \exp\left\{\frac{5}{6}+\frac{3\pi}{2\alpha}\right\}$$

the famous Landau−Pomerančuk[3] singularity, which corresponds to the huge energy scale

$$\hbar c\,|\,\mathbf{k}\,| = m_e\,c^2\ \exp\left\{\frac{5}{6}+\frac{3\pi}{2\alpha}\right\}\ \approx\ 10^{277}\ \mathrm{GeV}$$

However, well before we reach such an enormously large scale, much higher than the Planck scale $\sim\ 10^{19}$ GeV, the perturbative approximate equality (4.42) has to be amended by higher order corrections which are no longer negligible.

The combined vacuum polarization effects for e^-e^+ plus heavier charged leptons and quarks makes the value of $\alpha_{\text{eff}}(k)$ to increase by about 5% from $k=0$ to $k=30$ GeV, as observed in high energy experiments, with $\alpha_{\text{eff}}\,(0)\equiv\alpha$. More precisely from the Particle Data Group we find $\alpha_{\text{eff}}^{-1}\,(m_\tau)=133.452\pm 0.016$ while $\alpha_{\text{eff}}^{-1}\,(M_Z)\ =\ 127.923\pm 0.016$ with $m_\tau\ =\ 1776.84\pm 0.17$ MeV and $M_Z\ =\ 91.1876\pm 0.0021$ GeV. The idea of a distance dependent, or scale dependent or even running coupling parameter is the main result of the renormalization group invariance of perturbative renormalizable quantum field theories, as will be better focused in the sequel.

Exercise : calculate the contour integral leading to the Serber-Uehling radiative correction $\delta\,V(r)$ to the classical Coulomb potential. To this purpose, consider the functions of the complex variable $z=x+iy$

$$f(z)=\frac{z\,\mathrm{e}^{\,izr}}{z^2+\mu^2}\ g\,(z) \qquad\qquad r>0$$

$$g(z)=\int_0^1 \mathrm{d}x\ x\,(1-x)\ \ln\left[\frac{-\,z^2}{M^2}\left(x^2-x-\frac{M^2}{z^2}\right)\right]$$

in such a manner that we have

$$\delta\,V(r)=Z\,\frac{2\pi}{3r}\,\left(\frac{\alpha}{\pi}\right)^2\int_{-\infty}^{\infty}\mathrm{d}x\ f\,(x)$$

[3]L.D. Landau and I. Ja. Pomeranchuk, *Doklady Akad. Nauk USSR*, **102** (1955) 489.

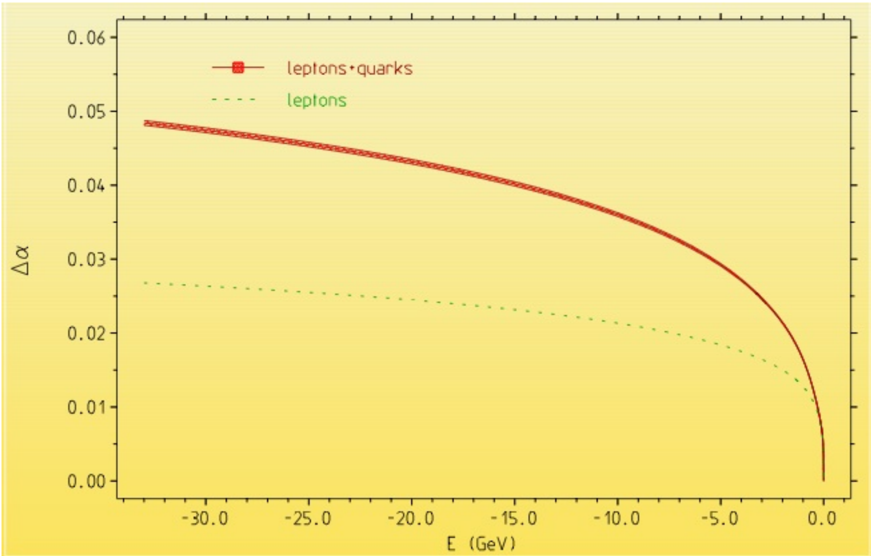


Figure 4.9: the running coupling in QED

The complex function $g(z)$ exhibits a branch point in the upper half-plane at $y = 2M$, leading to a cut along the positive imaginary axis from $2M$ to infinity. It turns out that the real part $\Re\mathrm{e}\, g(z)$ is continuous across the cut, while the imaginary part $\Im\mathrm{m}\, g(z)$ has a discontinuity across the cut which is given by

$$\Im\mathrm{m}\, g(0^+ + iy) - \Im\mathrm{m}\, g(0^- + iy) = \left(1 + \frac{2M^2}{y^2}\right)\sqrt{1 - \frac{4M^2}{y^2}} \qquad (y > 2M)$$

Consider now the oriented contour γ^+ so that

$$\oint_{\gamma^+} f(z)\,\mathrm{d}z = 2\pi i \lim_{\zeta \to i\mu} (\zeta - i\mu)\, f(\zeta) \tag{4.44}$$

The contributions from the two large arcs ($z = R\mathrm{e}^{i\theta}$, $\eta \to 0^+$) yield

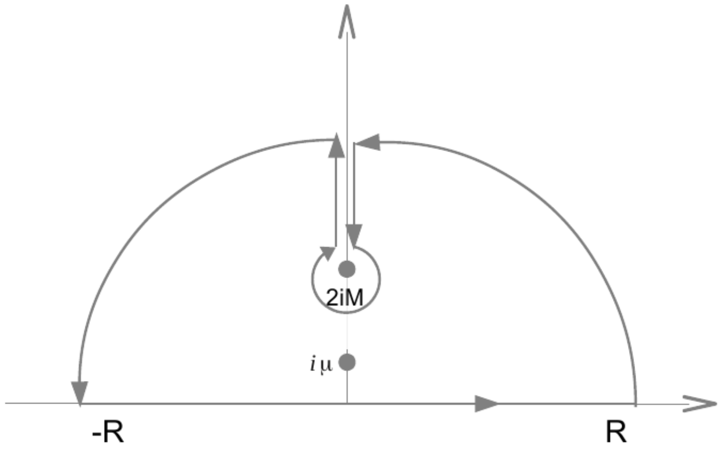
$$iR^2 \left(\int_0^{\pi/2-\eta} + \int_{\eta+\pi/2}^{\pi}\right) \left(R^2 + \mu^2 \mathrm{e}^{-2i\theta}\right)^{-1} \exp\{\, irR\cos\theta - rR\sin\theta\,\}$$
$$\times \left\{ \tfrac{1}{6}\, \ln \frac{R^2}{M^2} + \int_0^1 \mathrm{d}x\; x\,(1-x)\; \ln \left[\frac{M^2}{R^2} - x(1-x)\,\mathrm{e}^{2i\theta} \right] \right\} \mathrm{d}\theta \;\stackrel{R\to\infty}{\longrightarrow}\; 0 \tag{4.45}$$

which rapidly vanish when $R \to \infty$. Similarly, the contribution from the small circle around the branch point ($z = 2iM + \rho\,\mathrm{e}^{i\phi}$, $\eta \to 0^+$) does vanish, *viz.*,

$$i\rho \int_{\eta+\pi/2}^{\pi/2-\eta} \mathrm{d}\phi\; \mathrm{e}^{i\phi}\; f\left(2iM + \rho\,\mathrm{e}^{i\phi}\right) \;\stackrel{\rho\to 0}{\longrightarrow}\; 0 \tag{4.46}$$

For the contributions along the cut ($z = \pm\eta + iy$, $\mu < 2M < y < R$, $\eta \to 0^+$) we have

$$\int_{2M}^{R} \mathrm{d}y\; \frac{iy\,\mathrm{e}^{-ry}}{\mu^2 - y^2}\; \left[\, g(iy + 0^-) \;-\; g(iy + 0^+)\, \right] \tag{4.47}$$



The oriented contour γ^+ leading to the Serber-Uheling potential

It turns out that

$$\Re\mathrm{e}\big[\, g(iy+0^-) \;-\; g(iy+0^+)\,\big] = 0 \qquad\qquad (y > 2M)$$

while

$$\Im\mathrm{m}\big[\, g(0^+ + iy) - g(0^- + iy)\,\big] = \left(1 + \frac{2M^2}{y^2}\right)\sqrt{1 - \frac{4M^2}{y^2}} \qquad (y > 2M)$$

Putting altogether we eventually obtain

$$\begin{aligned}\lim_{R\to\infty}\lim_{\rho\to 0}\oint_{\gamma^+} f(z)\,\mathrm{d}z \;&=\; \int_{-\infty}^{\infty}\mathrm{d}x\; f(x) \\ &+\; \int_{2M}^{\infty}\mathrm{d}y\;\frac{y\,\mathrm{e}^{-\,ry}}{\mu^2 - y^2}\;\left(1 + \frac{2M^2}{y^2}\right)\sqrt{1 - \frac{4M^2}{y^2}} \\ &=\; \pi i\,\mathrm{e}^{-\,\mu r}\int_0^1 \mathrm{d}x\; x\,(1-x)\;\ln\left[\,\frac{\mu^2}{M^2}\left(x^2 - x + \frac{M^2}{\mu^2}\right)\right]\end{aligned}$$

in such a manner that in the limit $\mu \to 0$ we finally get Serber−Uehling correction to the classical Coulomb repulsive potential

$$\begin{aligned}\delta\, V(r) \;&=\; Z\,\frac{2\pi}{3r}\,\left(\frac{\alpha}{\pi}\right)^2\,\lim_{\mu\to 0}\int_{-\infty}^{\infty}\mathrm{d}x\; f(x) \\ &=\; Z\,\frac{2\pi}{3r}\,\left(\frac{\alpha}{\pi}\right)^2\int_{2M}^{\infty}\frac{\mathrm{d}y}{y}\;\mathrm{e}^{-\,ry}\;\left(1 + \frac{2M^2}{y^2}\right)\sqrt{1 - \frac{4M^2}{y^2}} \qquad (4.48)\end{aligned}$$

in accordance with eq. (4.38).

Chapter 5

Complements

In this Appendix I want to list some very useful technical tools, such as the values of many fundamental physical constants involved in Quantum Physics, the main formulæ of dimensional regularization, as well as the values of the 1-loop tensor integrals leading to the radiative corrections of basic physical quantities.

5.1 Physical Constants

Precisely known physical constants [1]

Speed of light in vacuum	$c = 299\,792\,458$ m s^{-1}
Planck constant, reduced	$\hbar = h/2\pi = 1.054\,571\,68(18) \times 10^{-34}$ J s
	$= 6.582\,119\,15(56) \times 10^{-22}$ MeV s
electron charge magnitude	$e = 4.803\,204\,41(41) \times 10^{-10}$ esu
fine-structure constant	$\alpha = e^2/4\pi\hbar c = 7.297\,352\,568(24) \times 10^{-3}$
Fermi coupling constant	$G_F/(\hbar c)^3 = 1.166\,37(1) \times 10^{-5}$ GeV^{-2}
electron mass	$m_e = 0.510\,998\,918(44)$ MeV/c^2
proton mass	$m_p = 938.272\,029(80)$ MeV/c^2
Bohr radius ($\hbar/\alpha m_e c$)	$a_\infty = 0.529\,177\,2108(18) \times 10^{-10}$ m
e^- Compton wavelength	$\lambda_e = \hbar/m_e c = 3.861\,592\,678(26) \times 10^{-13}$ m
classical electron radius	$r_e = \alpha\lambda_e = 2.817\,940\,325(28) \times 10^{-13}$ cm
Thomson cross section	$\sigma_T = \frac{8}{3}\pi r_e^2 = 0.665\,245\,873(13)$ barn
1 unit of R $\equiv \frac{4}{3}\pi(\alpha\hbar c/\sqrt{s}\,)^2 = 86.8$ nb	$\sqrt{s} = E_{CM} = 1$ GeV

Conversion factors :

$\hbar c = 197.326\ 968(17)$ MeV fm
$(\hbar c)$ 1 cm^{-1} $\simeq$ 2×10^{-14} GeV
$(\hbar c)$ 1 GeV^{-1} = 0.1973 fm 1 fm $\simeq (\hbar c)$ 5 GeV^{-1}
$(\hbar c)^2$ 1 GeV^{-2} = 0.3894 mb
1 barn = 10^{-28} m^2
$(\hbar c)$ 1 eV m^{-1} = 1.973×10^{-25} GeV^2
$(e\hbar c)$ 1 Tesla = $(e\hbar c)$ 10^4 Gauss = 5.916×10^{-25} GeV^2

Electromagnetic Units :

1 statvolt = 299.8 V
1 Gauß = 1 statvolt/cm = 299.8 V/cm = 29.98 KV/m
1 Tesla = 10^4 Gauß = 299.8 MV/m
$E_{\max}$ = 24 KV/cm = 2.4 MV/m is the typical laboratory field beyond which the Ohm law is no longer valid for metals ($\simeq$ 100 Gauß)

5.2 Dimensional Regularization

Here we list some useful identities concerning dimensional regularization. Let me first recall the the completely antisymmetric Levi Civita symbol in the four dimensional Minkowski space-time is normalized according to

$$\epsilon^{0123} = -\,\epsilon_{0123} \equiv 1 \tag{5.1}$$

in such a way that the following identity holds true in the four dimensional Minkowski space: namely,

$$\begin{aligned} \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu}^{\ \ \lambda\rho\sigma} &= g^{\nu\rho}g^{\alpha\lambda}g^{\beta\sigma} + g^{\alpha\rho}g^{\beta\lambda}g^{\nu\sigma} + g^{\beta\rho}g^{\nu\lambda}g^{\alpha\sigma} \\ &- g^{\nu\lambda}g^{\alpha\rho}g^{\beta\sigma} - g^{\alpha\lambda}g^{\beta\rho}g^{\nu\sigma} - g^{\beta\lambda}g^{\nu\rho}g^{\alpha\sigma} \end{aligned} \tag{5.2}$$

Concerning dimensional regularization, we collect here below the definitions and key properties [30] for the $2^\omega \times 2^\omega$ $\gamma-$matrices in a $2\omega-$dimensional space-time with a Minkowski signature

$$\gamma^{\mu} = \begin{cases} \bar\gamma^{\mu} & \mu = 0,1,2,3 \\ \hat\gamma^{\mu} & \mu = 4,\ldots,2\omega-4 \end{cases} \tag{5.3}$$

$$\{\bar\gamma^{\mu},\bar\gamma^{\nu}\} = 2\bar g^{\mu\nu}\,\mathbb{I} \qquad \{\hat\gamma^{\mu},\hat\gamma^{\nu}\} = 2\hat g^{\mu\nu}\,\mathbb{I} \qquad \{\bar\gamma^{\mu},\hat\gamma^{\nu}\} = 0 \tag{5.4}$$

$$\|\,\bar g\,\| = \mathrm{diag}\,(+,-,-,-) \qquad \|\,\hat g\,\| = -\,\hat{\mathbb{I}} \tag{5.5}$$

$$\gamma_5 \equiv i\bar\gamma^0\bar\gamma^1\bar\gamma^2\bar\gamma^3 \qquad \gamma_5^2 = \mathbb{I} \qquad \{\bar\gamma^{\mu},\gamma_5\} = 0 = [\,\hat\gamma^{\mu},\gamma_5\,] \tag{5.6}$$

where $\mathbb{I}$ denotes the identity $2^\omega \times 2^\omega$ square matrix, whereas $\hat{\mathbb{I}}$ denotes the identity matrix in the $2\omega - 4$ dimensional Euclidean space. Taking all the above listed equations into account, it is not difficult to check the following trace formulæ :

$$\mathrm{tr}\,(\gamma^{\mu}\gamma^{\nu}) = g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} = 2^{\omega}\,g^{\mu\nu} \tag{5.7}$$

$$2^{-\omega}\mathrm{tr}\,\left(\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\right) = g^{\kappa\lambda}\,g^{\mu\nu} - g^{\kappa\mu}\,g^{\lambda\nu} + g^{\kappa\nu}\,g^{\lambda\mu} \tag{5.8}$$

$$\begin{aligned} 2^{-\omega}\mathrm{tr}\,\left(\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= g^{\kappa\lambda}\,g^{\mu\sigma}\,g^{\nu\rho} - g^{\kappa\lambda}\,g^{\mu\rho}\,g^{\nu\sigma} - g^{\kappa\mu}\,g^{\lambda\sigma}\,g^{\nu\rho} \\ &+ g^{\kappa\mu}\,g^{\lambda\rho}\,g^{\nu\sigma} + g^{\kappa\nu}\,g^{\lambda\sigma}\,g^{\mu\rho} - g^{\kappa\nu}\,g^{\lambda\rho}\,g^{\mu\sigma} \\ &+ g^{\lambda\mu}\,g^{\kappa\sigma}\,g^{\nu\rho} - g^{\lambda\mu}\,g^{\kappa\rho}\,g^{\nu\sigma} - g^{\lambda\nu}\,g^{\kappa\sigma}\,g^{\mu\rho} \\ &+ g^{\lambda\nu}\,g^{\kappa\rho}\,g^{\mu\sigma} - g^{\mu\nu}\,g^{\kappa\rho}\,g^{\lambda\sigma} + g^{\mu\nu}\,g^{\kappa\sigma}\,g^{\lambda\rho} \\ &+ g^{\kappa\nu}\,g^{\lambda\mu}\,g^{\rho\sigma} - g^{\kappa\mu}\,g^{\lambda\nu}\,g^{\rho\sigma} + g^{\kappa\lambda}\,g^{\mu\nu}\,g^{\rho\sigma} \\ \mathrm{tr}\,\left(\bar\gamma^{\kappa}\bar\gamma^{\lambda}\hat\gamma^{\mu}\hat\gamma^{\nu}\right) &= 2^{\omega}\,\bar g^{\kappa\lambda}\,\hat g^{\mu\nu} \\ \mathrm{tr}\,\left(\gamma_5\bar\gamma^{\mu}\bar\gamma^{\lambda}\bar\gamma^{\rho}\bar\gamma^{\nu}\right) &= -\,i\,2^{\omega}\epsilon^{\mu\lambda\rho\nu} \\ \mathrm{tr}\,\left(\gamma_5\bar\gamma^{\mu}\bar\gamma^{\lambda}\bar\gamma^{\rho}\bar\gamma^{\nu}\bar\gamma^{\sigma}\bar\gamma^{\tau}\right) &= i\,2^{\omega}\left(\epsilon^{\nu\sigma\tau\mu}\,\bar g^{\lambda\rho} + \epsilon^{\nu\sigma\tau\rho}\,\bar g^{\lambda\mu} + \epsilon^{\mu\lambda\rho\sigma}\,\bar g^{\nu\tau}\right) \\ &- i\,2^{\omega}\left(\epsilon^{\nu\sigma\tau\lambda}\,\bar g^{\mu\rho} + \epsilon^{\mu\lambda\rho\nu}\,\bar g^{\sigma\tau} + \epsilon^{\mu\lambda\rho\tau}\,\bar g^{\nu\sigma}\right) \end{aligned} \tag{5.9}$$

Traces involving an odd number of Dirac matrices do vanish.

Remark : in $d = 2n\,,\ n \in \mathbb{N}$, the standard representation of the Dirac matrices has dimension 2^n, whereas in the dimensional regularization the Dirac matrices are infinite dimensional. Nevertheless, if we set $\mathrm{tr}\mathbf{1} \equiv f(\omega)$, it is not necessary to choose $f(\omega) = 2^\omega$. It is usually convenient to set $f(\omega) = f(2) = 4\,,\ \forall\omega \in \mathbf{C}$ [see J. Collins, *Renormalization*, Cambridge University Press (1984) p. 84]. We can definitely agree on that.

5.3 Glossary: 1-Loop Integrals

General Feynman parametric formula

$$
\begin{aligned}
D_1^{-a_1} D_2^{-a_2} \cdots D_k^{-a_k} &= \frac{\Gamma(a_1+a_2+\cdots+a_k)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_k)} \\
\int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \cdots &\int_0^1 \mathrm{d}x_k \, \delta(1-x_1-x_2-\cdots-x_k) \\
x_1^{a_1-1}\cdots x_k^{a_k-1} \, (x_1 D_1 + x_2 D_2 + &\cdots + x_k D_k)^{-a_1-a_2-\cdots-a_k}
\end{aligned} \tag{5.10}
$$

Definitions :

$$
R(x,a) = x^2 - x + \frac{m^2}{k^2} = x^2 - x + a \qquad \Delta = -1 + \frac{4m^2}{k^2}
$$

$$
\int_p = \mu^{4-2\omega} \int \frac{\mathrm{d}^{2\omega} p}{(2\pi)^{2\omega}}
$$

$$
I(r,s) = \int_p (p^2 - m^2 + i\varepsilon)^{-r} [(p-k)^2 - m^2 + i\varepsilon]^{-s}
$$

$$
I^{\mu}(r,s) = \int_p \frac{p^\mu}{(p^2 - m^2 + i\varepsilon)^r [(p-k)^2 - m^2 + i\varepsilon]^s}
$$

$$
I^{\mu\nu}(r,s) = \int_p \frac{p^\mu p^\nu}{(p^2 - m^2 + i\varepsilon)^r [(p-k)^2 - m^2 + i\varepsilon]^s}
$$

$$
I^{\mu\nu\rho}(r,s) = \int_p \frac{p^\mu p^\nu p^\rho}{(p^2 - m^2 + i\varepsilon)^r [(p-k)^2 - m^2 + i\varepsilon]^s}
$$

$$
I^{\mu\nu\rho\sigma}(r,s) = \int_p \frac{p^\mu p^\nu p^\rho p^\sigma}{(p^2 - m^2 + i\varepsilon)^r [(p-k)^2 - m^2 + i\varepsilon]^s}
$$

Parametric Integrals

$$
\begin{aligned}
\int_0^1 \frac{\mathrm{d}x}{R} &= \frac{4}{\sqrt{\Delta}} \operatorname{arcctg}\sqrt{\Delta} && \text{for} \quad 0 < k^2 < 4m^2 \\
&= -4 && \text{for} \quad k^2 = 4m^2 \\
&= \frac{-4}{\sqrt{-\Delta}} \operatorname{Arcth}\sqrt{-\Delta} \\
&= \frac{2}{\sqrt{-\Delta}} \ln \frac{\sqrt{-\Delta}-1}{1+\sqrt{-\Delta}} && \text{for} \quad k^2 > 4m^2 \vee k^2 < 0
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
I_0 &\equiv \int_0^1 \mathrm{d}x \;\ln\left(\frac{4\pi\mu^2}{R\,k^2}\right) \\
&= \ln\frac{4\pi\mu^2}{k^2} - \int_0^1 \mathrm{d}x \;\ln R \\
&= \ln\frac{4\pi\mu^2}{m^2} + \int_0^1 \mathrm{d}x \;\frac{2x^2 - x}{R} \\
&= 2 + \ln\frac{4\pi\mu^2}{m^2} + \frac{1}{2}\,(1-4a)\int_0^1 \frac{\mathrm{d}x}{R} \qquad (5.12) \\
I_1 &\equiv \int_0^1 \mathrm{d}x \;x \ln\left(\frac{4\pi\mu^2}{R\,k^2}\right) \\
&= 1 + \frac{1}{2}\,\ln\frac{4\pi\mu^2}{m^2} + \frac{1}{4}\,(1-4a)\int_0^1 \frac{\mathrm{d}x}{R} \qquad (5.13) \\
I_2 &\equiv \int_0^1 \mathrm{d}x \;x(1-x)\ln\left(\frac{4\pi\mu^2}{R\,k^2}\right) \\
&= \frac{1}{6}\,\ln\frac{4\pi\mu^2}{m^2} - \frac{1}{6}\int_0^1 \frac{\mathrm{d}x}{R}\;(\,4x^4 - 8x^3 + 3x^2\,) \\
&= \frac{5}{18} + \frac{1}{6}\,\ln\frac{4\pi\mu^2}{m^2} + \frac{2a}{3} + \frac{1-2a-8a^2}{12}\int_0^1 \frac{\mathrm{d}x}{R} \qquad (5.14) \\
& \quad a = m^2/k^2
\end{aligned}
$$

Scalar Integrals

$$
I(2,0) \;=\; I(0,2) \;=\; \frac{i}{16\pi^2}\,\Gamma(2-\omega)\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega} \qquad (5.15)
$$

$$
\lim_{\omega\to 2}\; m^2\, I(3,0) \;=\; -\,\frac{i}{32\pi^2} \qquad (5.16)
$$

$$
\begin{aligned}
I(1,1) &= \frac{i}{16\pi^2}\,\Gamma(2-\omega)\int_0^1 \mathrm{d}x\;\left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&\doteq \frac{i}{16\pi^2}\left\{\frac{1}{\epsilon} - \mathbf{C} + I_0\right\} \qquad (5.17)
\end{aligned}
$$

$$
I(2,1) \;=\; I(1,2) \;=\; -\,\frac{i}{16\pi^2}\int_0^1 \frac{\mathrm{d}x}{R\,k^2}\;x \qquad (5.18)
$$

$$
I(2,2) \;=\; \frac{i}{16\pi^2}\int_0^1 \frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\;x(1-x) \qquad (5.19)
$$

$$
I(3,1) \;=\; I(1,3) \;=\; \frac{i}{32\pi^2}\int_0^1 \frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\;x^2 \qquad (5.20)
$$

Vector Integrals

$$
\begin{aligned}
I_\nu(1,1) &= \frac{i}{16\pi^2}\, k_\nu\, \Gamma(2-\omega) \int_0^1 \mathrm{d}x\; x \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&\doteq \frac{i\,k_\nu}{32\pi^2} \left\{ \frac{1}{\epsilon} - \mathbf{C} + 2I_1 \right\} && (5.21)\\
I_\nu(2,1) &= -\frac{ik_\nu}{16\pi^2} \int_0^1 \frac{\mathrm{d}x}{R\,k^2}\; x(1-x) && (5.22)\\
I_\nu(1,2) &= -\frac{ik_\nu}{16\pi^2} \int_0^1 \frac{\mathrm{d}x}{R\,k^2}\; x^2 && (5.23)\\
I_\nu(2,2) &= \frac{ik_\nu}{16\pi^2} \int_0^1 \frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\; x^2\,(1-x) && (5.24)\\
I_\nu(3,1) &= \frac{ik_\nu}{32\pi^2} \int_0^1 \frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\; x^2(1-x) && (5.25)\\
I_\nu(1,3) &= \frac{ik_\nu}{32\pi^2} \int_0^1 \frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\; x^3 && (5.26)
\end{aligned}
$$

Rank Two Tensor Integrals

$$
\begin{aligned}
I_{\lambda\rho}(2,0) &= -\frac{i}{16\pi^2}\; m^2\, g_{\lambda\rho}\, \frac{\Gamma(2-\omega)}{2-2\omega} \left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega} && (5.27)\\
I_{\lambda\rho}(0,2) &= I_{\lambda\rho}(2,0) \;+\; k_\lambda k_\rho\, I(2,0) && (5.28)\\
I_{\lambda\rho}(3,0) &= \frac{i}{16\pi^2}\; g_{\lambda\rho}\, \frac{\Gamma(2-\omega)}{2\cdot 2!} \left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega} && (5.29)\\
\lim_{\omega\to 2}\; m^2\, I_{\lambda\rho}(4,0) &= \frac{-\,i}{192\pi^2}\; g_{\lambda\rho} && (5.30)\\
I_{\lambda\nu}(1,1) &= \frac{i}{16\pi^2}\, \Gamma(2-\omega) \int_0^1 \mathrm{d}x\; \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&\times \left\{ x^2 k_\lambda\, k_\nu - \frac{g_{\lambda\nu}}{2\omega-2} \left[\, x(1-x)k^2 - m^2 \,\right] \right\} \\
&\doteq \frac{i}{48\pi^2}\, k_\lambda\, k_\nu \left\{ \frac{1}{\epsilon} - \mathbf{C} + 3\,[\, I_1 - I_2 \,] \right\} \\
&- \frac{i}{32\pi^2}\, k^2 g_{\lambda\nu} \left\{ \frac{1}{6\epsilon} - \frac{\mathbf{C}}{6} + \frac{1}{6} + I_2 \right\} \\
&+ \frac{i}{32\pi^2}\, m^2 g_{\lambda\nu} \left\{ \frac{1}{\epsilon} - \mathbf{C} + 1 + I_0 \right\} && (5.31)
\end{aligned}
$$

$$
\begin{aligned}
I^{\lambda\nu}(2,1) &= \frac{i}{32\pi^2}\, g^{\lambda\nu}\,\Gamma(2-\omega)\int_0^1 \mathrm{d}x\ x\ \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&- \frac{i}{16\pi^2}\, k^\lambda\, k^\nu \int_0^1 \frac{\mathrm{d}x}{R\,k^2}\, x^2\,(1-x) \\
&\doteq \frac{i}{64\pi^2}\, g^{\lambda\nu}\left\{\frac{1}{\epsilon} - \mathbf{C} + 2I_1\right\} - \frac{i}{16\pi^2}\, k^\lambda\, k^\nu \int_0^1 \frac{\mathrm{d}x}{Rk^2}\,(1-x)\,x^2
\end{aligned}
\tag{5.32}
$$

$$
\begin{aligned}
I^{\lambda\nu}(1,2) &= \frac{i}{32\pi^2}\, g^{\lambda\nu}\,\Gamma(2-\omega)\int_0^1 \mathrm{d}x\ x\ \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&- \frac{i}{16\pi^2}\, k^\lambda\, k^\nu \int_0^1 \frac{\mathrm{d}x}{R\,k^2}\, x^3 \\
&\doteq \frac{i}{64\pi^2}\, g^{\lambda\nu}\left\{\frac{1}{\epsilon} - \mathbf{C} + 2I_1\right\} - \frac{i}{16\pi^2}\, k^\lambda\, k^\nu \int_0^1 \frac{\mathrm{d}x}{Rk^2}\, x^3
\end{aligned}
\tag{5.33}
$$

$$
I^{\lambda\nu}(2,2) = \frac{-i}{32\pi^2}\left\{ g^{\lambda\nu}\int_0^1 \frac{\mathrm{d}x}{Rk^2}\, x(1-x) - 2k^\lambda k^\nu \int_0^1 \frac{\mathrm{d}x}{[\,Rk^2\,]^2}\, x^3\,(1-x)\right\}
\tag{5.34}
$$

$$
I^{\lambda\nu}(3,1) = \frac{-i}{64\pi^2}\left\{ g^{\lambda\nu}\int_0^1 \frac{\mathrm{d}x}{Rk^2}\, x^2 - 2k^\lambda k^\nu \int_0^1 \frac{\mathrm{d}x}{[\,Rk^2\,]^2}\, x^2(1-x)^2\right\}
\tag{5.35}
$$

Rank Three Tensor Integrals

$$
\begin{aligned}
I^{\lambda\nu\rho}(1,1) &= \frac{i}{16\pi^2}\,\Gamma(2-\omega)\int_0^1 \mathrm{d}x\ \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega}\ (2\omega-2)^{-1} \\
&\times\ \left\{\left[\,xm^2 - x^2(1-x)k^2\,\right](g^{\lambda\nu}\,k^\rho + g^{\nu\rho}\,k^\lambda + g^{\rho\lambda}\,k^\nu) \right.\\
&+\ \left.(1-x)^3 k^\lambda\, k^\nu\, k^\rho\right\}
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
I^{\lambda\nu\rho}(2,2) &= \frac{i}{32\pi^2}\left\{ 2\,k^\lambda\,k^\nu\,k^\rho \int_0^1 \frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\ x^4\,(1-x)\right. \\
&- \left.(g^{\lambda\nu}\,k^\rho + g^{\nu\rho}\,k^\lambda + g^{\rho\lambda}\,k^\nu)\int_0^1 \frac{\mathrm{d}x}{R\,k^2}\ x^2\,(1-x)\right\}
\end{aligned}
\tag{5.37}
$$

$$
\begin{aligned}
I^{\lambda\nu\rho}(3,1) &= \frac{i}{64\pi^2}\left\{2\,k^\lambda\,k^\nu\,k^\rho\int_0^1\frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\;x^3\,(1-x)^2\right.\\
&- \left.(g^{\lambda\nu}\,k^{\,\rho}+g^{\nu\rho}\,k^\lambda+g^{\rho\lambda}\,k^\nu)\int_0^1\frac{\mathrm{d}x}{R\,k^2}\;x^2\,(1-x)\right\} \quad (5.38)
\end{aligned}
$$

Rank Four Tensor Integrals

$$
\begin{aligned}
I_{\lambda\rho\sigma\tau}(4,0) &= \frac{i}{384\pi^2}\,\Gamma(2-\omega)\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega}\\
&\times\;(g_{\lambda\rho}\,g_{\sigma\tau}+g_{\lambda\sigma}\,g_{\tau\rho}+g_{\lambda\tau}\,g_{\rho\sigma})\;\;; \quad (5.39)\\
I^{\lambda\nu\rho\sigma}(2,2) &= \frac{i}{64\pi^2}\,\Gamma(2-\omega)\,\left(g^{\lambda\nu}\,g^{\rho\sigma}+g^{\nu\rho}\,g^{\lambda\sigma}+g^{\rho\lambda}\,g^{\nu\sigma}\right)\\
&\times\;\int_0^1\mathrm{d}x\;x(1-x)\left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega}\\
&+\;\frac{i}{16\pi^2}\,k^\lambda\,k^\nu\,k^{\,\rho}\,k^\sigma\int_0^1\frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\;x^5\,(1-x)\\
&-\;\left(g^{\nu\lambda}\,k^{\,\rho}\,k^\sigma+\text{ cycl. perm.}\right)\;\frac{i}{32\pi^2}\int_0^1\frac{\mathrm{d}x}{R\,k^2}\;x^3\,(1-x)\\
&\doteq\;\frac{i}{384\pi^2}\,(g^{\lambda\nu}\,g^{\rho\sigma}+g^{\nu\rho}\,g^{\lambda\sigma}+g^{\rho\lambda}\,g^{\nu\sigma})\left\{\frac{1}{\epsilon}-\gamma+6I_2(\xi)\right\}\\
&+\;\frac{i}{16\pi^2}\,k^\lambda\,k^\nu\,k^{\,\rho}\,k^\sigma\int_0^1\frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\;x^5\,(1-x)\\
&-\;\left(g^{\nu\lambda}\,k^{\,\rho}\,k^\sigma+\text{ cycl. perm.}\right)\\
&\times\;\frac{i}{32\pi^2}\int_0^1\frac{\mathrm{d}x}{R\,k^2}\;x^3\,(1-x) \quad (5.40)
\end{aligned}
$$

$$
\begin{aligned}
I^{\lambda\nu\rho\sigma}(3,1) &= \frac{i}{128\pi^2}\,\Gamma(2-\omega)\,\left(g^{\lambda\nu}\,g^{\rho\sigma}+g^{\nu\rho}\,g^{\lambda\sigma}+g^{\rho\lambda}\,g^{\nu\sigma}\right)\\
&\times\;\int_0^1\mathrm{d}x\;x^2\left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega}\\
&+\;\frac{i}{32\pi^2}\,k^\lambda\,k^\nu\,k^{\,\rho}\,k^\sigma\int_0^1\frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\;x^4\,(1-x)^2\\
&-\;\left(g^{\nu\lambda}\,k^{\,\rho}\,k^\sigma+\text{ cycl. perm.}\right)\;\frac{i}{64\pi^2}\int_0^1\frac{\mathrm{d}x}{R\,k^2}\;x^2(1-x)^2
\end{aligned}
$$

$$
\begin{aligned}
&\doteq \frac{i}{384\pi^2}\left(g^{\lambda\nu}\, g^{\rho\sigma} + g^{\nu\rho}\, g^{\lambda\sigma} + g^{\rho\lambda}\, g^{\nu\sigma}\right) \\
&\times \left\{\frac{1}{\epsilon} - \mathbf{C} + 3I_1 - 3I_2\right\} \\
&+ \frac{i}{32\pi^2}\, k^\lambda\, k^\nu\, k^\rho\, k^\sigma \int_0^1 \frac{\mathrm{d}x}{[\,R\,k^2\,]^2}\ x^4(1-x)^2 \\
&- \left(g^{\nu\lambda}\, k^\rho\, k^\sigma + \text{ cycl. perm.}\right) \int_0^1 \frac{\mathrm{d}x}{R\,k^2}\ x^2(1-x)^2 \qquad (5.41)
\end{aligned}
$$

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