Supplement of Theoretical Physics
Advances in Quantum Field Theory

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December 11, 2019

Contents

1 **Quantum Effective Action** 4
- 1.1 Quantum Effective Action 4
  - 1.1.1 Classical Fields 4
  - 1.1.2 Proper Vertexes 8
- 1.2 Spontaneous Symmetry Breaking 15
  - 1.2.1 Goldstone's Bosons and Goldstone's Theorem 16
  - 1.2.2 Symmetry Breaking by Radiative Corrections 21
  - 1.2.3 Symmetry Breaking by the Higgs Mechanism 23
  - 1.2.4 The Schwinger-Dyson Equations 30
  - 1.2.5 Euclidean Effective Action 33
  - 1.2.6 1-Loop Correction to the Electron Self-Energy 34
  - 1.2.7 1-Loop Correction to the QED Vertex 38

2 **Renormalization** 44
- 2.1 Divergences of Feynman Diagrams 44
  - 2.1.1 Power Counting Criterion 44
  - 2.1.2 Renormalization 53
  - 2.1.3 The Physical Renormalization Condition . . . 62
- 2.2 The Renormalization Group 65
  - 2.2.1 Renormalization Prescriptions 69
  - 2.2.2 Fixed Points and Effective Couplings 77
  - 2.2.3 Asymptotic Behavior 79
- 2.3 1-Loop Renormalization of QED 84
  - 2.3.1 The Ward's Identities 89
- 2.4 Decay Rates of Unstable Particles 95

3 **Quantum Gauge Theories** 97
- 3.1 The Yang-Mills Quantum Theories 97

3.1.1 Gauge Invariance and Generating Functional 97
3.1.2 Faddeev-Popov Trick 101
3.1.3 Faddeev-Popov Determinant 105
3.1.4 Feynman Rules in Covariant Gauges 110
3.1.5 Becchi-Rouet-Stora-Tyutin Symmetry 115
3.2 The Standard Model 117
3.2.1 Weak Interactions Before Gauge Theories 118
3.2.2 Preliminary Building Program 121
3.2.3 Higgs Mechanism: Non-Abelian Case 126
3.2.4 Extension to Hadrons 131
3.2.5 The Standard Model Free Parameters 138
3.2.6 Unitarity versus Renormalizability: R_ξ gauges 152

Sono sempre di più quelli
che aspettano uno sguardo.
I nostri occhi
non sanno più vedere.
Per questo, un giorno, qualcosa cambierà.

Una luce

Soltanto le cose qui luccicano.
Il brillare è apparente
ed è lì per nascondere.
Nelle vite lontane dai clamori,
accanto a quotidiane sofferenze
nei discorsi mai sentiti,
nei pensieri repressi,
si cela una luce.

Maurizio Bacchilega

Chapter 1

Quantum Effective Action

1.1 Quantum Effective Action

We have already faced to the effective Action and the effective Lagrangian related to the fermion determinants in the presence of some background classical scalar or vector field. Those quantities encode the quantum effects due to the functional integration over the spinor matter. In what follows we aim to generalize this concept to the whole quantum field theory. To avoid unnecessary complications we restrict ourselves to the case of the quantum theory of a self-interacting real scalar field, the extension to other models being straightforward albeit nontrivial.

1.1.1 Classical Fields

Consider the generating functional of the connected Green's functions for the free scalar field theory, *i.e.*

$$W_0[\,J\,] = \tfrac{1}{2}\, i\hbar \int \mathrm{d}x \int \mathrm{d}y\;\, J(x)\, D_F\,(x-y)\, J(y) \equiv \tfrac{1}{2}\, i\hbar\, \langle\, J_x\, D_{xy}\, J_y\,\rangle$$

and define the so called classical field $\phi_{c\ell}\,(x)$ by the relation

$$\phi_{c\ell}\,(x) \stackrel{\mathrm{def}}{=} \frac{\delta W_0[\,J\,]}{\hbar\,\delta\, J(x)} = i \int \mathrm{d}y\;\, D_F\,(x-y)\, J(y) \tag{1.1}$$

Thus we immediately find

$$\left(\Box + m^2c^2/\hbar^2\right) \phi_{c\ell}\,(x) = \hbar c\, J(x) \tag{1.2}$$

that is the Klein$-$Gordon free wave equation in the presence of a classical external source $J(x)$. Notice that the classical field has canonical engineering dimension $[\,\phi_{c\ell}\,] = \sqrt{\mathrm{eV/cm}}$ in physical units, as expected, while $[\,J\,] = \mathrm{eV}^{-\frac{1}{2}}\ \mathrm{cm}^{-\frac{7}{2}}$. Let us perform the Legendre functional transformation and get

$$\begin{aligned}
\Gamma_0[\,\phi_{c\ell}\,] &\stackrel{\text{def}}{=} W_0[\,J\,] - \hbar\int \mathrm{d}x\ \phi_{c\ell}(x)\,J(x) \\
&= W_0[\,J\,] - \int\mathrm{d}t\int\mathrm{d}\mathbf{x}\ \phi_{c\ell}(x)\,\left(\Box + m^2c^2/\hbar^2\right)\phi_{c\ell}(x) \\
&= -\int\mathrm{d}t\int\mathrm{d}\mathbf{x}\ \phi_{c\ell}(x)\,\tfrac{1}{2}\left(\Box + m^2c^2/\hbar^2\right)\phi_{c\ell}(x) \\
&\doteq \int\mathrm{d}t\int\mathrm{d}\mathbf{x}\ \tfrac{1}{2}\left[\partial_\mu\,\phi_{c\ell}(x)\,\partial^\mu\,\phi_{c\ell}(x) - m^2c^2\,\phi_{c\ell}^2(x)/\hbar^2\right]
\end{aligned} \qquad (1.3)$$

which is nothing but the classical Action for a Klein-Gordon field written as a functional of the above introduced classical field $\phi_{c\ell}(x)$, once the classical field equations (1.2) have been employed.

A similar procedure can be closely carried out for a neutral spin-less field in the presence of the self-interaction term $V[\,\phi\,] = \lambda\int \mathrm{d}^4x\ \phi^4(x)/4!\hbar c^2$. To this concern let us still define the classical field $\phi_{c\ell}(x)$ by means of the equality

$$\begin{aligned}
\phi_{c\ell}(x) &\stackrel{\text{def}}{=} (1/\hbar)\,\delta W[\,J\,]/\delta J(x) \\
&= \mathrm{e}^{-(i/\hbar)W[J]}\langle\,0\,|\,\mathrm{T}\,\phi(x)\exp\{\,i\int\mathrm{d}y\,\phi(y)\,J(y)\,|\,0\,\rangle \\
&\equiv \mathrm{e}^{-(i/\hbar)W[\,J\,]}\ \langle\phi(x)\rangle_J = \frac{\langle\phi(x)\rangle_J}{\langle\,0\,|\,0\,\rangle_J}
\end{aligned} \qquad (1.4)$$

By the way, it should be admittedly recognized that the very denomination classical field for $\phi_{c\ell}(x)$ might appear a little bit misleading. To highlight this point, one has to acknowledge that, on the one hand, the classical field $\phi_{c\ell}(x)$ is the – normalized – vacuum expectation value of the self-interacting local quantum field operator $\phi(x)$ in the presence of an external classical source $J(x)$. On the other hand, as we shall see here below, the function $\phi_{c\ell}(x)$ does not satisfy at all the classical field equations of the $\lambda\phi_4^4$ theory in the presence of the external source. Hence the classical field $\phi_{c\ell}(x)$ is an average value of a local operator but does not fulfill classical wave field equation in the interacting case, at variance with the free Klein-Gordon case (1.2). Notice that in the limit of a vanishing external source we obtain

$$\frac{\langle 0|\phi(x)|0\rangle}{\langle 0|0\rangle} = \phi_{c\ell}(x)\rfloor_{J=0}$$

Translation invariance then requires

$$\frac{\langle 0|\phi(0)|0\rangle}{\langle 0|0\rangle} = \phi_{c\ell}(0)\rfloor_{J=0} = \text{constant}$$

If there are some reasons such that the above constant value turns out to be not zero, then it means that the important phenomenon of the spontaneous symmetry breaking does actually occur.

We can suitably define the Legendre functional transformation in the interacting case

$$\Gamma[\phi_{c\ell}] \stackrel{\text{def}}{=} W[J] - \hbar \int \mathrm{d}^4x\ \phi_{c\ell}(x)\, J(x) \equiv W[J] - \hbar\,\langle\, \phi_{c\ell}\, J\,\rangle \quad (1.5)$$

$$W[J] = \Gamma[\phi_{c\ell}] + \hbar\,\langle\, \phi_{c\ell}\, J\,\rangle \quad (1.6)$$

Taking the functional derivative of the very last relation with respect to the classical field we obtain the stationary configuration of the very last functional *viz.,*

$$\frac{\delta W[J]}{\delta\,\phi_{c\ell}(x)} = 0 \quad \Longleftrightarrow \quad \frac{\delta\Gamma[\phi_{c\ell}]}{\delta\,\phi_{c\ell}(x)} = -\hbar J(x) \quad (1.7)$$

as it is clear from (1.5) because $\Gamma[\phi_{c\ell}]$ depends solely upon the classical field $\phi_{c\ell}$ while the generating functional $W[J]$ of the connected Green's functions depends only on the external sources J. Just like we did in the free field case, once $Z[J]$ is known we can use (1.4) to determine $J(x)$ in terms of $\phi_{c\ell}(x)$ so that the right hand side of (1.5) can be written as a functional of $\phi_{c\ell}(x)$ which determines $\Gamma[\phi_{c\ell}]$. The value of (1.4) when the external source is turned off, *i.e.* $J(x) = 0$, is the vacuum expectation value of the quantum scalar field $\langle\, 0\,|\,\phi(x)\,|\,0\,\rangle$ which is assumed to be zero

$$\left.\frac{\delta W[J]}{\hbar\delta\, J(x)}\right\rfloor_{J=0} = \langle\, 0\,|\,\phi(0)\,|\,0\,\rangle = \phi_{c\ell}(0)\rfloor_{J=0} = 0 \quad (1.8)$$

unless the phenomenon called spontaneous symmetry breaking did actually take place. Equation (1.7) expresses $J(x)$ in terms of $\phi_{c\ell}(x)$ and in this sense it is the inverse of (1.4). In particular, taking equations (1.4) and (1.8) into account, we come to the equality

$$\hbar\,\Gamma^{(1)}(x) \equiv \left.\frac{\delta\Gamma[\phi_{c\ell}]}{\delta\,\phi_{c\ell}(x)}\right\rfloor_{\phi_{c\ell}=0} = 0 \quad (1.9)$$

because $\phi_{c\ell} = 0$ when $J = 0$ and *vice versa.* From the expression

$$\begin{aligned} Z[J] &= \exp\{(i/\hbar) W[J]\} \\ &= \exp\{(-i/\hbar) V[\delta/i\,\delta J]\}\ Z_0[J] \\ &= \exp\{(-i/\hbar) V[\delta/i\,\delta J]\}\ \exp\left\{-\tfrac{1}{2}\,\langle\, J_x\, D_{xy}\, J_y\,\rangle\right\} \quad (1.10) \end{aligned}$$

we obtain

$$\begin{aligned}\frac{\delta Z}{\delta J_x} &= \exp\{(-i/\hbar)V[\delta/i\,\delta J]\}\,\langle -D_{xy}\,J_y\rangle\,Z_0[J]\\ &= \exp\left\{-\frac{i}{\hbar}V[\delta/i\,\delta J]\right\}\,\langle -D_{xy}\,J_y\rangle\,\exp\left\{\frac{i}{\hbar}V[\delta/i\,\delta J]\right\}\,Z[J]\end{aligned}$$

Hence it follows that we can write the functional equation

$$\frac{-i}{\hbar c}\left(\Box_x + m^2c^2/\hbar^2\right)\frac{\delta Z}{\delta J_x} = O_x\,Z[J] \tag{1.11}$$

in which I have set

$$\begin{aligned} O_x &\stackrel{\text{def}}{=} \exp\{(-i/\hbar)V[\delta/i\,\delta J]\}\;J_x\,\exp\{(i/\hbar)V[\delta/i\,\delta J]\}\\ &= J_x - \frac{\lambda}{3!\hbar^2c^2}\left(-i\,\frac{\delta}{\delta J_x}\right)^3 \end{aligned} \tag{1.12}$$

`Proof`. Let me go back to natural units and set

$$O_x(a) = \exp\{-\,i\,a\,V[\delta/i\,\delta J]\}\;J_x\,\exp\{i\,a\,V[\delta/i\,\delta J]\}$$

where a is a real parameter. Differentiating we find

$$\frac{\mathrm{d}}{\mathrm{d}a}\,O_x(a) = \exp\{-\,i\,a\,V[\delta/i\,\delta J]\}\;[-\,i\,V[\delta/i\,\delta J]\,,\,J_x]\,\exp\{i\,a\,V[\delta/i\,\delta J]\}$$

On the other side we get

$$\begin{aligned}[-\,i\,V[\delta/i\,\delta J]\,,\,J_x] &= -\frac{i\lambda}{4!}\int \mathrm{d}^4y\,\left[\frac{\delta^4}{\delta J_y^4}\,,\,J_x\right]\\ = \;-\frac{i\lambda}{3!}\int \mathrm{d}^4y\,\frac{\delta^3}{\delta J_y^3}\,\delta^{(4)}(x-y) &= -\frac{\lambda}{3!}\,(-i)^3\,\frac{\delta^3}{\delta J_x^3}\end{aligned} \tag{1.13}$$

and thereby

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\,O_x(a) &= -\frac{\lambda}{3!}\,(-i)^3\,\frac{\delta^3}{\delta J_x^3}\\ \int_0^1 \mathrm{d}a\,\frac{\mathrm{d}}{\mathrm{d}a}\,O_x(a) = O_x(1) - J_x &= -\frac{i\lambda}{3!}\cdot\frac{\delta^3}{\delta J_x^3}\\ O_x &= J_x - \frac{\lambda}{3!}\left(-i\,\frac{\delta}{\delta J_x}\right)^3\end{aligned} \tag{1.14}$$

which is what we had to prove. $\Box$

Hence, after setting $\mathcal{K}_x = \Box_x + m^2$, we get

$$\begin{aligned}\mathcal{K}_x\,\frac{\delta Z}{\delta J_x} &= \left(i\,J_x + \frac{\lambda}{3!}\cdot\frac{\delta^3}{\delta J_x^3}\right)Z[J]\\ \mathcal{K}_x\,\frac{\delta W}{\delta J_x} &= J_x - \frac{i\lambda}{3!}\,\mathrm{e}^{-\,i\,W[J]}\,\frac{\delta^3}{\delta J_x^3}\,Z[J]\\ \mathcal{K}_x\,\phi_{c\ell}(x) &= J_x - \lambda\,\mathrm{e}^{-\,i\,W[J]}\,\frac{1}{6}\left(-i\,\frac{\delta}{\delta J_x}\right)^3 Z[J]\end{aligned} \tag{1.15}$$

the very last term just looking like a kind of driving functional force. Now we have

$$
\begin{aligned}
& \frac{i\lambda}{6}\, \mathrm{e}^{-iW[J]}\, \frac{\delta^3}{\delta J_x^3}\, \mathrm{e}^{iW[J]} \\
= \; & \frac{\lambda}{6} \left[\phi_{c\ell}^2(x) - \frac{\delta^2}{\delta J_x^2} - 3i\, \phi_{c\ell}(x)\, \frac{\delta}{\delta J_x} \right] \phi_{c\ell}(x)
\end{aligned} \tag{1.16}
$$

and finally

$$
\left(\Box + \frac{m^2c^2}{\hbar^2} \right) \phi_{c\ell} + \frac{\lambda}{6\hbar c}\, \phi_{c\ell}^3 - \hbar c\, J = \frac{\lambda}{6\hbar c} \left[\frac{\delta^2 \phi_{c\ell}}{\delta J^2} + 3i\, \phi_{c\ell}\, \frac{\delta \phi_{c\ell}}{\delta J} \right] \tag{1.17}
$$

whence it clearly appears that the right hand side just corresponds to the quantum corrections to the classical field equations for a spin-less neutral field with $\lambda\phi_4^4$ self-interaction and in the presence of a classical external source. Note that the canonical or engineering dimensions of the classical field and of the source density are $[\,\phi_{c\ell}\,] = \mathrm{eV}$ and $[\,J\,] = \mathrm{cm}^{-3}$ in natural units, so that the generating functional $\Gamma[\,\phi_{c\ell}\,]$ shares the very same dimensions with a classical Action, as expected.

1.1.2 Proper Vertexes

The effective Action in the interacting case can not be written in closed form and turns out to be a non-local functional of the classical field $\phi_{c\ell}(x)$. To the aim of being a little bit more general, consider here below the spin-less massive neutral field with the most general stable and power counting renormalizable self-interaction potential: namely,

$$
\begin{aligned}
V[\phi] &= \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}\mathbf{x} \left\{ \frac{mc}{\hbar} \cdot \frac{\varkappa}{3!\sqrt{\hbar c}}\, \phi^3(t,\mathbf{x}) + \frac{\lambda}{4!\hbar c}\, \phi^4(t,\mathbf{x}) \right\} \\
&= \int \mathrm{d}^4x \left\{ \frac{1}{6}\, m\varkappa\, \phi^3(x) + \frac{\lambda}{4!}\, \phi^4(x) \right\} \qquad (\,\varkappa \in \mathbb{R}\,,\ \lambda > 0\,)
\end{aligned}
$$

which breaks the $\mathbb{Z}_2-$symmetry, in such a manner that the Green's functions with an odd number of legs do not vanish. We can write

$$
\Gamma[\,\phi_{c\ell}\,] = \sum_{n=2}^{\infty} \frac{\hbar}{n!} \prod_{\jmath=1}^{n} \int \mathrm{d}^4x_\jmath\ \phi_{c\ell}(x_\jmath)\ \Gamma^{(n)}(x_1\,,\,\ldots\,,\,x_n) \tag{1.18}
$$

with

$$
\hbar\, \Gamma^{(n)}(x_1\,,\,\ldots\,,\,x_n) = \delta^{(n)}\Gamma[\,\phi_{c\ell}\,] / \delta\phi_{c\ell}(x_1)\ \ldots\ \delta\phi_{c\ell}(x_n)\big]_{\,\phi_{c\ell}=0} \tag{1.19}
$$

where the non-local coefficients $\Gamma^{(n)}(x_1, \ldots, x_n)$ are named the $n-$point proper vertexes, or strongly connected Green's functions, or 1-particle irreducible (1PI) Green's functions and turn out to be translation invariant, in such a manner that their Fourier transforms read

$$\begin{aligned} \Gamma^{(n)}(x_1, \ldots, x_n) \; &\equiv \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4}\, \mathrm{e}^{-ik_1 x_1} \ldots \int \frac{\mathrm{d}^4 k_n}{(2\pi)^4}\, \mathrm{e}^{-ik_n x_n} \\ \widetilde{\Gamma}^{(n)}&(k_1, \ldots, k_n)\,(2\pi)^4\,\delta\,(k_1 + k_2 + \cdots + k_n) \end{aligned} \tag{1.20}$$

so that

$$\begin{aligned} \Gamma[\phi_{c\ell}] \;&=\; \sum_{n=2}^{\infty} \frac{\hbar}{n!} \prod_{\jmath=1}^{n} \int \frac{\mathrm{d}^4 k_\jmath}{(2\pi)^4}\, \widetilde{\phi}_{c\ell}(k_\jmath) \\ &\times\; \widetilde{\Gamma}^{(n)}(-k_1, \ldots, -k_n)\,(2\pi)^4\,\delta\,(k_1 + \cdots + k_n) \end{aligned} \tag{1.21}$$

In particular – see (1.9) – we get the special values

$$\Gamma^{(0)} \equiv 0 \qquad\qquad \hbar\,\Gamma^{(1)}(x) = \left.\frac{\delta\,\Gamma[\phi_{c\ell}]}{\delta\,\phi_{c\ell}(x)}\right\rfloor_{\phi_{c\ell}=0} = 0$$

$$\begin{aligned} \Gamma^{(2)}(x-y) \;&=\; \hbar^{-1}\delta^{(2)}\Gamma[\phi_{c\ell}]/\delta\,\phi_{c\ell}(x)\,\delta\,\phi_{c\ell}(y)\rfloor_{\phi_{c\ell}=0} \\ &=\; \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \widetilde{\Gamma}^{(2)}(k)\,\mathrm{e}^{-ik(x-y)} \end{aligned} \tag{1.22}$$

Notice that the functional derivative of the inverse functional leads to the following remarkable relation: namely,

$$\begin{aligned} \hbar^{-1}\delta^{(2)} W[J]/\delta J(y)\,\delta J(x) \;&=\; \delta\phi_{c\ell}(x)/\delta J(y) \\ = \; [\delta J(y)/\delta\phi_{c\ell}(x)]^{-1} \;&=\; -\,\hbar\,\left[\delta^{(2)}\Gamma/\delta\phi_{c\ell}(x)\,\delta\phi_{c\ell}(y)\right]^{-1} \end{aligned} \tag{1.23}$$

that means

$$\frac{1}{i}\int \mathrm{d}^4 y\, \frac{\delta^{(2)} W[J]}{\delta J(x)\,\delta J(y)} \cdot \frac{\delta^{(2)}\Gamma[\phi_{c\ell}]}{\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(z)} \;=\; i\,\delta\,(x-z) \tag{1.24}$$

and after setting the external sources and the classical fields equal to zero

$$\int \mathrm{d}^4 y\; G_c^{(2)}(x-y)\,\Gamma^{(2)}(y-z) \;=\; i\,\delta\,(x-z) \tag{1.25}$$

or in momentum space

$$\widetilde{G}_c^{(2)}(k)\,\widetilde{\Gamma}^{(2)}(k) \;=\; i \tag{1.26}$$

where use have been made of the definition

$$G_c^{(n)}(x_1\,,\,\dots\,,\,x_n) = \frac{1}{\hbar}\,(-\,i\,)^{n-1}\,\delta^{(n)}W\,[\,J\,]\,/\,\delta\,J\,(x_1)\ \dots\ \delta\,J\,(x_n)\Big\rfloor_{\,J=0}$$

The engineering dimensions of the $n-$point 1PI vertexes are, in natural units, $[\,\Gamma^{(n)}\,] = (\mathrm{eV})^{3n}$ in the Minkowski space and $[\,\widetilde{\Gamma}^{(n)}\,] = (\mathrm{eV})^{4-n}$ in the Fourier space [1]. If we pass to the $3-$point connected Green's functions, from the main relation (1.24) taking one more functional derivative with respect to the source density $J(\xi)$ we can write

$$\begin{aligned}&\int \mathrm{d}y\ \frac{\delta^{(3)}W\,[\,J\,]}{\delta J(x)\,\delta J(y)\,\delta J(\xi)}\cdot\frac{\delta^{(2)}\Gamma\,[\,\phi_{c\ell}\,]}{\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(z)}\ +\\ &\int \mathrm{d}y\ \frac{\delta^{(2)}W\,[\,J\,]}{\delta J(x)\,\delta J(y)}\cdot\frac{\delta^{(3)}\Gamma\,[\,\phi_{c\ell}\,]}{\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(z)\,\delta J(\xi)}\ =\ 0\end{aligned}\tag{1.27}$$

and using the functional relation

$$\begin{aligned}\frac{\delta}{\delta J(\xi)}\left(\frac{\delta^{(2)}\Gamma\,[\,\phi_{c\ell}\,]}{\delta\phi_{c\ell}(x)\,\delta\phi_{c\ell}(y)}\right) &= \int \mathrm{d}w\ \frac{\delta^{(3)}\Gamma\,[\,\phi_{c\ell}\,]}{\delta\phi_{c\ell}(x)\,\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(w)}\ \cdot\ \frac{\delta\phi_{c\ell}(w)}{\delta J(\xi)}\\ &= \int \mathrm{d}w\ \frac{\delta^{(3)}\Gamma\,[\,\phi_{c\ell}\,]}{\delta\phi_{c\ell}(x)\,\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(w)}\ \cdot\ \frac{\delta^{(2)}W\,[\,J\,]}{\delta J(w)\,\delta J(\xi)}\end{aligned}$$

we eventually obtain, after setting sources equal to zero,

$$\frac{\delta^{(3)}W}{\delta J_x\,\delta J_y\,\delta J_\xi}\ *\ \frac{\delta^{(2)}\Gamma}{\delta\varphi_y\,\delta\varphi_z}\ +\ \frac{\delta^{(2)}W}{\delta J_x\,\delta J_y}\ *\ \frac{\delta^{(2)}W}{\delta J_\xi\,\delta J_w}\ *\ \frac{\delta^{(3)}\Gamma}{\delta\varphi_w\,\delta\varphi_y\,\delta\varphi_z}\cdot = 0\tag{1.28}$$

where I have suitably introduced a discrete type index notation, while the convolution product $*$ over repeated indexes is understood, together with $\phi_{c\ell}\equiv\varphi$. Hence, if we remember that

$$\frac{\delta^{(n)}W}{\delta J_1\,\dots\,\delta J_n}\Bigg\rfloor_{J=0} =\ \hbar\,i^{\,n-1}\,G_c^{(n)}(x_1\,,\,\dots\,,\,x_n) \qquad G_{xy}^{(2)}\ *\ \Gamma_{yz}^{(2)}\ =\ i\,\delta_{xz}$$

$$\frac{\delta^{(n)}\Gamma}{\delta\varphi_1\,\dots\,\delta\varphi_n}\Bigg\rfloor_{\varphi=0} =\ \hbar\,\Gamma^{(n)}(x_1\,,\,\dots\,,\,x_n)$$

it readily follows

$$G_{xy\xi}^{(3)}\ *\ \Gamma_{yz}^{(2)}\ =\ -\,G_{xy}^{(2)}\ *\ G_{\xi w}^{(2)}\ *\ \Gamma_{wyz}^{(3)}\tag{1.29}$$

[1] In the physical C.G.S. system of units we have instead $[\,\Gamma^{(n)}\,] = \mathrm{cm}^{-7n/2}\ \mathrm{eV}^{-n/2}$ while $\left[\,\widetilde{\Gamma}^{(n)}\,\right] = \mathrm{eV}^{-n/2}\ \mathrm{cm}^{-4+n/2}$.

and thereby

$$\Gamma^{(2)}_{vx} \, * \, \Gamma^{(2)}_{u\xi} \, * \, G^{(3)}_{xy\xi} \, * \, \Gamma^{(2)}_{yz} \;\; = \;\; \Gamma^{(3)}_{vzu} \tag{1.30}$$

This equality allows us to identify $\Gamma^{(3)}_{uvz}$ with the 3−point proper vertex

$$\begin{aligned} \frac{\delta^{(3)}\Gamma}{\delta\varphi_w \, \delta\varphi_y \, \delta\varphi_z}\bigg\rfloor_{\varphi=0} \;\; &= \;\; \hbar\,\Gamma^{(3)}(w\,,\,y\,,\,z) \\ &= \;\; \delta^{(3)}\,\Gamma\,[\,\phi_{c\ell}\,]\,/\,\delta\,\phi_{c\ell}\,(w)\,\delta\,\phi_{c\ell}\,(y)\,\delta\,\phi_{c\ell}\,(z)\big\rfloor_{\,\phi_{c\ell}=0} \end{aligned} \tag{1.31}$$

Actually, the 3−point proper vertex $\Gamma^{(3)}(w, y, z)$ is nothing but the connected 3−point Green's function in which the external complete propagators have been amputated, *i.e.* the 1PI strongly connected 3−point Green's function. By iterating the above described procedure, it can be shown by induction that the effective Action is the generating functional of all the proper vertexes. Consider for example the $\lambda\phi_4^4$ theory, where all the Green's functions with an odd number of arguments do vanish owing to the $\mathbb{Z}_2$−symmetry, and let's take one more functional derivative of eq. (1.28) and finally set all the sources equal to zero. Then we get

$$0 = \frac{\delta^{(4)}W}{\delta J_x \, \delta J_y \, \delta J_\xi \, \delta J_\eta} \, * \, \frac{\delta^{(2)}\Gamma}{\delta\varphi_y \, \delta\varphi_z} \; + \; \frac{\delta^{(2)}W}{\delta J_x \, \delta J_y} \, * \, \frac{\delta^{(2)}W}{\delta J_\xi \, \delta J_w} \, * \, \frac{\delta^{(4)}\Gamma}{\delta\varphi_w \, \delta\varphi_y \, \delta\varphi_z \, \delta J_\eta}$$

and taking into account that

$$\begin{aligned} \frac{\delta^{(4)}\Gamma}{\delta\varphi_w \, \delta\varphi_y \, \delta\varphi_z \, \delta J_\eta} \;\; &= \;\; \frac{\delta^{(4)}\Gamma}{\delta\varphi_w \, \delta\varphi_y \, \delta\varphi_z \, \delta\varphi_v} \, * \, \frac{\delta\varphi_v}{\delta J_\eta} \\ &= \;\; \frac{\delta^{(4)}\Gamma}{\delta\varphi_w \, \delta\varphi_y \, \delta\varphi_z \, \delta\varphi_v} \, * \, \frac{\delta^{(2)}W}{\delta J_\eta \, \delta J_v} \end{aligned}$$

we come to the equality

$$\begin{aligned} 0 \;\; = \;\; & \frac{\delta^{(4)}W}{\delta J_x \, \delta J_y \, \delta J_\xi \, \delta J_\eta} \, * \, \frac{\delta^{(2)}\Gamma}{\delta\varphi_y \, \delta\varphi_z} \\ + \;\; & \frac{\delta^{(2)}W}{\delta J_x \, \delta J_y} \, * \, \frac{\delta^{(2)}W}{\delta J_\xi \, \delta J_w} \, * \, \frac{\delta^{(4)}\Gamma}{\delta\varphi_w \, \delta\varphi_y \, \delta\varphi_z \, \delta\varphi_v} \, * \, \frac{\delta^{(2)}W}{\delta J_\eta \, \delta J_v} \end{aligned}$$

and thereby

$$\begin{aligned} G^{(4)}_{xy\xi\eta} * \Gamma^{(2)}_{xw} * \Gamma^{(2)}_{yz} * \Gamma^{(2)}_{\xi u} * \Gamma^{(2)}_{\eta v} + i^3\,\Gamma^{(4)}_{wzuv} &= 0 \\ i\,\Gamma^{(4)}_{yzwv} = G^{(4)}_{xy\xi\eta} * \Gamma^{(2)}_{xw} * \Gamma^{(2)}_{yz} * \Gamma^{(2)}_{\xi u} * \Gamma^{(2)}_{\eta v} & \end{aligned}$$

Taking the Fourier transform of the very last equality we find

$$i\widetilde{\Gamma}^{(4)}(k_1,k_2,k_3,k_4) = \widetilde{G}_c^{(4)}(k_1,k_2,k_3,k_4)\prod_{\jmath=1}^{4}\widetilde{\Gamma}^{(2)}(k_\jmath) \tag{1.32}$$

which yields, using the Feynman rules to the lowest order,

$$\begin{aligned}\widetilde{\Gamma}^{(4)}(k_1,k_2,k_3,k_4) &= -i\left(-\frac{i\lambda}{\hbar^2 c^2}\right) i^4 + O(\lambda^2) \\ &= -\frac{1}{\hbar^2 c^2}\left[\lambda + O(\lambda^2)\right]\end{aligned} \tag{1.33}$$

The above procedure can be eventually iterated and extended to $n-$point connected and strongly connected Green's functions. Let me summarize here below the canonical engineering dimensions of the various quantities we have met so far for our future convenience :

$\ddots$	C.G.S.	$\hbar = c = 1$
$\phi = \varphi$	$\sqrt{\mathrm{eV/cm}}$	$\mathrm{eV} = \mathrm{cm}^{-1}$
J	$\mathrm{eV}^{-1/2}\ \mathrm{cm}^{-7/2}$	cm^{-3}
$G_c^{(n)}(x)$	$\mathrm{eV}^{n/2}\ \mathrm{cm}^{-n/2}$	eV^{n}
$\widetilde{G}_c^{(n)}(k)$	$\mathrm{eV}^{n/2}\ \mathrm{cm}^{-4+7n/2}$	eV^{4-3n}
$\Gamma^{(n)}(x)$	$\mathrm{eV}^{-n/2}\ \mathrm{cm}^{-7n/2}$	eV^{3n}
$\widetilde{\Gamma}^{(n)}(k)$	$\mathrm{eV}^{-n/2}\ \mathrm{cm}^{-4+n/2}$	eV^{4-n}

Effective Potential

The concept of effective potential is a useful tool in order to understand and analyze the phenomenon of the spontaneous symmetry breaking. We have already met this topic in the discussion of the effective Action for the Yukawa determinant, see Chapter **2** § **2.3.1.** of the II semester course. As a matter of fact, the effective potential is nothing but the effective Action in the limit of constant classical field configurations $\phi_{c\ell}(x) = \varphi = \text{constant}$. Thus we have

$$\hbar^{-1}\Gamma[\varphi] = \sum_{n=2}^{\infty}\frac{\varphi^n}{n!}\int \mathrm{d}x_1 \ldots \int \mathrm{d}x_n\, \Gamma^{(n)}(x_1, \ldots, x_n)$$

From the definition of the Fourier transform (1.20) we can write

$$
\begin{aligned}
\hbar^{-1}\,\Gamma[\,\varphi\,] \;&=\; \sum_{n=2}^{\infty}\frac{\varphi^{\,n}}{n\,!}\,\prod_{\jmath=1}^{n}\int \mathrm{d}^4x_\jmath\int\frac{\mathrm{d}^4k_\jmath}{(2\pi)^4}\;\exp\{-\,i\,k_\jmath\,x_\jmath\}\\
&\times\;\widetilde{\Gamma}^{(n)}\,(\,k_1\,,\,\dots\,,\,k_n\,)\,(2\pi)^4\,\delta\,(\,k_1\,+\,k_2\,+\,\cdots\,+\,k_n\,)\\
&=\;\sum_{n=2}^{\infty}\frac{\varphi^{n}}{n\,!}\int \mathrm{d}^4k_1\;\delta(k_1)\dots\int \mathrm{d}^4k_n\;\delta(k_n)\\
&\times\;\widetilde{\Gamma}^{(n)}\,(\,k_1\,,\,\dots\,,\,k_n\,)\,(2\pi)^4\,\delta\,(\,k_1\,+\,k_2\,+\,\cdots\,+\,k_n\,)\\
&=\;\sum_{n=2}^{\infty}\frac{\varphi^{\,n}}{n\,!}\;\widetilde{\Gamma}^{(n)}\,(\,0\,,\,\dots\,,\,0\,)\,(2\pi)^4\,\delta(0)\\
&\equiv\;-\,(2\pi)^4\,\delta(0)\,\mathcal{V}_{\mathrm{eff}}(\varphi)/\hbar c
\end{aligned}
$$

whence

$$
\mathcal{V}_{\mathrm{eff}}(\varphi)\;=\;-\,\hbar c\sum_{n=2}^{\infty}\frac{\varphi^{\,n}}{n\,!}\;\widetilde{\Gamma}^{(n)}\,(\,0\,,\,\dots\,,\,0\,) \tag{1.34}
$$

In the case of the $\lambda\phi_4^4$ theory all the proper vertexes with odd n are null, owing to the $\mathbb{Z}_2-$symmetry, in such a manner that we can write

$$
\mathcal{V}_{\mathrm{eff}}(\varphi)\;=\;-\,\hbar c\sum_{n=1}^{\infty}\frac{\varphi^{\,2n}}{(2n)\,!}\;\widetilde{\Gamma}^{(2n)}\,(\,0\,,\,\dots\,,\,0\,)
$$

Hence to the lowest order, *i.e.* at the so called tree level, the only non-vanishing 1PI Green's functions are just

$$
\widetilde{\Gamma}^{(2)}(p)=\frac{p^2}{\hbar c}-\frac{m^2c}{\hbar^3}\qquad\qquad \widetilde{\Gamma}^{(4)}\;=\;-\;\frac{\lambda}{\hbar^2c^2}
$$

so that we obtain in natural units

$$
\mathcal{V}_{\mathrm{eff}}^{(0)}(\varphi)=\tfrac{1}{2}\,m^2\varphi^2+(\lambda/4!)\,\varphi^4
$$

To the higher orders we can write the loop expansions

$$
\widetilde{\Gamma}^{(2)}\,(\,0\,)=\sum_{\ell=0}^{\infty}\lambda^{\ell}\,\widetilde{\Gamma}_{\ell}^{(2)}\,(\,0\,)\,,\qquad\qquad \widetilde{\Gamma}_{0}^{(2)}\,(\,0\,)=-\,m^2 \tag{1.35}
$$

$$
\widetilde{\Gamma}^{(4)}\,(\,0\,)=\lambda\sum_{\ell=0}^{\infty}\lambda^{\ell}\,\widetilde{\Gamma}_{\ell}^{(4)}\,(\,0\,)\,,\qquad \widetilde{\Gamma}_{0}^{(4)}\,(\,0\,)=-1 \tag{1.36}
$$

$$
\widetilde{\Gamma}^{(2n)}\,(\,0\,)=\lambda^{n-1}\sum_{\ell=0}^{\infty}\lambda^{\ell}\,\widetilde{\Gamma}_{\ell}^{(2n)}\,(\,0\,)\,,\qquad \widetilde{\Gamma}_{0}^{(2n)}\,(\,0\,)=0 \tag{1.37}
$$

so that we finally come to the double series expansion

$$
\begin{aligned}
\mathcal{V}_{\mathrm{eff}}(\varphi)\;&=\;-\;\sum_{n=1}^{\infty}\frac{\varphi^{2n}}{(2n)\,!}\;\lambda^{n-1}\sum_{\ell=0}^{\infty}\lambda^{\ell}\,\widetilde{\Gamma}_{\ell}^{(2n)}\,(\,0\,)\\
&=-\sum_{\ell=0}^{\infty}\sum_{n=1}^{\infty}\frac{\varphi^{2n}}{(2n)\,!}\;\lambda^{n+\ell-1}\,\widetilde{\Gamma}_{\ell}^{(2n)}\,(\,0\,)
\end{aligned} \tag{1.38}
$$

In particular, for $\ell = 1$ we obtain the 1-loop effective potential

$$\mathcal{V}_{\rm eff}^{(1)}(\varphi) = -\sum_{n=1}^{\infty} \frac{\varphi^{2n}}{(2n)\,!}\;\lambda^n\,\widetilde{\Gamma}_1^{(2n)}(0) \tag{1.39}$$

where the 1-loop quantity $\lambda^n\,\widetilde{\Gamma}_1^{(2n)}(0)$ just corresponds to a closed cycle with n vertexes and n propagators at zero external momenta. From the Feynman rules, we obtain

$$\begin{aligned}\lambda^n\,\widetilde{\Gamma}_1^{(2n)}(0) \quad &= \quad \tfrac{1}{2}\,(2n-1)!!\,(n-1)!\,i\,(-\,i\lambda)^n \\ \times \quad &\int \frac{{\rm d}^4\ell}{(2\pi)^4}\left(\frac{i}{\ell^{\,2}-m^2+i0}\right)^n\end{aligned}$$

The symmetry factor can be easily understood as follows: there is the factor $\frac{1}{2}$ for one pair of indistinguishable scalar particles, the number of indistinguishable pairs one can form with $2n$ particles is $(2n-1)!! = (2n-1)(2n-3)\ldots 1$, and finally there are $(n-1)!$ permutations of the independent internal lines for overall momentum conservation. Moreover, it is worthwhile to remark that for $n = 1, 2$ the integral is divergent by superficial power counting, while for $n \geq 3$ it is convergent. Nonetheless, let us proceed formally and suppose to introduce some regularization for the integrals with $n = 1, 2$ and to make the transition to the Euclidean formulation, *viz.,*

$$\texttt{reg}\int \frac{{\rm d}^4\ell}{(2\pi)^4}\left[\frac{1}{\ell^2-m^2+i0}\right]^n = i(-1)^n\;\texttt{reg}\int \frac{{\rm d}^4\ell_E}{(2\pi)^4}\left(\ell_E^2+m^2\right)^{-n} \tag{1.40}$$

Now, taking into account that

$$\frac{1}{(2n)!}\cdot(2n-1)!!(n-1)!\,\tfrac{1}{2} = \frac{1}{2n}\left(\tfrac{1}{2}\right)^n$$

we can write the formal equality

$$\begin{aligned}(2\pi)^4\,\delta(0)\,\mathcal{V}_{\rm eff}^{(1)}(\varphi) \quad &= \quad -\,(2\pi)^4\,\delta(0)\,\tfrac{1}{2}\,\hbar c\;\texttt{reg}\int \frac{{\rm d}^4\ell_E}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{\lambda\varphi^2/2\hbar c}{\ell_E^2+m^2c^2/\hbar^2}\right)^n \\ &= \quad (2\pi)^4\,\delta(0)\,\tfrac{1}{2}\,\hbar c\;\texttt{reg}\int \frac{{\rm d}^4\ell_E}{(2\pi)^4}\;\ln\left(1+\frac{\lambda\varphi^2/2\hbar c}{\ell_E^2+m^2c^2/\hbar^2}\right) \\ &= \quad \tfrac{1}{2}\,\hbar c\;\texttt{reg}\;{\rm Tr}\;\ln\left(1+\frac{\lambda\varphi^2/2\hbar c}{-\,\partial_E^2+m^2c^2/\hbar^2}\right) \\ &= \quad \tfrac{1}{2}\,\hbar c\;\texttt{reg}\;{\rm Tr}\;\ln\left(-\,\partial_E^2+\frac{m^2c^2}{\hbar^2}+\frac{\lambda\varphi^2}{2\hbar c}\right) \\ &- \quad \tfrac{1}{2}\,\hbar c\;\texttt{reg}\;{\rm Tr}\;\ln\left(-\,\partial_E^2+\frac{m^2c^2}{\hbar^2}\right) \\ &= \quad -\,\hbar c\;\ln\;\texttt{reg}\;\left[\,{\rm det}\;\|-\partial_E^2+(m^2c^2/\hbar^2)+(\lambda\varphi^2/2\hbar c)\,\|\,\right]^{-\frac{1}{2}} \\ &+ \quad \hbar c\;\ln\;\texttt{reg}\;\left[\,{\rm det}\;\|-\partial_E^2+(m^2c^2/\hbar^2)\,\|\,\right]^{-\frac{1}{2}}\end{aligned}$$

At this point it appears rather clear that we can safely and suitably make use of the $\zeta-$function regularization technique to obtain a finite result. From the knowledge of the determinant of the Klein-Gordon operator – see Chapter **1** § **1.1.3.** of the II semester course – we get

$$\begin{aligned}\mathcal{V}_{\text{eff}}^{(1)}(\varphi) &= -\frac{\hbar c}{64\pi^2}\left\{\left(\frac{mc}{\hbar}\right)^4\left[\ln\frac{m^2}{\mu^2}-\frac{3}{2}\right]\right.\\ &+ \left.\left(\frac{m^2c^2}{\hbar^2}+\frac{\lambda\varphi^2}{2\hbar c}\right)^2\left[\ln\left(\frac{m^2}{\mu^2}+\frac{\hbar\,\lambda\varphi^2}{2\mu^2c^3}\right)-\frac{3}{2}\right]\right\} \qquad (1.41)\end{aligned}$$

in such a manner that we can finally recast the effective potential for the $\lambda\phi_4^4$ theory up to the 1-loop approximation in the form

$$\begin{aligned}\mathcal{V}_{\text{eff}}(\varphi) &= \frac{m^2c^2}{2\hbar^2}\,\varphi^2+\frac{\lambda\,\varphi^4}{4!\hbar c}-\frac{\hbar c}{64\pi^2}\left\{\left(\frac{mc}{\hbar}\right)^4\left[\ln\frac{m^2}{\mu^2}-\frac{3}{2}\right]\right.\\ &+ \left.\left(\frac{m^2c^2}{\hbar^2}+\frac{\lambda\varphi^2}{2\hbar c}\right)^2\left[\ln\left(\frac{m^2}{\mu^2}+\frac{\hbar\lambda\varphi^2}{2\mu^2c^3}\right)-\frac{3}{2}\right]\right\}\\ &+ \text{ higher loop corrections} \qquad (1.42)\end{aligned}$$

1.2 Spontaneous Symmetry Breaking

The conventional or customary way in which any symmetry in quantum field theory can be broken is by means of a suitable, explicit symmetry breaking term in the Lagrangian and/or in the field equations. However, a truly revolutionary breakthrough occurred in the early sixties of the last Century, when it was realized that *e.g.* an internal symmetry can be broken by a much more economical and fruitful mechanism, the so called spontaneous symmetry breaking, in such a manner that the Action and the field equations keep the very symmetry untouched. The main idea has been probably suggested by the long standing Heisenberg model for ferromagnetism and by the discovery of a theory for the superconductivity.

Werner Heisenberg
Zur Theorie des Ferromagnetismus
Zeitschrift der Physik **49** (1928) 619-636;

Yoichiro Nambu
Axial Vector Current Conservation in Weak Interaction
Physical Review Letters **4** (1960) 380-382;
Quasiparticles and Gauge Invariance in the Theory of Superconductivity
Physical Review **117** (1960) 648-663;

Jeoffrey Goldstone
Field Theories with Superconductor Solutions

Il Nuovo Cimento **19** (1961) 154-164;

Jeoffrey Goldstone, Abdus Salam and Steven Weinberg
Broken Symmetries
Physical Review **127** (1962) 965-970.

The simplest and most economical way to place and frame the phenomenon of the spontaneous symmetry breaking within the context of perturbative quantum field theory is probably by means of the above described formalism of the effective potential. Moreover, the original non-perturbative Coleman-Weinberg mechanism, as well as the quantization of Higgs-Brout-Englert Abelian model for the spontaneous symmetry breaking in gauge theories will be shortly analyzed. However, let me start by the statement of the Goldstone theorem, which encodes the essential ingredients of the subject and unravels the origin of the so called Goldstone bosons.

1.2.1 Goldstone's Bosons and Goldstone's Theorem

Consider a general theory described by a classical Lagrangian $\mathcal{L}$ which is invariant under a global $n-$dimensional Lie group G of internal symmetry transformations. Then, according to Nöther theorem, there are n conserved charges Q_a $(\,a = 1,2,\ldots,n\,)$ which act on the fields appearing in the Lagrangian as generators of of the symmetry transformations in some finite dimensional representation. It turns out that, at least to the lowest order in perturbation theory, in a theory with a spontaneously broken symmetry the mass matrix of the scalar sector does always admit zero modes, *i.e.* null eigenstates: these are the Goldstone's bosons. Suppose in fact that the theory involves in general a number N of real Poincaré invariant classical fields $\Phi_\imath(x) : \mathcal{M} \longrightarrow \mathbb{R}$ $(\,\forall\,\imath = 1,2,\ldots,N\,)$ belonging to some finite dimensional representation of the internal symmetry Lie group G. Then the specific part of the Action that involves the scalar fields can be constructed out of the Lagrangian

$$\mathcal{L}_{\,\mathrm{scalar}} = \tfrac{1}{2}\,\partial_\mu\Phi_\imath\,\partial^{\,\mu}\Phi_\imath - \mathcal{V}[\,\Phi_\imath\,] \;+\; \cdots \tag{1.43}$$

where the dots stand for unspecified couplings of the scalar field content with other fields with spin of the theory, those couplings being irrelevant for the present discussion, at least up to the lowest order approximation of the theory, *i.e.* as long as radiative corrections can be neglected. The potential density $\mathcal{V}$ is a fourth order real polynomial in the scalar fields, at most, if the theory has to be power counting renormalizable. By assumption the multiplet of the real scalar fields does transform according to a finite dimensional real and orthogonal representation of the global – *i.e.* space-time point independent – internal symmetry group: namely,

$$\Phi_\imath^{\prime}(x) = \mathbb{A}_{\imath\jmath}(\alpha)\,\Phi_\jmath(x) \qquad\qquad \mathbb{A}^{\top} = \mathbb{A}^{-1} \tag{1.44}$$

$$\mathbb{A}_{\imath\jmath}(\alpha) = \|\,\exp\{\alpha^a I^{\,a}\}\,\|_{\imath\jmath} = \;\delta_{\imath\jmath} + \alpha^a I^{\,a}_{\imath\jmath} \;+\; \cdots \tag{1.45}$$

where α^a $(\,a = 1,2,\ldots,n\,)$ are a set of canonical coordinates for the compact Lie group G, while the generators I^a $(\,a = 1,2,\ldots,n\,)$ are real anti-symmetric matrices. In the quantum theory the generators of the symmetry transformations become the Hermitean operators corresponding to the conserved Nöther charges Q_a $(\,a = 1,2,\ldots,n\,)$ in such a manner that the infinitesimal transformations on the operator valued tempered distributions of the scalar fields just read

$$\delta\Phi_\imath(t,\mathbf{x}) = i\,[\,Q^a(t)\,,\Phi_\imath(t,\mathbf{x})\,]\,\delta\alpha^a = i\,\delta\alpha^a I^a_{\imath\jmath}\Phi_\jmath(t,\mathbf{x}) \tag{1.46}$$

Here we are interested in theories where the phenomenon of the spontaneous symmetry breaking of the global Lie group G of the internal symmetry does actually occur: namely, it turns out that some of the quantum scalar fields exhibit in fact a non-vanishing constant vacuum expectation value, as required by the invariance of the theory under space-time translations, *viz.,*

$$\langle\, 0\,|\,\Phi_\imath(x)\,|\,0\,\rangle = \upsilon_\imath \neq 0 \qquad\qquad \text{for some } \imath = 1,2,\ldots,N \tag{1.47}$$

Now, to the zeroth order or semi-classical approximation, in which the energy density of the vacuum state is provided by the minimum of the effective potential $\mathcal{V}(\varphi_\imath)$ for constant – *i.e.* space-time point independent – field configurations, one can readily realize that the spontaneous breaking of the global internal symmetry G may occur iff the condition for a minimum

$$0 = \frac{\delta\,\mathcal{V}(\varphi_\imath)}{\delta\alpha_a} = \frac{\partial\mathcal{V}}{\partial\varphi_\imath}\cdot\frac{\delta\varphi_\imath}{\delta\alpha_a} \Leftrightarrow \frac{\partial\mathcal{V}}{\partial\varphi_\imath}\;I^a_{\imath\jmath}\varphi_\jmath = 0 \qquad\qquad \forall\, a = 1,2,\ldots,n \tag{1.48}$$

admits a solution $\varphi_\imath = \upsilon_\imath$ ($\imath = 1,2,\ldots,N$) with at least a non-vanishing $\upsilon_\imath$. Notice that if $\Phi_\imath(x) = \varphi_\imath = \upsilon_\imath$ is any solution of the above minimum condition (1.48) then $\upsilon'_\imath(\alpha) = \mathbb{A}_{\imath\jmath}(\alpha)\,\upsilon_\jmath$ is also a solution, owing to the invariance of the potential density under the global symmetry group G. Hence, for any given potential density leading to the spontaneous breaking of a global symmetry group G the vacuum energy is infinitely and continuously degenerate: any choice of it is equally possible and in turn any choice of the nontrivial minimum $\upsilon = (\upsilon_1,\ldots,\upsilon_N)$ of the semi-classical effective potential just corresponds to some definite choice of the quantum vacuum state of the very same finite zero-point energy density $\mathcal{V}_0 \equiv \mathcal{V}(\upsilon_\imath)$. As a matter of fact one can set up a correspondence between the different choices of the potential minimum and of the vacuum state: namely,

$$\upsilon\,' = \exp\{\alpha^a I^{\,a}\}\,\upsilon \quad\Longleftrightarrow\quad |\,0\,'\,\rangle \;=\; \exp\{i\alpha^a Q^{\,a}\}\,|\,0\,\rangle \tag{1.49}$$

in accordance with equation (1.47). Hence, the occurrence of the spontaneous symmetry breaking is nothing but any specific choice of the minimum of the potential (semi-classical level) and of the vacuum state (quantum level). This very choice of the minimum of the potential density does not spoil at all the symmetry of the classical Lagrangian and Hamiltonian: this is the reason why this kind of breaking has been called *spontaneous*. As a matter of fact, in the theory of ferromagnetism the spontaneous and permanent magnetization occurs in spite of the fact that the quantum Hamiltonian operators for the iron or nichel atoms still exhibit the O(3) rotation symmetry. Once some definite solution υ has been chosen, it is convenient to define the shifted and physical field $\Phi\,'$ according to

$$\Phi'_\imath(x) \equiv \Phi_\imath(x) - \upsilon_\imath \tag{1.50}$$

and to suitably rewrite the potential density in the form

$$\mathcal{V}[\,\Phi\,'\,] = \mathcal{V}(\upsilon) + \tfrac{1}{2}\,M^{\,2}_{\imath\jmath}\,\Phi'_\imath\Phi'_\jmath + \cdots \tag{1.51}$$

where

$$M^{\,2}_{\imath\jmath} \equiv \left(\frac{\partial^{\,2}\mathcal{V}}{\partial\,\Phi_\imath\partial\,\Phi_\jmath}\right)_{\Phi=\upsilon} \tag{1.52}$$

are the matrix elements of the square mass real symmetric matrix for the physical scalar fields. Notice that the term linear in the fields does not appear because of the minimum condition (1.48). A further differentiation of equation (1.48) yields

$$\begin{aligned} 0 = \frac{\partial^2\, \mathcal{V}(\varphi_\imath)}{\partial\varphi_\jmath\, \partial\alpha_a}\bigg]_{\varphi=\upsilon} &= \frac{\partial^2 \mathcal{V}}{\partial\varphi_\jmath\, \partial\varphi_\imath}\bigg]_{\varphi=\upsilon} \cdot \frac{\partial\varphi_\jmath}{\partial\alpha_a} \\ \Leftrightarrow\ M^2_{\imath\jmath} I^a_{\jmath k}\upsilon_k = 0 &\qquad \forall\, a = 1, 2, \ldots, n \end{aligned} \tag{1.53}$$

which means that the square mass matrix necessarily admits some zero modes, *i.e.* some null eigenstates of the kind $I^{\,a} \cdot \upsilon$ ($a = 1, 2, \ldots, n$). Those scalar zero modes that appear in the presence of the spontaneous symmetry breaking are called the Goldstone's bosons.

Let me discuss this point more carefully and precisely. We can suitably separate the generators of the spontaneously broken internal symmetry group G into two subsets. The first one consists in the generators of the Lie sub-algebra $\mathfrak{h}$ of the residual symmetry subgroup $H \subset G$ with $\dim(H) = \nu < n$. This sub-algebra $\mathfrak{h}$ involves the generators Y_ℓ ($\ell = 1, 2, \ldots, \nu < n$) which are called unbroken with respect to our potential density minimum choice, that means $Y_\ell \cdot \upsilon = 0$ ($\forall\, \ell = 1, \ldots, \nu$) which corresponds in turn to the relic symmetry subgroup that leaves the vacuum invariant, *i.e.* $Q_\ell\, |\, 0\, \rangle = 0$ ($\forall\, \ell = 1, \ldots, \nu < n$). The second and complementary subset of the Lie algebra $\mathfrak{g}$ is provided by the so called spontaneously broken generators. Denoting their matrix representations by X_l ($l = \nu + 1, \ldots, n$) we find $X_l \cdot \upsilon \neq 0$, so that the condition (1.53) implies the existence, for each broken generator of the Lie algebra $\mathfrak{g}$ of a zero mode $X_l \cdot \upsilon$ ($l = \nu + 1, \ldots, n$) of the square mass matrix. In turn, at the quantum level one necessarily finds $Q_l\, |\, 0\, \rangle \neq 0$ for $l = \nu + 1, \ldots, n$. The Goldstone's boson fields are therefrom identified by the $(n - \nu)$ scalar mass-less fields $\chi_l(x) \equiv (X_l \cdot \upsilon)_\imath \Phi_\imath(x) / \left(\upsilon^\top \cdot \upsilon\right)^{\frac{1}{2}}$ for $l = \nu + 1, \ldots, n$. At this point, it is worthwhile to remark that the present discussion does actually concern the spontaneously broken **global** internal symmetries. For spontaneously broken **gauge** internal symmetries, the group normal coordinates of which are space-time points dependent, the Goldstone's bosons do not appear in the physical spectrum, for they can always be gauged away, according to the conventional Particle Physics jargon. As a matter of fact, it turns out that the corresponding degrees of freedom precisely give rise to the longitudinal polarization of the massive gauge vector bosons: this is the so called Higgs mechanism, that will be discussed in the sequel. Let me conclude this Section with three quite useful remarks.

1. If some of the quantum scalar fields $\Phi_\imath(x)$ have non vanishing vacuum expectation values $\langle\, 0\, |\, \Phi_\imath(x)\, |\, 0\, \rangle = \upsilon_\imath \neq 0$ for some $\imath = 1, \ldots, N$, then some generators of the internal global symmetry group G must be necessarily broken for $X_l \cdot \upsilon \neq 0$ if $l = \nu + 1, \ldots, n$. It turns out that the unbroken generators Y_ℓ ($\ell = 1, \ldots, \nu$) must necessarily form the Lie algebra of a subgroup $H \subset G$ (of course, it may happen that $H = \mathbb{I}$). Indeed, if it were not the case then we get

   $$[\, Y_a, Y_b\,] = f_{abc}\, Y_c + f'_{abl}\, X_l \quad \text{with } a, b, c \le \nu\ \vee\ l > \nu$$

   with at least some non-vanishing coefficients f'_{abl}. Under application of both sides of the above relation to the nontrivial minimum υ, taking into account that we have by assumption $Y_\ell \cdot \upsilon = 0$ ($\forall\, \ell = 1, 2, \ldots \nu$) we immediately find a contradiction unless $f'_{abl} = 0$ for $a, b \le \nu$ and $l > \nu$. Hence the unbroken generators Y_ℓ ($\ell = 1, 2, \ldots, \nu$) must necessarily form a matrix representation of the Lie sub-algebra $\mathfrak{h}$ of the residual symmetry subgroup $H \subset G$.

2. It is also easy to see that the broken generators X_l $(\, l = \nu + 1, \ldots, n\,)$ do actually transform under some representation of the residual symmetry subgroup $H \subset G$. As a matter of fact one finds

$$[\, Y_a, X_b\,] = f_{abl}\, X_l \qquad \text{for } a \le \nu \,\vee\, b, l > \nu$$

owing to the previously proved relations $f'_{abl} = 0$ for $a, b \le \nu$ and $l > \nu$, as well as to the complete anti-symmetry of the structure constants of H that forbids the presence of terms like $f'_{alb}\, Y_b$ with $a, b \le \nu \,\vee\, l > \nu$. Thus, the system of relations

$$\begin{cases} \chi_l(x) \equiv (X_l \cdot \upsilon)_\imath \Phi_\imath(x) / \left(\upsilon^\top \cdot \upsilon\right)^{\frac{1}{2}} & l = \nu + 1, \ldots, n \\ \\ [\, Y_a, X_b\,] = f_{abl}\, X_l & a \le \nu \,\vee\, b, l > \nu \end{cases}$$

taken together tell us that the Goldstone's boson fields transform according to some finite dimensional representation of the residual unbroken symmetry subgroup $H \subset G$.

3. Finally, it turns out that in some physical settings it is possible to build up an internal parity operator $\mathbb{P}$, which leaves invariant the Lie algebra $\mathfrak{g}$ of the global internal symmetry group G and such that

$$\mathbb{P} = \mathbb{P}^\dagger = \mathbb{P}^{-1}$$

$$[\mathbb{P}, Y_\ell\,] = 0 \quad (\,\ell = 1, \ldots, \nu < n\,) \qquad \{\mathbb{P}, X_l\} = 0 \quad (\, l = \nu + 1, \ldots, n\,)$$

In such a case - a well known example being the chiral symmetry group for QCD - there is one more obvious but quite useful relation, *viz.*

$$[\, X_a, X_b\,] = f_{ab\ell}\, Y_\ell \qquad a, b > \nu \,\vee\, \ell \le \nu$$

We are now ready to face the general formal proof of the Goldstone's theorem

Jeffrey Goldstone, Abdus Salam & Steven Weinberg (1962)
Broken Symmetries
The Physical Review **127**, 965-970.

Those authors showed for the first time that the spontaneous breakdown of an internal global symmetry group G for a classical Lagrangian of a quantum field theory implies the existence of mass-less poles in certain Green's functions of the theory and consequently entails the appearance of mass-less bosons in the physical spectrum. Consider in general the Green's function

$$G^{\,a}_{\mu,\imath}(x - y) = \langle\, 0\,|\, T\, J^a_\mu(x)\, \Phi_\imath(y)\,|\, 0\,\rangle \tag{1.54}$$

where $J^a_\mu(x)$ are the quantum operators associated to the the classical Nöther's currents of the global internal symmetry group G of the classical Lagrangian and which thereby satisfy the continuity operator equations $\partial^{\,\mu} J^a_\mu(x) = 0$ $(\, a = 1, 1, \ldots, n\,)$, while $\Phi_\imath(y)$ $(\,\imath = 1, 2\ldots, N\,)$ is a multiplet of Hermitean scalar quantum fields which transform according to some N-dimensional irreducible representation of G. In such a situation, owing to the chronological ordering, the Green's functions will satisfy the differential identities

$$\partial^{\,\mu}_x G^{\,a}_{\mu,\imath}(x - y) = \delta(x_0 - y_0)\,\langle\, 0\,|\,[\, J^a_0(x)\,, \Phi_\imath(y)\,]\,|\, 0\,\rangle \tag{1.55}$$

From the assumed transformation law

$$[\,Q^a\,,\Phi_\imath(t,\mathbf{y})\,] = \int \mathrm{d}\mathbf{x}\,[\,J_0^a(t,\mathbf{x})\,,\Phi_\imath(t,\mathbf{y})\,] = \boldsymbol{\tau}^a_{\imath\jmath}\,\Phi_\jmath(t,\mathbf{y})$$

where $\boldsymbol{\tau}^a$ ($a = 1,2,\ldots,n$) are the infinitesimal operators of the irreducible N-dimensional representation of the internal global symmetry group G, under the assumptions of locality and continuity we can derive the relation

$$[\,J_0^a(t,\mathbf{x})\,,\Phi_\imath(t,\mathbf{y})\,] = \boldsymbol{\tau}^a_{\imath\jmath}\,\Phi_\jmath(t,\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y}) \tag{1.56}$$

which follows from the assumed field transformation rules (1.46). Then, under the general requirement of space-time translation invariance, we eventually come to the final relation

$$\partial_x^\mu G^{\,a}_{\mu,\imath}(x-y) = \delta(x-y)\,\boldsymbol{\tau}^a_{\imath\jmath}\,\langle\,0\,|\,\Phi_\jmath(0)\,|\,0\,\rangle \tag{1.57}$$

Taking the Fourier transform

$$G^{\,a}_{\mu,\imath}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\;\exp\{-\,ik\cdot(x-y)\}\;\widetilde{G}^{\,a}_{\mu,\imath}(k)$$

we obtain

$$k^\mu\,\widetilde{G}^{\,a}_{\mu,\imath}(k) = i\boldsymbol{\tau}^a_{\imath\jmath}\,\langle\,0\,|\,\Phi_\jmath(0)\,|\,0\,\rangle \tag{1.58}$$

and from the Lorentz invariance we necessarily get the general structure

$$\widetilde{G}^{\,a}_{\mu,\imath}(k) = k_\mu\,\tilde{g}_\imath^{\,a}(k^2)$$

that eventually yields

$$k^2\,\tilde{g}_\imath^{\,a}(k^2) = i\boldsymbol{\tau}^a_{\imath\jmath}\,\langle\,0\,|\,\Phi_\jmath(0)\,|\,0\,\rangle \qquad\qquad \widetilde{G}^{\,a}_{\mu,\imath}(k) = k_\mu\,\boldsymbol{\tau}^a_{\imath\jmath}\,\langle\,0\,|\,\Phi_\jmath(0)\,|\,0\,\rangle\;\frac{i}{k^2} \tag{1.59}$$

The very last equality implies that, if some of the vacuum expectation values of the fields do not vanish, then there are mass-less poles in the corresponding Green's functions. It turns out that the very last relation (1.59) is the quantum counterpart of the semi-classical relation (1.53): namely, to each broken generator of the internal global symmetry group G there is a zero mode of the classical square mass symmetric matrix and a pole in the corresponding quantum Green's function. THIS IS THE GENERAL STATEMENT OF THE GOLDSTONE'S THEOREM. There is an important remark to be added. The above derivation of the Goldstone's theorem deals with quantum Green's functions which, generally speaking, are plagued by ultra-violet as well as infrared divergences, as we shall thoroughly discuss in the next chapters. The latter are due to the field interactions at the higher orders of perturbation theory, the so called radiative corrections. Now it turns out that the Goldstone's theorem can be proven to hold true for the renormalized Green's functions, which are finite and free from the ultraviolet divergences to any order of perturbation theory. Thus the above derivation can be suitably generalized, *mutatis mutandis*, to renormalized fields, masses and couplings leaving untouched the correspondence among broken generators, zero modes of the classical square mass matrix and mass-less poles of the quantum renormalized Green's functions.

1.2.2 Symmetry Breaking by Radiative Corrections

It is very interesting to find the stationary points of the effective potential up to the 1-loop approximation. For the sake of simplicity, to better grasp the role of the symmetries and to obtain analytic solutions in closed form, it is convenient to consider the mass-less case. Turning back for the sake of simplicity to the natural unit system, for $m = 0$ we get

$$\mathcal{V}_{\mathrm{eff}}(\varphi) = \frac{\lambda\varphi^4}{4!}\left\{1 + \frac{3\lambda}{32\pi^2}\left[\ln\left(\frac{\lambda\,\varphi^2}{2\mu^2}\right) - \frac{3}{2}\right]\right\} + O(\lambda^3)$$

because in the spin-less, charge-less and mass-less case we can write

$$\begin{aligned}\Gamma[\varphi] \;&\approx\; (2\pi)^4\,\delta(0)\left[\mathcal{V}_{\mathrm{eff}}^{(0)}(\varphi) + \mathcal{V}_{\mathrm{eff}}^{(1)}(\varphi)\right] \\ &=\; (2\pi)^4\,\delta(0)\left\{\frac{\lambda\varphi^4}{4!} + \frac{1}{2}\,\ln\,\mathtt{reg}\,\det\,\|\,\mathbb{I} - \lambda\varphi^2/2\partial_E^2\,\|\right\} \qquad (1.60)\end{aligned}$$

where it appears to be clear that the first 1-loop quantum correction is a very small quantity of order $\lambda\ln\lambda$ for $0 < \lambda \ll 1$, *i.e.* in the perturbative regime. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\,\mathcal{V}_{\mathrm{eff}}(\varphi) = \frac{\lambda}{6}\,\varphi^3\left\{1 + \frac{3\lambda}{32\pi^2}\left[\ln\left(\frac{\lambda\,\varphi^2}{2\mu^2}\right) - 1\right]\right\} + O(\lambda^3)$$

so that we find the extreme values, up to the 1-loop approximation,

$$\varphi_0 = 0 \;\vee\; \ln\left(\frac{\hbar\lambda\,\varphi_*^2}{2\mu^2c^3}\right) = -\,\frac{32\pi^2}{3\lambda} + 1 + O(\lambda)$$

Since we are within the perturbative regime we must require $0 < \lambda < 1$ and consequently

$$\varphi_*^{(\pm)} \;\simeq\; \pm\,\frac{\mu c}{\hbar}\,\sqrt{\frac{2\mathrm{e}\hbar c}{\lambda}}\;\exp\left\{-\,\frac{16\pi^2}{3\lambda}\right\}$$

$$\mathcal{V}_{\mathrm{eff}}(\varphi_*) \simeq -\,\frac{\hbar c}{4!}\left(\frac{\mu c}{\hbar}\right)^4\,\frac{\mathrm{e}^2}{16\pi^2}\;\exp\left\{-\,\frac{64\pi^2}{3\lambda}\right\} < \mathcal{V}_{\mathrm{eff}}(0) = 0$$

Hence, if we take the first quantum corrections into account, a new pair of $\mathbb{Z}_2-$symmetric minima do appear, while the classical null value of the massless scalar field becomes a local maximum. This is an example of the the phenomenon of spontaneous breaking of the $\mathbb{Z}_2$ symmetry, or internal reflection symmetry $\phi(x) \to -\,\phi(x)$, induced by radiative corrections. Notice that the minima $\varphi_*^{(\pm)}$ are non-analytic in λ and exponentially close to the null local maximum. **Thus they cannot be reached within the framework of the perturbative approach** so that the perturbative vacuum expectation value of the quantum Hamiltonian keeps standing at $\varphi_0 = 0$. It seems at a first sight that the 1-loop effective potential is arbitrary, owing to the presence of the unknown mass μ, which is due in turn to the $\zeta-$function regularization technique. Needless to say, any other regulator, such as UV cut-off, Pauli-Villars, dimensional etc., necessarily entails the appearance of some arbitrary mass or length scale. However, to remove that arbitrariness and to recover the physical meaning of the effective potential, we can introduce the concept of **effective** or **running** coupling parameter as follows. Suppose to fix the coupling parameter of the theory by the requirement

$$\frac{\mathrm{d}^4}{\mathrm{d}\varphi^4}\,\mathcal{V}_{\mathrm{eff}}(\varphi) = \lambda(M) \qquad \text{at } \varphi = \frac{Mc}{\hbar}\,\sqrt{\hbar c} \qquad (1.61)$$

that yields

$$\ln\left(M^2\lambda(M)/2\mu^2\right) = -\,\frac{8}{3} \qquad\qquad \mu = M\mathrm{e}^{\frac{4}{3}}\sqrt{\tfrac{1}{2}\lambda(M)}$$

Thus we can truly remove the arbitrary regularization mass μ in favor of some actual, empirical energy scale $\sqrt{s} = Mc^2$, that could represent the center of momentum energy of some laboratory collision of two spin-less mass-less particles. In so doing we get

$$\mathcal{V}_{\mathrm{eff}}(\varphi) = \frac{\lambda(M)}{4!\hbar c}\,\varphi^4\left\{1 + \frac{3\lambda(M)}{32\pi^2}\left[\ln\left(\frac{\hbar\varphi^2}{M^2c^3}\right) - \frac{25}{6}\right]\right\} + O(\lambda^3(M)) \tag{1.62}$$

in full agreement with the famous result of Sidney Coleman and Eric Weinberg, *The Physical Review* D**7** (1973) 1888. From the condition

$$\frac{\mathrm{d}}{\mathrm{d}M}\,\mathcal{V}_{\mathrm{eff}}(\varphi) = 0$$

we immediately derive

$$\begin{aligned}\beta(\lambda) \equiv M\,\frac{\mathrm{d}\lambda}{\mathrm{d}M} \quad &\approx \quad \frac{3\lambda^2}{16\pi^2}\left\{1 + \frac{3\lambda}{16\pi^2}\left[\ln\left(\frac{\hbar\varphi^2}{M^2c^3}\right) - \frac{25}{6}\right]\right\}^{-1} \\ &= \quad \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)\end{aligned} \tag{1.63}$$

and after integration

$$\lambda(M) \approx \frac{\lambda(M_0)}{1 - \dfrac{3\lambda(M_0)}{16\pi^2}\cdot\ln\dfrac{M}{M_0}} \tag{1.64}$$

Thus we see that the requirement for the effective potential to be an invariant physical quantity just drives to the concept of the running coupling, *i.e.* a scale dependent coupling $\lambda(M)$. Notice that by taking the upper bound of the validity for the perturbation theory $\lambda(M_0) = 1$, then we find a singularity – the famous Landau-Pomerančhuk pole – at $M \simeq 10^{23} \times M_0$, which is a huge energy scale even for M_0 of the Standard Model order, *i.e.* hundred of GeV. Then the pair of $\mathbb{Z}_2-$symmetric and scale independent minima of the effective potential occur at

$$\lambda(M)\,\ln\frac{\varphi^2}{M^2} \;=\; -\,\tfrac{32}{3}\pi^2 + \tfrac{8}{3}\lambda(M) + O(\lambda^2(M)) \tag{1.65}$$

$$\varphi_*^{(\pm)} = \pm\,M\,\exp\left\{\frac{4}{3}\left[1 - \frac{4\pi^2}{\lambda(M)}\right]\right\} \tag{1.66}$$

$$\mathcal{V}(\varphi_*) \;=\; -\,\frac{3\lambda^2(M)}{512\pi^2}\cdot\frac{M^4c^5}{\hbar^3}\;\exp\left\{\frac{16}{3}\left[1 - \frac{4\pi^2}{\lambda(M)}\right]\right\} \tag{1.67}$$

It is worthwhile to remark that in the mass-less case the classical Action turns out to be invariant under the scale transformations

$$x^\mu \longrightarrow y^\mu = x^\mu\,\mathrm{e}^\alpha \qquad\qquad \phi(x) \longrightarrow \overline{\phi}(y) = \mathrm{e}^{-\alpha}\,\phi(x) \qquad\qquad (\,\alpha \in \mathbb{R}\,)$$

because evidently

$$\int \mathrm{d}^4y\;\left[\frac{1}{2}\,\partial_\mu\overline{\phi}_y\partial^\mu\overline{\phi}_y - \frac{\lambda}{4!}\overline{\phi}_y^{\,4}\right] = \int \mathrm{d}^4x\;\left[\frac{1}{2}\,\partial_\mu\phi_x\partial^\mu\phi_x - \frac{\lambda}{4!}\phi_x^4\right]$$

On the contrary, the 1-loop effective Action for a constant field configuration is no longer invariant under scale transformations and so does, *a fortiori*, the quantum effective Action as well. Actually, for $\varphi \longrightarrow \overline{\varphi} = \mathrm{e}^{-\alpha}\,\varphi$ and $\alpha = \ln(M_0/M)$, from (1.62) we find

$$\mathrm{d}^4y\;\mathcal{V}_{\mathrm{eff}}(\overline{\varphi}) = \mathrm{d}^4x\,\left[\,\mathcal{V}_{\mathrm{eff}}(\varphi) - \alpha\cdot\frac{\lambda^2}{16\pi^2}\cdot\frac{\varphi^4}{8}\,\right]$$

and thus we see that the sole effect of the dilatation transformations - up to the 1-loop approximation - is to change the running coupling λ according to

$$\lambda(M_0)\;\;\longrightarrow\;\;\lambda(M_0) + \ln(M/M_0)\,\frac{3\lambda^2(M_0)}{16\pi^2}$$

or even

$$\lambda(M_0)\;\;\longrightarrow\;\;\lambda(M) \approx \lambda(M_0)\left\{1 + \frac{3\lambda(M_0)}{16\pi^2}\cdot\ln\frac{M}{M_0}\right\}$$

which coincides with the previously obtained relationship (1.64). This very important example tells us that even though the coupling is a constant small number at the classical level, it always develops a nontrivial scale dependence as a result of the quantum effects. In this particular case, it turns out that at large space-time scales the running coupling decreases, which means that the free field theory is a good zeroth order approximation. Hence, we have learned that in Quantum Field Theories generally speaking the running couplings **must be defined** at some specific scale, for they usually develop quantum scale dependence.

1.2.3 Symmetry Breaking by the Higgs Mechanism

Consider a charged scalar field interacting with a neutral mass-less Abelian vector field, the dynamics being governed by the classical Lagrangian[2]

$$\mathcal{L} = g^{\mu\nu}\,D_\mu\phi\,(\,D_\nu\phi\,)^* + \mu^2\,\phi\phi^* - \lambda\,(\phi\phi^*)^2 - \tfrac{1}{4}\,\mathrm{F}^{\mu\nu}\,\mathrm{F}_{\mu\nu}$$

$$D_\mu\phi \equiv (\,\partial_\mu - igA_\mu\,)\,\phi \qquad\qquad \mathrm{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

with λ and g positive coupling constants. The Lagrangian is invariant under the full non-homogeneous Lorentz group IO(1,3) as well as with respect to the gauge group U(1) of local phase transformations

$$\phi(x)\;\mapsto\;\phi'(x) = \phi(x)\,\exp\{ig\theta\,(x)\}$$

$$A_\mu(x)\;\mapsto\;A'_\mu(x) = A_\mu(x) + \partial_\mu\theta\,(x)$$

[2]Peter Ware Higgs, *Broken Symmetries, Massless Particles and Gauge Fields*, Physics Letters B**12** (1964) 132-133; *Broken Symmetries and the Masses of Gauge Bosons*, Physical Review Letters **13** (1964) 508-509; Francois Englert and Robert Brout, *Broken Symmetries and the Mass of Gauge Vector Mesons*, Physical Review Letters **13** (1964) 321-323; Gerald Guralnik, Carl Richard Hagen and Tom Kibble, *Global Conservation Laws and Massless Particles*, Physical Review Letters **13** (1964) 585-587.

The Lagrangian can be rewritten in the more explicit form

$$\begin{aligned}\mathcal{L} &= \partial^{\mu}\phi\,\partial_{\mu}\phi^{*} + \mu^{2}\,\phi^{*}\,\phi - \tfrac{1}{2}\,\partial^{\mu}A^{\nu}\,\partial_{\mu}A_{\nu} + \tfrac{1}{2}\,\partial^{\mu}A^{\nu}\,\partial_{\nu}A_{\mu} \\ &- \lambda\,(\,\phi^{*}\,\phi\,)^{2} + igA^{\mu}\,(\,\phi^{*}\,\partial_{\mu}\phi - \phi\,\partial_{\mu}\phi^{*}\,) + g^{2}\,A^{\mu}\,\phi\,A_{\mu}\,\phi^{*} \qquad (1.68)\end{aligned}$$

Then we evidently have

$$\delta\mathcal{L}/\delta\,\partial^{\mu}\phi = (\,D_{\mu}\phi\,)^{*} = \partial_{\mu}\phi^{*} + igA_{\mu}\phi^{*}$$
$$\delta\mathcal{L}/\delta\,\partial^{\mu}\phi^{*} = D_{\mu}\phi = \partial_{\mu}\phi - igA_{\mu}\phi$$
$$\delta\mathcal{L}/\delta\,\partial_{\mu}A_{\nu} = -\,\mathrm{F}^{\mu\nu}$$

$$\delta\mathcal{L}/\delta\phi = -\,igA\cdot(\,D\phi\,)^{*} + \mu^{2}\phi^{*} - 2\lambda\,\phi^{*}\,(\,\phi\phi^{*}\,)$$
$$\delta\mathcal{L}/\delta\phi^{*} = igA\cdot D\phi + \mu^{2}\phi - 2\lambda\,\phi\,(\,\phi\phi^{*}\,)$$

$$\begin{aligned}\delta\mathcal{L}/\delta A^{\nu} &= g\,\phi^{*}\,i\,\overleftrightarrow{\partial}_{\nu}\,\phi + 2g^{2}A_{\nu}\,\phi\phi^{*} \\ &= ig\,(\,\phi^{*}\,D_{\nu}\phi - \phi\,D_{\nu}^{*}\phi^{*}\,) \equiv ig\,\phi^{*}\,\overleftrightarrow{D}_{\nu}\;\phi\end{aligned}$$

so that the Euler-Lagrange classical field equations read

$$D_{\mu}^{*}\,(\,D^{\mu}\phi\,)^{*} - \mu^{2}\,\phi^{*} + 2\lambda\,\phi^{*}\phi\phi^{*} = 0 \qquad (1.69)$$
$$D^{\mu}D_{\mu}\phi - \mu^{2}\,\phi + 2\lambda\,\phi\,\phi^{*}\,\phi = 0 \qquad (1.70)$$
$$\partial^{\mu}\mathrm{F}_{\mu\nu} + ig\phi^{*}\,\overleftrightarrow{D}_{\nu}\;\phi = 0 \qquad (1.71)$$

Notice that, by taking the tetra-divergence of the very last equation of motion we immediately get that the U(1) tetra-current does satisfy the continuity equation

$$\partial_{\mu}J^{\mu} = 0 = \partial^{\mu}\left(\phi^{*}\,i\,\overleftrightarrow{D}_{\mu}\;\phi\right) \qquad (1.72)$$

which guarantees U(1) Abelian charge conservation

$$Q \equiv (\,-\,g\,)\int \mathrm{d}\mathbf{r}\;\phi^{*}(t,\mathbf{r})\,i\,\overleftrightarrow{D}_{t}\,\phi(t,\mathbf{r}) \qquad \dot{Q} = 0 \qquad (1.73)$$

It is also important to realize that the covariant derivative $D_{\mu}\,\phi$ of the charged scalar field transforms homogeneously with respect to the gauge transformations

$$\begin{cases}\phi(x)\;\mapsto\;\phi'(x) = \phi(x)\,\exp\{ig\,\theta(x)\} \\ \\ A_{\mu}(x)\;\mapsto\;A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu}\,\theta(x)\end{cases}$$

In fact we obtain

$$\begin{aligned}D'_{\mu}\phi'(x) &= \partial_{\mu}\,[\,\phi(x)\,\exp\{ig\theta(x)\}\,] \\ &- ig\,[\,A_{\mu}(x) + \partial_{\mu}\,\theta(x)\,]\,\phi(x)\,\exp\{ig\theta(x)\} \\ &= \exp\{ig\theta(x)\}\,D_{\mu}\,\phi(x) \qquad (1.74)\end{aligned}$$

which entails that the U(1) Abelian tetra-current is also gauge invariant, *viz.,*

$$J'_\mu(x) = \Big(-\, ig\,\phi^*(x)\,\overset{\leftrightarrow}{D}_\mu\,\phi(x)\Big)' = J_\mu(x)$$

The canonical energy and momentum tensor is provided by the Noether theorem expression

$$\begin{aligned} T_{\mu\nu} &\equiv (\,\delta\mathcal{L}/\delta\,\partial^\mu\phi\,)\,\partial_\nu\,\phi + (\,\delta\mathcal{L}/\delta\,\partial^\mu\phi^*\,)\,\partial_\nu\,\phi^* \\ &+ (\,\delta\mathcal{L}/\delta\,\partial^\mu A_\lambda\,)\,\partial_\nu\,A_\lambda - \mathcal{L}\,g_{\mu\nu} \\ &= (\,D_\mu\,\phi\,)^*\,\partial_\nu\,\phi + (\,D_\mu\,\phi\,)\;\partial_\nu\,\phi^* - \mathrm{F}_{\mu\lambda}\,\partial_\nu\,A^\lambda - \mathcal{L}\,g_{\mu\nu} \end{aligned} \tag{1.75}$$

Then we obtain the energy of the classical system

$$\begin{aligned} P_0 &= \int \mathrm{d}\mathbf{r}\,\Big\{\dot\phi\,(t,\mathbf{r})\,D_0^*\phi^*(t,\mathbf{r}) + \dot\phi^*(t,\mathbf{r})\,D_0\,\phi\,(t,\mathbf{r})\Big] \\ &- \;\mathbf{E}\,(t,\mathbf{r})\cdot\dot{\mathbf{A}}\,(t,\mathbf{r}) - \mathcal{L}\,(t,\mathbf{r})\Big\} \end{aligned} \tag{1.76}$$

where we have set

$$A^\mu(t,\mathbf{r}) \equiv \Big(\varphi\,(t,\mathbf{r})\,,\;\mathbf{A}(t,\mathbf{r})\Big)$$

and taking into account that we have the canonical field momenta

$$\begin{aligned} \delta\,\mathcal{L}/\delta\,\dot\phi\,(t,\mathbf{r}) \equiv \Pi^*(t,\mathbf{r}) &= D_0^*\phi^*(t,\mathbf{r}) \\ \delta\,\mathcal{L}/\delta\,\dot\phi^*(t,\mathbf{r}) \equiv \Pi\,(t,\mathbf{r}) &= D_0\,\phi\,(t,\mathbf{r}) \\ \delta\,\mathcal{L}/\delta\,\dot\varphi\,(t,\mathbf{r}) &= 0 \\ \delta\,\mathcal{L}/\delta\,\dot{\mathbf{A}}(t,\mathbf{r}) \;&\equiv\; -\,\mathbf{E}\,(t,\mathbf{r}) \end{aligned}$$

so that

$$\begin{aligned} \dot\phi\,(t,\mathbf{r}) &= \Pi\,(t,\mathbf{r}) + ig\,\varphi\,(t,\mathbf{r})\,\phi\,(t,\mathbf{r}) \\ \dot{\mathbf{A}}(t,\mathbf{r}) &= -\,\mathbf{E}\,(t,\mathbf{r}) - \boldsymbol{\nabla}\varphi\,(t,\mathbf{r}) \end{aligned}$$

we find the classical Hamiltonian functional

$$\begin{aligned} H &= \int \mathrm{d}\mathbf{r}\,\Big\{\Big[\,\Pi\,(t,\mathbf{r}) + ig\,\varphi\,(t,\mathbf{r})\,\phi\,(t,\mathbf{r})\Big]\,\Pi^*(t,\mathbf{r}) \\ &+ \;\Big[\,\Pi^*(t,\mathbf{r}) - ig\,\varphi\,(t,\mathbf{r})\,\phi^*(t,\mathbf{r})\Big]\,\Pi\,(t,\mathbf{r}) \\ &+ \;\mathbf{E}^2(t,\mathbf{r}) + \mathbf{E}\,(t,\mathbf{r})\cdot\boldsymbol{\nabla}\varphi\,(t,\mathbf{r}) - \mathcal{L}\,(t,\mathbf{r})\Big\} \end{aligned} \tag{1.77}$$

Since we can write

$$\mathcal{V}(\phi\,\phi^*) \equiv -\,\mu^2\,\phi\,\phi^* + \lambda\,(\,\phi\,\phi^*\,)^2 \qquad \mathbf{B} = (\,F_{32}, F_{13}, F_{21}\,)$$

$$\begin{aligned} \mathcal{L}\,(t,\mathbf{r}) &= \;|\;\Pi\,(t,\mathbf{r})|^2 - |\;D\phi\,(t,\mathbf{r})|^2 - \mathcal{V}(\phi\,\phi^*) \\ &+ \;\tfrac{1}{2}\,\mathbf{E}^2(t,\mathbf{r}) - \tfrac{1}{2}\,\mathbf{B}^2(t,\mathbf{r}) \end{aligned} \tag{1.78}$$

we eventually obtain the total momentum

$$\mathbf{P} = -\int \mathrm{d}\mathbf{r} \Big[\mathrm{E}^k(t,\mathbf{r}) \boldsymbol{\nabla} A^k(t,\mathbf{r}) + \Pi^*(t,\mathbf{r}) \boldsymbol{\nabla} \phi(t,\mathbf{r}) + \Pi(t,\mathbf{r}) \boldsymbol{\nabla} \phi^*(t,\mathbf{r}) \Big]$$

and the classical Hamiltonian functional of the fields and their conjugate momenta

$$\begin{aligned} H &= \int \mathrm{d}\mathbf{r} \,\Big\{ \Pi\,(t,\mathbf{r})\,\Pi^*(t,\mathbf{r}) + D_k\,\phi\,(t,\mathbf{r})\,D^*_k\,\phi^*(t,\mathbf{r}) + \mathcal{V}(\phi\,\phi^*) \\ &+ \; ig\,\varphi\,(t,\mathbf{r})[\,\phi\,(t,\mathbf{r})\,\Pi^*(t,\mathbf{r}) - \phi^*(t,\mathbf{r})\,\Pi\,(t,\mathbf{r})\,] \\ &+ \; \tfrac{1}{2}\,\mathbf{E}^2(t,\mathbf{r}) + \tfrac{1}{2}\,\mathbf{B}^2(t,\mathbf{r}) + \mathbf{E}\,(t,\mathbf{r}) \cdot \boldsymbol{\nabla}\varphi\,(t,\mathbf{r}) \Big\} \\ &= \int \mathrm{d}\mathbf{r} \,\Big\{ \Pi\,(t,\mathbf{r})\,\Pi^*(t,\mathbf{r}) + D_k\,\phi\,(t,\mathbf{r})\,D^*_k\,\phi^*(t,\mathbf{r}) + \mathcal{V}(\phi\,\phi^*) \\ &+ \; \tfrac{1}{2}\,\mathbf{E}^2(t,\mathbf{r}) + \tfrac{1}{2}\,\mathbf{B}^2(t,\mathbf{r}) + \mathbf{E}\,(t,\mathbf{r}) \cdot \boldsymbol{\nabla}\varphi\,(t,\mathbf{r}) + \varphi\,(t,\mathbf{r})\,J_0\,(t,\mathbf{r}) \Big\} \end{aligned} \tag{1.79}$$

From the Gauß law

$$\partial^k\,\mathrm{F}_{k0} + ig\,\phi^*\,\overset{\leftrightarrow}{D}_0\,\phi = 0 \quad \Leftrightarrow \quad \boldsymbol{\nabla}\cdot\mathbf{E}\,(t,\mathbf{r}) = J_0\,(t,\mathbf{r})$$

we see that the last two terms of the classical Hamiltonian can be rewritten as

$$\int \mathrm{d}\mathbf{r}\,\Big\{ \mathbf{E}\,(t,\mathbf{r}) \cdot \boldsymbol{\nabla}\varphi\,(t,\mathbf{r}) + \varphi\,(t,\mathbf{r})\,J_0\,(t,\mathbf{r}) \Big\} = \int \mathrm{d}\mathbf{r}\;\boldsymbol{\nabla}\cdot\Big(\mathbf{E}\,(t,\mathbf{r})\,\varphi\,(t,\mathbf{r})\Big)$$

in such a manner that we can eventually write – up to a boundary term

$$\begin{aligned} H &\doteq \int \mathrm{d}\mathbf{r}\,\Big\{ \Pi\,(t,\mathbf{r})\,\Pi^*(t,\mathbf{r}) + D_k\,\phi\,(t,\mathbf{r})\,D^*_k\,\phi^*(t,\mathbf{r}) \\ &+ \; \tfrac{1}{2}\,\mathbf{E}^2(t,\mathbf{r}) + \tfrac{1}{2}\,\mathbf{B}^2(t,\mathbf{r}) - \mu^2\,\phi\,\phi^* + \lambda\,(\phi\,\phi^*)^2 \Big\} \end{aligned} \tag{1.80}$$

which in not positive semi-definite, owing to the unconventional mass term for the charged scalar field. The extreme field configurations of the classical Hamiltonian functional do correspond to a constant and homogeneous scalar fields $\phi(t,\mathbf{r}) = \phi_o$ and to a null vector potential $A^\mu_o(t,\mathbf{r}) = 0$ that yields

$$H_o = \mathrm{V}\,\big\{\lambda\,(\phi_o\,\phi^*_o)^2 - \,\mu^2\,\phi_o\,\phi^*_o\big\}$$

where V is the volume of a very large spatial box in which the system is supposed to be enclosed. The extreme constant field configurations read

$$\frac{\delta\,H_o}{\delta\,\phi_o} = \mathrm{V}\,\phi^*_o\,\big\{2\lambda\,(\phi_o\,\phi^*_o) - \,\mu^2\big\} = 0 \quad \Leftrightarrow \quad \phi_o = 0\;\vee\;\phi_o\,\phi^*_o = \frac{\mu^2}{2\lambda}$$

and therefore

$$H_o(0) = 0 \qquad\qquad H_o(\phi_o\,\phi^*_o) \;=\; -\,\mathrm{V}\,\mu^4\,c^5/4\lambda\,\hbar^3$$

so that the minimal constant field configurations do describe a U(1) symmetric circle of ray $\mu/\sqrt{2\lambda}$ in the complex ϕ_o plane.

Quantum Theory and the Feynman Rules in the R_ξ gauges

In order to set up the quantum version of the present gauge invariant Abelian model and to derive the general Feynman rules, one has:

i) to redefine the field functions as follows: namely,

$$|\langle 0\,|\,\phi(x)\,|\,0\,\rangle\,| = \frac{\upsilon}{\sqrt{2}} \;\equiv\; \frac{\mu}{\sqrt{2\lambda}}$$

$$\phi(x) = \frac{1}{\sqrt{2}}\,[\,\upsilon + \eta(x)\,]\;\exp\{i\chi(x)/\upsilon\}$$

$$A_\mu(x) \equiv V_\mu(x) + \partial_\mu\,\chi(x)/\upsilon g$$

ii) to add a gauge fixing: for example, the simplest choice is the linear and Lorentz invariant gauge fixing Lagrangian

$$\mathcal{L}_{\,\mathrm{g.f.}} = V^{\,\nu}\partial_\nu\,B + \tfrac{1}{2}\,\xi B^2 = \left(A^\nu - \frac{\sqrt{\lambda}}{g\mu}\,\partial^{\,\nu}\chi\right)\partial_\nu B + \tfrac{1}{2}\,\xi B^2$$

leading to the non-homogeneous Lorenz condition $\partial\cdot V = \xi B$, where $B(x)$ is the auxiliary scalar field and $\xi\in\mathbb{R}$ the gauge-fixing parameter. This one-parameter family of gauge choices is often called in the Literature the R_ξ gauges [3]. Since the covariant derivative of the charged scalar field transforms homogeneously under gauge transformations we can write

$$\begin{aligned} D_\mu\,\phi(x) &= \exp\{i\chi(x)/\upsilon\}\,[\,\partial_\mu - igV_\mu(x)\,]\;[\,\upsilon+\eta(x)\,]/\sqrt{2} \\ &= \exp\{i\chi(x)/\upsilon\}\,[\,\partial_\mu\,\eta(x) - igV_\mu(x)\,\eta(x) - ig\upsilon V_\mu(x)\,]\,/\sqrt{2} \end{aligned} \tag{1.81}$$

and consequently

$$\begin{aligned} D^{\,\mu}\,\phi(x)\,D^*_\mu\,\phi^*(x) &= \tfrac{1}{2}\,[\,\partial^{\,\mu}\,\eta(x) - igV^{\,\mu}(x)\,\eta(x) - ig\upsilon V^{\,\mu}(x)\,] \\ &\times\;\; [\,\partial_\mu\,\eta(x) + igV_\mu(x)\,\eta(x) + ig\upsilon V_\mu(x)\,] \\ &= \tfrac{1}{2}\,g^{\,\mu\nu}\,\partial_\mu\eta\,(x)\,\partial_\nu\,\eta\,(x) + \tfrac{1}{2}\,g^2\,V^{\,2}(x)\,\eta^2(x) \\ &+\;\; \upsilon\,g^2\,V^{\,2}(x)\,\eta\,(x) + \tfrac{1}{2}\,g^2\,\upsilon^2\,V^{\,2}(x) \end{aligned} \tag{1.82}$$

Moreover, since we have $\upsilon^2 = \mu^2/\lambda$, we eventually obtain

$$\begin{aligned} \mathcal{V}(\phi\phi^*) &= -\,\tfrac{1}{2}\,\mu^2\,[\,\upsilon+\eta\,(x)\,]^2 + \tfrac{1}{4}\,\lambda\,[\,\upsilon+\eta\,(x)\,]^4 \\ &= -\,\frac{\mu^4}{4\lambda} + \mu^2\,\eta^2(x) + \mu\,\sqrt{\lambda}\,\eta^3(x) + \tfrac{1}{4}\,\lambda\,\eta^4(x) \\ -\,\tfrac{1}{4}\,\mathrm{F}^{\,\mu\nu}\,\mathrm{F}_{\,\mu\nu} &= -\,\tfrac{1}{2}\,\partial^{\,\mu}\,V^{\,\nu}(x)\,\partial_\mu\,V_\nu(x) + \tfrac{1}{2}\,\partial^{\,\mu}\,V^{\,\nu}(x)\,\partial_\nu\,V_\mu(x) \end{aligned} \tag{1.83}$$

[3] Gerard 't Hooft & Martin J.G. Veltman (1972) *Regularization and Renormalization of Gauge Fields*, Nucl. Phys. **B44**, 189-213; Benjamin W. Lee & Jean Zinn-Justin (1972) *Spontaneously Broken Gauge Symmetries*, Phys. Rev. D**5**, 3121-3160.

It follows that the new Lagrange density can be recast in the form

$$\mathcal{L}' = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}} + \mathcal{L}_{\mathrm{g.f.}} + \frac{\mu^4}{4\lambda}$$

where the very last term is the opposite of a finite zero-point energy density, while

$$\begin{aligned} \mathcal{L}_0 &= \tfrac{1}{2}\, g^{\mu\nu}\, \partial_\mu \eta(x)\, \partial_\nu \eta(x) - \mu^2\, \eta^2(x) \\ &- \tfrac{1}{4}\, \mathrm{F}^{\mu\nu}(x)\, \mathrm{F}_{\mu\nu}(x) + \tfrac{1}{2}\, m^2\, V^{\mu}(x)\, V_\mu(x) \end{aligned} \tag{1.84}$$

in which we keep the dimensionless couplings g, λ as well as the mass scale μ as independent parameters and set

$$m^2 \equiv g^2\, \mu^2/\lambda \qquad\qquad \upsilon \equiv \frac{\mu}{\sqrt{\lambda}}$$

together with

$$\begin{aligned} \mathcal{L}_{\mathrm{I}} &= \tfrac{1}{2}\, g^2\, V^{\nu}(x)\, V_\nu(x)\, \eta^2(x) + \frac{\mu g^2}{\sqrt{\lambda}}\, \eta(x)\, V^{\mu}(x)\, V_\mu(x) \\ &- \mu\sqrt{\lambda}\, \eta^3(x) - \tfrac{1}{4}\, \lambda\, \eta^4(x) \end{aligned} \tag{1.85}$$

$$\mathcal{L}_{\mathrm{g.f.}} = V^{\nu}(x)\, \partial_\nu B(x) + \tfrac{1}{2}\, \xi B^2(x) \tag{1.86}$$

Notice that the so called Goldstone field $\chi(x)$ is completely disappeared from the R_ξ gauge-fixed Lagrangian, while a Proca mass term arose for the Abelian vector field $V^{\mu}(x)$. In this manner, the total number of field degrees of freedom does not vary, for we pass from 2 + 2 to 1 + 3 real independent field components, once we restrict ourselves to the physical polarization of the massless and massive vector potentials A^μ and V_ν respectively. This is known as the Higgs mechanism to generate the vector boson mass from the spontaneous breaking of the gauge invariance. The Euler-Lagrange field equations now read

$$(\Box + m^2)\, V^{\nu} = -(1-\xi)\, \partial^{\nu} B - g^2\, \eta \left(\frac{2\mu}{\sqrt{\lambda}} + \eta \right) V^{\nu} \tag{1.87}$$

$$(\Box + 2\mu^2)\, \eta = g^2\, V \cdot V \left(\frac{\mu}{\sqrt{\lambda}} + \eta \right) - 3\mu\sqrt{\lambda}\, \eta^2 - \lambda\, \eta^3 \tag{1.88}$$

$$\xi B = \partial \cdot V \qquad\qquad \left(\Box + \xi m^2\right) B = 0 \tag{1.89}$$

and by taking the tetra-divergence of the first equation we get

$$\left[\partial \cdot V \left(\eta + \frac{\mu}{\sqrt{\lambda}} \right) + 2V \cdot \partial\eta \right] \left(\eta + \frac{\mu}{\sqrt{\lambda}} \right) = 0 \tag{1.90}$$

which is nothing but the continuity equation (1.72) for the U(1) Abelian current. If we substitute the first equation (1.89) for the auxiliary field into the gauge fixing Lagrangian (1.86) we find

$$\mathcal{L}_{\mathrm{g.f.}} = \frac{1}{2\xi}\, V^{\rho}(x)\, \partial_\rho \partial_\nu V^{\nu}(x) \tag{1.91}$$

in such a manner that we can eventually recast the full kinetic term for the vector and scalar fields in the form

$$
\begin{aligned}
\mathcal{L}_0 + \mathcal{L}_{\text{g.f.}} &= \tfrac{1}{2}\, V^{\nu}(x)\,\mathcal{K}_{\nu\rho} V^{\rho}(x) + \tfrac{1}{2}\, g^{\rho\nu}\, \partial_\rho \eta(x)\, \partial_\nu \eta(x) - \mu^2\, \eta^2(x) \\
\mathcal{K}_{\nu\rho} &= \left(\Box + g^2\mu^2/\lambda\right)\, g_{\rho\nu} - \partial_\rho\, \partial_\nu \left(1 - \frac{1}{\xi}\right) \qquad (1.92) \\
\mathcal{L}_{\mathrm{I}} &= \tfrac{1}{2}\, g^2\, V^{\nu}(x)\, V_\nu(x)\, \eta^2(x) + \frac{\mu g^2}{\sqrt{\lambda}}\, \eta(x)\, V^{\mu}(x)\, V_\mu(x) \\
&- \mu\sqrt{\lambda}\, \eta^3(x) - \tfrac{1}{4}\, \lambda\, \eta^4(x) \qquad (1.93)
\end{aligned}
$$

It follows that the nonphysical Goldstone field is decoupled and non-interacting, while the momentum space propagator for the massive vector field in the general linear and Lorentz invariant R_ξ gauges is provided, as usual, by the opposite of the inverse of the kinetic term multiplied by the imaginary unit: namely,

$$
\widetilde{D}_{\rho\nu}(k\,;\xi) = -\,i\,\widetilde{\mathcal{K}}^{-1}_{\rho\nu} = \frac{i\lambda}{\lambda k^2 - g^2\mu^2 + i\varepsilon}\,\left\{ -\, g_{\rho\nu} + \frac{\lambda\,(1-\xi)\, k_\rho\, k_\nu}{\lambda k^2 - \xi\, g^2\mu^2 + i0} \right\} \qquad (1.94)
$$

The Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\mathrm{I}}$ is invariant under the non-homogeneous Lorentz group and also under the discrete $\mathbb{Z}_2$ internal symmetry $V_\mu \mapsto -V_\mu$.

The Feynman rules can be obtained in a straightforward manner from the classical Action multiplied by the imaginary unit

$$
iS[\eta, V_\mu] = i \int \mathrm{d}^4 x\ \mathcal{L}(x)
$$

Actually we get the scalar propagator in momentum space

$$
\widetilde{D}(p) = \frac{i}{p^2 - 2\mu^2 + i\varepsilon}
$$

the Stückelberg propagator for the neutral massive vector particle

$$
\widetilde{D}_{\rho\nu}(k\,;\xi) = \frac{i\lambda}{\lambda k^2 - g^2\mu^2 + i\varepsilon}\,\left\{ -\, g_{\rho\nu} + \frac{\lambda\,(1-\xi)\, k_\rho\, k_\nu}{\lambda k^2 - \xi\, g^2\mu^2 + i0} \right\}
$$

Furthermore, after setting as usual

$$
\eta(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \widetilde{\eta}(\,p\,)\, \mathrm{e}^{-\,i\,p\,x} \qquad V_\mu(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \widetilde{V}_\mu(k)\, \mathrm{e}^{-\,i\,k\,x}
$$

we obtain the three point scalar vertex

$$
-\,6i\,\mu\,\sqrt{\lambda} \qquad (\,p_1 + p_2 + p_3 = 0\,)
$$

the four point scalar vertex

$$
-\,6i\,\lambda \qquad (\,p_1 + p_2 + p_3 + p_4 = 0\,)
$$

the three point vertex with one spinless and two massive vector neutral bosons

$$\frac{\mu g^2}{\sqrt{\lambda}}\, 2i\, g^{\mu\nu} \qquad\qquad (\, p + k_1 + k_2 = 0\,)$$

and the four point vertex with two vector and two neutral spin-less bosons

$$2i\, g^2\, g^{\mu\nu} \qquad\qquad (\, k_1 + k_2 + p_1 + p_2 = 0\,)$$

all the momenta being incoming. As a final remark, it is worthwhile to recall that the interacting neutral massive vector field in the R_ξ gauges propagates four polarization modes and the Fock space of states is of indefinite metric. Hence the Hilbert subspace of the physical states can be selected by the usual subsidiary condition which removes the nonphysical ghost-like Stückelberg quanta, *viz.*,

$$B^{(-)}(x)|\,\mathrm{phys}\,\rangle = 0$$

Moreover, in the limit $\xi \to \infty$ one recovers the unitary gauge with the non-renormalizable Proca propagator for the neutral massive vector field.

1.2.4 The Schwinger-Dyson Equations

In the non-interacting case the relation (1.26) reduces to the trivial identity

$$\widetilde{G}_0^{(2)}(\,k\,)\,\widetilde{\Gamma}_0^{(2)}(\,k\,) \;\equiv\; \frac{i}{k^2 - \left(\dfrac{mc}{\hbar}\right)^2 + i\varepsilon} \cdot \left[k^2 - \left(\frac{mc}{\hbar}\right)^2 \right] \;=\; i$$

When the interaction is switched on, the Fourier transform $\widetilde{G}^{(2)}(k)$ of the 2-point function $G^{(2)}(\,x-y\,)$ is customarily named the full or exact, or even dressed propagator

$$G^{(2)}(\,x-y\,) = \langle\, 0\,|\, T\,\phi\,(x)\,\phi\,(y)\,|\, 0\,\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\; \widetilde{G}^{(2)}(\,k\,)\; \mathrm{e}^{-\,i\,k\,x} \qquad (1.95)$$

In the interacting case it is customary [4] to introduce the self-energy invariant function by means of the relationships

$$\widetilde{G}^{(2)}(\,k\,) \stackrel{\mathrm{def}}{=} \frac{i\hbar c}{k^2 - \left(\dfrac{mc}{\hbar}\right)^2 - \hbar c\,\Sigma(\,k^2\,) + i\varepsilon}$$

$$\hbar c\,\widetilde{\Gamma}^{(2)}(\,k\,) = k^2 - \left(\frac{mc}{\hbar}\right)^2 - \hbar c\,\Sigma\,(\,k^2\,)$$

[4] Here I follow the convention and notation of [12] §**6-2-2** eq. (6-78) p. 291.

It is important to realize that we can write the famous Schwinger−Dyson equation for the full or exact or even dressed scalar propagator, *i.e.*

$$\begin{aligned}
\widetilde{G}^{(2)}(k) &\stackrel{\text{def}}{=} \widetilde{G}_0^{(2)}(k) + \widetilde{G}_0^{(2)}(k)\,\frac{1}{i}\,\Sigma(k^2)\,\widetilde{G}_0^{(2)}(k) + \cdots \\
&= \widetilde{G}_0^{(2)}(k)\sum_{n=0}^{\infty}\left[\frac{1}{i}\,\Sigma(k^2)\,\widetilde{G}_0^{(2)}(k)\right]^{n} \\
&= i\left[\widetilde{\Gamma}_0^{(2)}(k) - \Sigma(k^2)\right]^{-1} = \left[\frac{1}{i}\,\widetilde{\Gamma}^{(2)}(k)\right]^{-1}
\end{aligned} \tag{1.96}$$

It turns out that the self-energy invariant function $\Sigma(k^2)$ corresponds by construction to the sum of all the 1PI 2−point diagrams amputated by their two external free propagators, *i.e.* the sum of all the 2−point proper vertexes (the sum of the sausage chain)

$$\widetilde{G}^{(2)}(k) = \widetilde{G}_0^{(2)}(k) + \widetilde{G}_0^{(2)}(k)\,[\,(-i)\,\Sigma(k^2)\,]\,\widetilde{G}^{(2)}(k) \tag{1.97}$$

$$\Sigma(k^2) = i\sum\{\text{ all 1PI } 2-\text{point diagrams }\} \tag{1.98}$$

or more explicitly and turning to natural units

$$\begin{aligned}
\widetilde{G}^{(2)}(k) &= \frac{i}{k^2 - m^2 + i\varepsilon} \\
&+ \frac{i}{k^2 - m^2 + i\varepsilon}\,[\,(-i)\,\Sigma(k^2)\,]\,\frac{i}{k^2 - m^2 + i\varepsilon} + \cdots \\
&= \frac{i}{k^2 - m^2 + i\varepsilon}\left\{\frac{1}{1 + i\,\Sigma(k^2)\,\dfrac{i}{k^2 - m^2 + i\varepsilon}}\right\} \\
&= \frac{i}{k^2 - m^2 - \Sigma(k^2) + i\varepsilon}
\end{aligned} \tag{1.99}$$

It is very important to remark that we can always write the self-energy invariant function in terms of its loop expansion

$$\Sigma(k^2) = \sum_{\ell=1}^{\infty}\lambda^{\ell}\,\Sigma_{\ell}(k^2) \tag{1.100}$$

where $-\lambda^{\ell}\Sigma_{\ell}(k^2)$ is nothing but the divergent expression of the 2-point proper vertex $\widetilde{\Gamma}^{(2)}(k^2)$ at $\ell-$loops. For instance

$$\widetilde{\Gamma}_1^{(2)}(0) \equiv -\,\lambda\,\Sigma_1(0) = -\,\frac{\lambda}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 + i0} \tag{1.101}$$

Thus the loop expansion is a development in powers of the dimensionless small coupling parameter λ, as expected [5].

In a quite analogous way we define the spinor self-energy matrix

$$\widetilde{S}^{(2)}(\not p) \stackrel{\text{def}}{=} \frac{i\hbar c}{\not p - (Mc/\hbar) - \hbar c\,\Sigma(\not p) + i\varepsilon} \tag{1.102}$$

$$\hbar c\,\widetilde{\Gamma}^{(2)}(\not p) = \not p - \frac{Mc}{\hbar} - \hbar c\,\Sigma(\not p) \tag{1.103}$$

In the photon field case, the exact photon propagator, *i.e.*, the exact 2−point Green's function, in the Feynman gauge and in natural units, is defined by

$$\begin{aligned} \widetilde{G}^{\mu\nu}(k) &\stackrel{\text{def}}{=} \widetilde{G}_0^{\mu\nu}(k) + \widetilde{G}_0^{\mu\rho}(k)\left[i\left(k^2 g_{\rho\sigma} - k_\rho k_\sigma\right)\Pi(k^2)\right]\widetilde{G}_0^{\sigma\nu}(k) \\ &+ \cdots \\ &= \widetilde{G}_0^{\mu\nu}(k) + \widetilde{G}_0^{\mu\rho}(k)\,P_\rho^{\ \nu}\,\Pi(k^2) + \widetilde{G}_0^{\mu\rho}(k)\,P_\rho^{\ \sigma}\,P_\sigma^{\ \nu}\,\Pi^2(k^2) \\ &+ \cdots \end{aligned} \tag{1.104}$$

where the momentum space free photon propagator is

$$\widetilde{G}_0^{\mu\nu}(k) = \frac{-\,i\,g^{\mu\nu}}{k^2 + i\varepsilon}$$

whereas I have introduced the off-mass-shell transverse projector

$$P_\rho^{\ \nu} \stackrel{\text{def}}{=} \delta_\rho^{\ \nu} - k_\rho\,k^\nu/k^2 \qquad (k^2 \neq 0) \tag{1.105}$$

which satisfy $P_\rho^{\ \sigma}\,P_\sigma^{\ \nu} = P_\rho^{\ \nu}$. Then we can reduce the above expression to the sum of the formal series

$$\begin{aligned} \widetilde{G}^{\mu\nu}(k) &= \widetilde{G}_0^{\mu\nu}(k) + \widetilde{G}_0^{\mu\rho}(k)\,P_\rho^{\ \nu}\sum_{n=1}^{\infty}\left[\Pi(k^2)\right]^n \\ &= \frac{-\,i}{k^2 + i\varepsilon}\left\{\frac{1}{1-\Pi(k^2)}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{k^\mu k^\nu}{k^2}\right\} \end{aligned} \tag{1.106}$$

Thus, in the case of the photon field, we are eventually led to the gauge invariant polarization function $\Pi(k^2)$, so that we can identify the 2−point transverse gauge invariant proper vertex with

$$\begin{aligned} \widetilde{\Gamma}_{\mu\nu}(k) &\stackrel{\text{def}}{=} (-\,k^2 g_{\mu\nu} + k_\mu k_\nu)\left[1 - \Pi(k^2)\right] \\ &= k_\mu k_\nu - k^2 g_{\mu\nu} + \Pi_{\mu\nu}(k) \end{aligned} \tag{1.107}$$

$$i\,\Pi_{\mu\nu}(k) = \sum\{\,\text{all 1PI photon self}-\text{energy diagrams}\,\} \tag{1.108}$$

[5] For some unknown reason, in many textbooks it is wrongly claimed that the loop expansion is a development in powers of $\hbar$, a statement which is manifestly untrue owing to the physical dimensions [erg s] of the reduced Planck constant.

and consequently

$$\widetilde{G}^{\mu\rho}(k)\,\widetilde{\Gamma}_{\rho\nu}(k) \;=\; iP^{\mu}_{\nu} \tag{1.109}$$

1.2.5 Euclidean Effective Action

In going to the Euclidean formulation, it is convenient to define

$$Z_E = \mathrm{e}^{-W_E} \qquad \varphi_E(\bar{x}) \;=\; -\,\frac{\delta W_E[\,J_E\,]}{\delta J_E(x_E)}$$

$$\Gamma_E[\,\varphi_E\,] = \langle\, J_E\,\varphi_E\,\rangle + W_E[\,J_E\,] = \int \mathrm{d}\bar{x}\;\; J_E(\bar{x})\,\varphi_E(\bar{x}) + W_E[\,J_E\,]$$

$$J_E(\bar{x}) = \frac{\delta\,\Gamma_E[\,\varphi_E\,]}{\delta\,\varphi_E(\bar{x})}$$

so that the Euclidean effective Action is nothing but the Gibbs free enthalpy in natural units $kT = 1 = 1/\beta$, where k is the Boltzmann constant and T the absolute temperature. As a matter of fact, we nicely eventually come to the following correspondences among Euclidean functional and statistical mechanics entities in natural units : namely,

Z_E	canonical partition function (*Zustandsumme*)
W_E	Helmoltz free energy
J_E	generalized external parameter (*volume*)
φ_E	generalized external force (*pressure*)
Γ_E	Gibbs free enthalpy

We can directly obtain

$$\frac{\delta\,\varphi_E(\bar{x})}{\delta J_E(\bar{y})} \;=\; -\,\frac{\delta^{(2)}\,W_E[\,J_E\,]}{\delta J_E(\bar{x})\,\delta J_E(\bar{y})} \;=\; \left[\,\frac{\delta^{(2)}\,\Gamma_E[\,\varphi_E\,]}{\delta\,\varphi_E(\bar{x})\,\delta\,\varphi_E(\bar{y})}\,\right]^{-1} \tag{1.110}$$

and taking into account the definition of the Schwinger's functions we come to the simple relation

$$\int \mathrm{d}\bar{y}\;G^{(2)}_E(\bar{x}-\bar{y})\,\Gamma^{(2)}_E(\bar{y}-\bar{z}) \;=\; \delta(\bar{x}-\bar{z}) \tag{1.111}$$

$$\widetilde{G}^{(2)}_E(\bar{k})\,\widetilde{\Gamma}^{(2)}_E(\bar{k}) \;=\; 1 \tag{1.112}$$

where I have set

$$\Gamma^{(n)}_E(\bar{x}_1\,,\,\ldots\,,\,\bar{x}_n) \;=\; \delta^{(n)}\,\Gamma_E[\,\varphi_E\,]/\,\delta\,\varphi_E(\bar{x}_1)\,\ldots\,\delta\,\varphi_E(\bar{x}_n)\,\Big]_{\varphi_E=0} \tag{1.113}$$

Figure 1.1: the spinor self-energy 1-loop diagram

1.2.6 1-Loop Correction to the Electron Self-Energy

Exercise 1. Calculate the spinor self-energy at one loop

$$\Sigma_2(\not p) = i(ie)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \gamma^{\mu}\, D_{\mu\nu}(k)\, S(p+k)\, \gamma^{\nu}$$

in quantum electrodynamics using dimensional regularization, where $D_{\mu\nu}(k)$ is the photon propagator in the Feynman gauge, while $S(p)$ is the Dirac propagator. Segregate the structure of the divergent and finite parts.

Solution. It is convenient to set $\epsilon \equiv 2-\omega$ together with

$$\Sigma_2(\not p) = m\, A(p^2) - B(p^2)\,\not p = -\, ie^2 \mu^{\,2\epsilon} \int \frac{\mathrm{d}^{2\omega} k}{(2\pi)^{2\omega}}\, \gamma^{\,\mu}\, D_{\mu\nu}(\,p-k\,)\, S(k)\, \gamma^{\,\nu}$$

with

$$\begin{aligned} D^{\mu\nu}(\,p-k\,) \quad &= \quad \frac{-\, i g^{\mu\nu}}{(\,p-k\,)^2 + i\varepsilon} \qquad\qquad \text{Feynman gauge} \\ S(k) \quad &= \quad \frac{i\,(\not k + m)}{k^2 - m^2 + i\varepsilon} \end{aligned}$$

Taking the traces over the gamma matrices we obtain

$$\begin{aligned} A(p^2)\,\mathrm{tr}\,\mathbb{I} \quad &= \quad -\, ie^2 \mu^{2\epsilon} (2\pi)^{-\,2\omega} \int \mathrm{d}^{2\omega} k\, \frac{\mathrm{tr}\,\{\gamma^{\mu}\,\gamma_{\mu}\}}{[\,(p-k)^2 + i\varepsilon\,]\;(\,k^2 - m^2 + i\varepsilon\,)} \\ \mathrm{tr}\,\{\,\not p\,\Sigma_2(\not p)\,\} \quad &= \quad -\,p^2\, B(p^2)\,\mathrm{tr}\,\mathbb{I} \\ &= \quad -\, ie^2 \mu^{2\epsilon}\,(2\pi)^{-\,2\omega} \int \mathrm{d}^{2\omega} k\, \frac{\mathrm{tr}\,\{\,\not p\,\gamma^{\mu}\not k\gamma_{\mu}\,\}}{[\,(p-k)^2 + i\varepsilon\,]\;(\,k^2 - m^2 + i\varepsilon\,)} \end{aligned}$$

$$\begin{aligned} \mathrm{tr}\,\{\gamma^{\mu}\,\gamma^{\nu}\} &= 2^{\omega}\, g^{\mu\nu} \\ \mathrm{tr}\,\{\gamma^{\mu}\,\gamma^{\nu}\,\gamma^{\rho}\,\gamma^{\sigma}\} &= 2^{\omega}\,(\,g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\rho}\,g^{\nu\sigma} + g^{\mu\sigma}\,g^{\nu\rho}\,) \\ \mathrm{tr}\,\{\gamma^{\mu}\,\gamma_{\mu}\} &= 2^{\omega}\,g^{\mu\nu}\,g_{\mu\nu} = 2^{\omega}\,2\omega \\ \mathrm{tr}\,\{\gamma^{\mu}\,\gamma^{\nu}\,\gamma^{\rho}\,\gamma_{\nu}\} &= 2^{\omega}(\,2-2\omega\,)\,g^{\mu\rho} \end{aligned}$$

and thereby

$$A(p^2) \;=\; 2ie^2\mu^{2\epsilon}\,\frac{\epsilon-2}{(2\pi)^{2\omega}}\int\frac{\mathrm{d}^{2\omega}k}{[\,(p-k)^2+i\varepsilon\,]\,(\,k^2-m^2+i\varepsilon\,)} \tag{1.114}$$

$$p^2\,B(p^2) \;=\; 2ie^2\mu^{2\epsilon}\,\frac{\epsilon-1}{(2\pi)^{2\omega}}\int\frac{p\cdot k\;\mathrm{d}^{2\omega}k}{[\,(p-k)^2+i\varepsilon\,]\,(\,k^2-m^2+i\varepsilon\,)} \tag{1.115}$$

From the general Feynman parametric formula [22] §**5.3** we get

$$\begin{aligned} A(p^2) &= 2ie^2\mu^{2\epsilon}\,\frac{\epsilon-2}{(2\pi)^{2\omega}} \\ &\times \int_0^1\mathrm{d}x\int\frac{\mathrm{d}^{2\omega}k}{\{\,x\,[\,(p-k)^2+i\varepsilon\,]+(1-x)\,(\,k^2-m^2+i\varepsilon\,)\,\}^2} \\ &= 2ie^2\mu^{2\epsilon}\,\frac{2-\epsilon}{(2\pi)^{2\omega}} \\ &\times \int_0^1\mathrm{d}x\int\frac{\mathrm{d}^{2\omega}k}{[\,k^2-2x\,p\cdot k+xp^2-(1-x)\,m^2+i\varepsilon\,]^2} \\ p^2\,B(p^2) &= 2ie^2\mu^{2\epsilon}\,\frac{\epsilon-1}{(2\pi)^{2\omega}} \\ &\times \int_0^1\mathrm{d}x\int\frac{p\cdot k\;\mathrm{d}^{2\omega}k}{[\,k^2-2x\,p\cdot k+xp^2-(1-x)\,m^2+i\varepsilon\,]^2} \end{aligned}$$

Completing the square in the denominator and after defining the shifted momentum $\ell\equiv k-xp$, dropping the linear term in ℓ in the numerator owing to symmetric integration, we have

$$\begin{aligned} A(p^2) &= 2ie^2\mu^{2\epsilon}\,\frac{\epsilon-2}{(2\pi)^{2\omega}}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^{2\omega}\ell}{[\,\ell^2-p^2\varrho(x,a)+i\varepsilon\,]^2} \\ B(p^2) &= 2ie^2\mu^{2\epsilon}\,\frac{\epsilon-1}{(2\pi)^{2\omega}}\int_0^1\mathrm{d}x\,x\int\frac{\mathrm{d}^{2\omega}\ell}{[\,\ell^2-p^2\varrho(x,a)+i\varepsilon\,]^2} \end{aligned}$$

where

$$\varrho(x,a)\equiv\;(1-x)(\,a-x\,)\qquad\qquad a\equiv m^2/p^2$$

in such a manner that

$$p^2\varrho(x,a)>0\;\Leftrightarrow\;0\le x\le 1\,\vee\,p^2<m^2$$

so that we can perform the Wick rotation. From [22] eq. (5.15) we find

$$\begin{aligned} A(p^2) &= \frac{\alpha}{\pi}\,(4\pi\mu^2)^{\epsilon}\,\left[\,\Gamma(\epsilon)-\tfrac{1}{2}\,\Gamma(1+\epsilon)\,\right]\int_0^1\mathrm{d}x\;\left[\,p^2\,\varrho(x,a)\,\right]^{-\epsilon} \\ B(p^2) &= \frac{\alpha}{2\pi}\,(4\pi\mu^2)^{\epsilon}\,[\,\Gamma(\epsilon)-\Gamma(1+\epsilon)\,]\int_0^1\mathrm{d}x\,x\;\left[\,p^2\,\varrho(x,a)\,\right]^{-\epsilon} \end{aligned}$$

Expanding in powers of ϵ we eventually obtain

$$A(p^2) \;=\; \frac{\alpha}{\pi}\left\{1+\epsilon\,\ln(4\pi\mu^2)+O(\epsilon)\right\}\left[\,\frac{1}{\epsilon}+\psi(1)-\frac{1}{2}+O(\epsilon)\right]$$

$$\times \quad \left\{1-\epsilon\int_0^1 \mathrm{d}x\ \ln[\,x\,m^2-x(1-x)\,p^2\,]+O(\epsilon)\right\}$$
$$\doteq \quad \frac{\alpha}{\pi}\left\{\frac{1}{\epsilon}-\mathbf{C}-\frac{1}{2}-\int_0^1 \mathrm{d}x\ \ln\,\frac{x\,m^2-x(1-x)\,p^2}{4\pi\mu^2}\right\}$$

$$\begin{aligned}
B(p^2) &= \frac{\alpha}{4\pi}\left\{1+\epsilon\,\ln(4\pi\mu^2)+O(\epsilon)\right\}\left[\frac{1}{\epsilon}+\psi(1)-1+O(\epsilon)\right]\\
&\times \left\{1-2\epsilon\int_0^1 \mathrm{d}x\,x\ \ln[\,(1-x)\,m^2-x(1-x)\,p^2\,]+O(\epsilon)\right\}\\
&\doteq \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon}-\mathbf{C}-1-2\int_0^1 \mathrm{d}x\,x\ \ln\,\frac{(1-x)\,m^2-x(1-x)\,p^2}{4\pi\mu^2}\right\}
\end{aligned}$$

Notice that we have the elementary integral

$$\begin{aligned}
I &= \int_0^1 \mathrm{d}x\ \ln[\,x\,m^2-x(1-x)\,p^2\,]\\
&= \int_0^1 \mathrm{d}x\ \ln\,x+\int_0^1 \mathrm{d}x\ \ln[\,m^2-x\,p^2\,]\\
&= -2+\ln m^2+\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)
\end{aligned}$$

that yields

$$A(p^2) \doteq \frac{\alpha}{\pi}\left\{\frac{1}{\epsilon}+\psi(2)+\frac{1}{2}+\ln\frac{4\pi\mu^2}{m^2}-\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)\right\}$$

Similarly we obtain

$$\begin{aligned}
&\int_0^1 \mathrm{d}x\,x\ \ln[\,(1-x)\,m^2-x(1-x)\,p^2\,]\\
&= \int_0^1 \mathrm{d}x\,(1-x)\ \ln[\,x\,m^2-x(1-x)\,p^2\,]\\
&= I-\int_0^1 \mathrm{d}x\,x\ \ln[\,x\,m^2-x(1-x)\,p^2\,]
\end{aligned}$$

so that we need to compute the further elementary integral

$$\begin{aligned}
I' &= \int_0^1 \mathrm{d}x\,x\ \ln[\,x\,m^2-x(1-x)\,p^2\,]\\
&= \int_0^1 \mathrm{d}x\,x\ \ln x+\int_0^1 \mathrm{d}x\,x\ \ln[\,m^2-(1-x)\,p^2\,]\\
&= -1+\ln m^2+\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)\\
&- \frac{1}{2}\ln(m^2-p^2)+\frac{m^2}{2p^2}\left[1+\frac{m^2}{p^2}\ln\left(1-\frac{p^2}{m^2}\right)\right]
\end{aligned}$$

Thus we eventually get

$$2I-2I'=-2+\ln m^2+\left[1-\left(\frac{m^2}{p^2}\right)^2\right]\ln\left(1-\frac{p^2}{m^2}\right)-\frac{m^2}{p^2}$$

and consequently

$$A(\varkappa) \doteq \frac{\alpha}{\pi}\left\{\frac{1}{\epsilon} + \psi(2) + \ln\frac{4\pi\mu^2}{m^2} + \frac{1}{2} - \left(1 - \frac{1}{\varkappa}\right)\ln(1-\varkappa)\right\} \tag{1.116}$$

$$B(\varkappa) \doteq \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} + \psi(2) + \ln\frac{4\pi\mu^2}{m^2} + \frac{1}{\varkappa} - \left(1 - \frac{1}{\varkappa^2}\right)\ln(1-\varkappa)\right\} \tag{1.117}$$

where

$$\varkappa \equiv \frac{p^2}{m^2}$$

It is important to realize that on the one hand we get

$$\lim_{\varkappa\to 1} A(\varkappa) = \frac{\alpha}{\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \frac{3}{2} + \ln\frac{4\pi\mu^2}{m^2}\right\}$$

$$\lim_{\varkappa\to 1} B(\varkappa) = \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + 2 + \ln\frac{4\pi\mu^2}{m^2}\right\}$$

which shows that the regularized quantities $A(\varkappa)$ and $B(\varkappa)$ are ultraviolet divergent for $\epsilon \downarrow 0$ but finite on the mass shell $p^2 = m_e^2$. On the other hand we find

$$\varkappa\,\frac{\mathrm{d}A}{\mathrm{d}\varkappa} = -\,\frac{\alpha}{\pi}\left\{1 + \frac{1}{\varkappa}\,\ln(1-\varkappa)\right\}$$

$$\varkappa\,\frac{\mathrm{d}B}{\mathrm{d}\varkappa} = -\,\frac{\alpha}{4\pi}\left\{1 + \frac{2}{\varkappa} + \frac{2}{\varkappa^2}\ln(1-\varkappa)\right\}$$

which turn out to be ultraviolet finite but logarithmic divergent for $p^2 \to m^2$. This is an example of the physical, on the mass shell, infrared divergences which plague QED. Those are ultimately related to the null photon mass, for they could be removed by the introduction of a tiny photon mass in (1.114) and (1.115), *viz.*,

$$\frac{-\,ig^{\mu\nu}}{(k-p)^2 + i\varepsilon} \quad\longmapsto\quad \frac{-\,ig^{\mu\nu}}{(k-p)^2 - \mu_\gamma^2 + i\varepsilon} \tag{1.118}$$

The present days experimental bound on the photon mass, *i.e.* on gauge invariance and on the second postulate of the Special Relativity Theory, is the very stringent benchmark value $\mu_\gamma < 1\times 10^{-18}$ eV. It follows that we can understand the infrared regularization as

$$\ln\left(1 - \frac{p^2}{m^2}\right) \quad\longmapsto\quad \ln\left(1 - \frac{p^2 - \mu_\gamma^2}{m^2}\right) \quad\xrightarrow{\varkappa\downarrow 1}\quad \ln\frac{\mu_\gamma^2}{m^2} \tag{1.119}$$

that yields

$$\lim_{p^2\to m^2}\quad p^2\,\frac{\partial A}{\partial p^2} \quad=\quad -\,\frac{\alpha}{\pi}\left(1 + \ln\frac{\mu_\gamma^2}{m^2}\right) \tag{1.120}$$

$$\lim_{p^2\to m^2}\quad p^2\,\frac{\partial B}{\partial p^2} \quad=\quad -\,\frac{\alpha}{2\pi}\left(\tfrac{3}{2} + \ln\frac{\mu_\gamma^2}{m^2}\right) \tag{1.121}$$

Moreover we find another infrared divergence for the coefficient function or form factor $B(p^2)$ when $p^2 \to m^2 \to 0$, the so called mass singularity. Actually, for $m \neq 0$ we find

$$\begin{aligned}
\lim_{p^2 \to 0} A(p^2) &= \frac{\alpha}{\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \frac{1}{2} + \ln\frac{4\pi\mu^2}{m^2}\right\} \\
\lim_{p^2 \to 0} B(p^2) &= \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \tfrac{1}{2} + \ln\frac{4\pi\mu^2}{m^2}\right\} \\
\lim_{p^2 \to 0} p^2\,\frac{\partial A}{\partial p^2} &= \lim_{p^2 \to 0} p^2\,\frac{\partial B}{\partial p^2} \;=\; 0 \\
\lim_{p^2 \to 0}\left[B(p^2) + \frac{1}{2}\,p^2\,\frac{\partial B}{\partial p^2}\right] &= \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \frac{1}{2} + \ln\frac{4\pi\mu^2}{m^2}\right\}
\end{aligned}$$

Notice in particular the infrared singular behavior of the combination

$$\lim_{p^2 \to m^2}\left[B(p^2) + \frac{1}{2}\,p^2\,\frac{\partial B}{\partial p^2}\right] = \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{4\pi\mu^2}{m^2} + \frac{1}{2} - \ln\frac{\mu_\gamma^2}{m^2}\right\}$$

1.2.7 1-Loop Correction to the QED Vertex

Exercise 2. Evaluate the one loop correction to the so called vertex function or the 3−point proper vertex in quantum electrodynamics, which is defined to be

$$\Gamma^\mu(p,q) = \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\; ie\gamma^\lambda\, S(\ell+p)\,\gamma^\mu\, S(\ell+q)\, ie\gamma^\nu\, D_{\lambda\nu}(\ell)$$

using dimensional regularization. Here k denotes the four momentum of the incoming photon, while p is the four momentum of the incoming electron and q the tetra-momentum of the outgoing electron, in such a manner that the energy and momentum conservation reads $p + k = q$. Moreover where $D_{\lambda\nu}(\ell)$ is the photon propagator in the Feynman gauge, while $S(\ell)$ is the Dirac propagator. Segregate the structure of the divergent and finite parts.

Solution. It is convenient to set $\epsilon \equiv 2 - \omega$ together with

$$D_{\lambda\nu}(\ell) = \frac{-\,ig_{\lambda\nu}}{\ell^2 + i\varepsilon} \qquad\qquad S(\ell+p) = \frac{i(\not\ell + \not p + m)}{(\ell+p)^2 - m^2 + i\varepsilon}$$

so that we can write

$$\Gamma^\mu(p,q) = \frac{e^2\mu^{2\epsilon}}{i(2\pi)^{2\omega}} \int \frac{\mathrm{d}^{2\omega}\ell\;[\,\gamma^\lambda\,(\not\ell + \not p + m)\,\gamma^\mu\,(\not\ell + \not q + m)\,\gamma_\lambda\,]}{[\,(\ell+p)^2 - m^2 + i\varepsilon\,][\,(\ell+q)^2 - m^2 + i\varepsilon\,](\ell^2 + i\varepsilon)}$$

Now we make use of the Feynman parametric formula [22] §**5.3** to reduce the products in the denominator of the integrated function, *viz.,*

$$\begin{aligned}
\frac{1}{abc} &= \int_0^1 \mathrm{d}x\;\frac{1}{[\,ax + b\,(1-x)\,]^2}\cdot\frac{1}{c} \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\;\frac{2y}{[\,axy + by\,(1-x) + c\,(1-y)\,]^3}
\end{aligned}$$

Changing the variables according to

$$
\begin{array}{ll}
x = \dfrac{v}{1-u} & \qquad y = 1 - u \\
0 \leqslant u \leqslant 1 & \qquad 0 \leqslant v \leqslant 1 - u \\
\left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| = \dfrac{1}{1-u} &
\end{array}
$$

then we also find

$$
\frac{1}{abc} = \int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v \; \frac{2}{[\, av + b(1-u-v) + cu \,]^3}
$$

and thereby

$$
\begin{aligned}
& \Big\{ [\,(\ell+q)^2 - m^2 + i\varepsilon\,](\ell^2 + i\varepsilon)[\,(\ell+p)^2 - m^2 + i\varepsilon\,] \Big\}^{-1} \\
= \;\; & 2 \int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v \; \frac{2}{D^3}
\end{aligned}
$$

where the new denominator D reads

$$
D = \ell^2 + 2\ell \cdot (up + qv) - (u+v)\,m^2 + up^2 + vq^2 + i\varepsilon
$$

Now a shift is in order to complete the square at the denominator

$$
\ell \;\longmapsto\; \ell - up - qv
$$

that yields

$$
D = \ell^2 - (u+v)\,m^2 + p^2\,u\,(1-u) + q^2\,v\,(1-v) - 2\,p \cdot q\,uv + i\varepsilon
$$

Moreover the numerator of the fraction in the Feynman integral becomes

$$
\begin{aligned}
& \gamma^\lambda \; [\, \not{\ell} + \not{p}(1-u) - v\not{q} + m \,]\; \gamma^\mu \; [\, \not{\ell} + \not{q}(1-v) - u\not{p} + m \,]\; \gamma_\lambda \\
= \;\; & \gamma^\lambda \not{\ell}\, \gamma^\mu \not{\ell}\, \gamma_\lambda + \gamma^\lambda \; [\, \not{p}(1-u) - v\not{q} \,]\; \gamma^\mu \; [\, \not{q}(1-v) - u\not{p} \,]\; \gamma_\lambda \\
+ \;\; & m \; \big\{ \gamma^\lambda \; [\, \not{p}(1-u) - v\not{q} \,]\; \gamma^\mu\, \gamma_\lambda + \gamma^\lambda\, \gamma^\mu \; [\, \not{q}(1-v) - u\not{p} \,]\; \gamma_\lambda \big\} \\
+ \;\; & m^2\, \gamma^\lambda\, \gamma^\mu\, \gamma_\lambda \; + \; \text{irrelevant}
\end{aligned}
$$

where irrelevant stands for terms linear in ℓ^μ that vanish owing to symmetric integration. Putting altogether we can write

$$
\Gamma^\mu(p,q) = \Gamma^\mu_{\mathrm{div}}(p,q) + \Gamma^\mu_{\mathrm{fin}}(p,q)
$$

where

$$
\begin{aligned}
\Gamma^\mu_{\mathrm{div}}(p,q) \;\; = \;\; & \frac{2e^2 \mu^{2\epsilon}}{i(2\pi)^{2\omega}} \int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v \; \gamma^\lambda\, \gamma^\kappa\, \gamma^\mu\, \gamma^\nu\, \gamma_\lambda \\
\times \;\; & \int \frac{\ell^\kappa\, \ell^\nu \; \mathrm{d}^{2\omega}\ell}{[\, \ell^2 - (u+v)\,m^2 + p^2\,u\,(1-u) + q^2\,v\,(1-v) - 2\,p \cdot q\,uv + i\varepsilon \,]^3} \\
\Gamma^\mu_{\mathrm{fin}}(p,q) \;\; = \;\; & \int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v \; \gamma^\lambda \; [\, \not{p}(1-u) - v\not{q} + m \,]\; \gamma^\mu \; [\, \not{q}(1-v) - u\not{p} + m \,]\; \gamma_\lambda \\
\times \;\; & \frac{2e^2}{i(2\pi)^4} \int \frac{\mathrm{d}^4\ell}{[\, \ell^2 - (u+v)\,m^2 + p^2\,u\,(1-u) + q^2\,v\,(1-v) - 2\,p \cdot q\,uv + i\varepsilon \,]^3}
\end{aligned}
$$

It is important to grasp that only $\Gamma^{\mu}_{\text{div}}(p,q)$ contains the divergent part of the Feynman integral, while $\Gamma^{\mu}_{\text{fin}}(p,q)$ does represent a convergent integral for we have directly set $\omega = 2$. From the basic one loop Feynman integrals [22] (5.29) and (5.16) we obtain

$$\begin{aligned} &\Gamma^{\mu}_{\text{fin}}(p,q) \;= \\ &\frac{\alpha}{4\pi}\int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v\; \frac{\gamma^{\lambda}\,[\,\not p(1-u) - v\not q + m\,]\,\gamma^{\mu}\,[\,\not q(1-v) - u\not p + m\,]\,\gamma_{\lambda}}{p^2\,u\,(1-u) + q^2\,v\,(1-v) - (u+v)\,m^2 - 2p\cdot q\,uv} \end{aligned} \tag{1.122}$$

$$\begin{aligned} &\Gamma^{\mu}_{\text{div}}(p,q) \;= \\ &\int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v\; \frac{\Gamma(2-\omega)\,(\alpha/8\pi)\,\gamma^{\lambda}\,\gamma^{\kappa}\,\gamma^{\mu}\,\gamma_{\kappa}\,\gamma_{\lambda}\,(4\pi\mu^2)^{2-\omega}}{[\,(u+v)\,m^2 + 2p\cdot q\,uv - p^2\,u\,(1-u) - q^2\,v\,(1-v)\,]^{2-\omega}} \end{aligned} \tag{1.123}$$

To the purpose of recasting the numerators of the above fraction in a simpler form the following identities in 2ω dimensions are quite useful: namely,

$$\gamma^{\lambda}\,\gamma_{\lambda} = 2\omega\,\mathbb{I}_{2\omega} \qquad \gamma^{\lambda}\,\gamma^{\mu}\,\gamma_{\lambda} = (2-2\omega)\,\gamma^{\mu} \tag{1.124}$$

$$\gamma^{\lambda}\,\gamma^{\mu}\,\gamma^{\nu}\,\gamma_{\lambda} = 4\,g^{\mu\nu}\,\mathbb{I}_{2\omega} - (4-2\omega)\,\gamma^{\mu}\,\gamma^{\nu} \tag{1.125}$$

$$\gamma^{\lambda}\,\gamma^{\kappa}\,\gamma^{\mu}\,\gamma^{\nu}\,\gamma_{\lambda} = (4-2\omega)\,\gamma^{\kappa}\,\gamma^{\mu}\,\gamma^{\nu} - \,2\gamma^{\nu}\,\gamma^{\mu}\,\gamma^{\kappa} \tag{1.126}$$

Using those identities we can write

$$\begin{aligned} \Gamma^{\mu}_{\text{div}}(p,q) &= \Gamma(2-\omega)\,\gamma^{\mu}\,\frac{\alpha}{2\pi}\,(1-\omega)^2\,(4\pi\mu^2)^{2-\omega} \\ &\times \int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v\;\left[\,(u+v)\,m^2 + 2p\cdot q\,uv - p^2\,u\,(1-u) - q^2\,v\,(1-v)\,\right]^{\omega-2} \\ &= \left(\frac{1}{\epsilon} - \mathbf{C} + \cdots\right)\gamma^{\mu}\,\frac{\alpha}{2\pi}\,(\epsilon-1)^2 \\ &\times \int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v\;\left[\frac{(u+v)\,m^2 + 2p\cdot q\,uv - p^2\,u\,(1-u) - q^2\,v\,(1-v)}{4\pi\mu^2}\right]^{-\epsilon} \\ &= \gamma^{\mu}\,\frac{\alpha}{4\pi}\,\left[\frac{1}{\epsilon} - \mathbf{C} - 2\right. \\ &- \left. 2\int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v\;\ln\,\frac{(u+v)\,m^2 + 2p\cdot q\,uv - p^2\,u\,(1-u) - q^2\,v\,(1-v)}{4\pi\mu^2}\right] \end{aligned}$$

up to evanescent terms for $2-\omega = \epsilon \to 0$. Thus we find

$$\Gamma^{\mu}_{\text{div}}(p,q) = \gamma^{\mu}\,\left[\frac{\alpha}{4\pi}\cdot\frac{1}{\epsilon} + \text{finite}\right]$$

Let us compute the above ultraviolet divergent quantity on the particles mass shell $p^2 = q^2 = m^2$ with $k = q - p$ and $k^2 = 0$. We find

$$\begin{aligned} \overline{\Gamma}^{\mu}_{\text{div}} &= \gamma^{\mu}\,\frac{\alpha}{4\pi}\,\left\{\frac{1}{\epsilon} - \mathbf{C} - 2 + 2\int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}v\;\left[\ln\left(\frac{4\pi\mu^2}{m^2}\right) - 2\ln(u+v)\right]\right\} \\ &= \gamma^{\mu}\,\frac{\alpha}{4\pi}\,\left\{\frac{1}{\epsilon} - \mathbf{C} - 1 + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right\} \end{aligned}$$

To go further on it's worth to consider the on-mass-shell and spin-averaged quantity

$$\widetilde{\Gamma}^{\mu}(p,q) \equiv \lim_{p^2\to m^2}\ \lim_{q^2\to m^2}\ \bar{u}(p)\,\Gamma^{\mu}(p,q)\,u(q)$$

where $\bar{u}(p)$ and $u(q)$ are the usual spin states which satisfy $(\not{q}-m)u(q)=0=\bar{u}(p)(\not{p}-m)$. In such a circumstance we can rearrange the numerator appearing in $\Gamma^{\mu}_{\rm fin}(p,q)$ by making use of the identities (1.124)-(1.126) with $\omega=2$ and of the Dirac equation. In this process of reduction the Gordon identities will prove to be useful: namely,

$$\gamma^{\mu}\not{q} = \tfrac{1}{2}\Big(\{\gamma^{\mu},\not{q}\}+[\,\gamma^{\mu},\not{q}\,]\Big) = q^{\mu}-2i\sigma^{\mu\nu}\,q_{\nu} = m\gamma^{\mu}+\gamma^{\mu}(\not{q}-m)$$
$$\not{p}\,\gamma^{\mu} = \tfrac{1}{2}\Big(\{\gamma^{\mu},\not{p}\}-[\,\gamma^{\mu},\not{p}\,]\Big) = p^{\mu}+2i\sigma^{\mu\nu}\,p_{\nu} = m\gamma^{\mu}+(\not{p}-m)\gamma^{\mu}$$

where $\sigma^{\mu\nu}=\frac{1}{4}\,\mathrm{i}\,[\,\gamma^{\mu},\gamma^{\nu}\,]$. Moreover we obtain

$$\begin{aligned}\not{q}\,\gamma^{\mu}\not{p} \ &\doteqdot\ (m+\not{k})\,\gamma^{\mu}\,(m-\not{k}) = m^2\gamma^{\mu}-m\,[\,\gamma^{\mu},\not{k}\,]-\not{k}\,\gamma^{\mu}\not{k}\\ &=\ m^2\gamma^{\mu}+4im\sigma^{\mu\nu}\,k_{\nu}-2\not{k}\,k^{\mu}+k^2\gamma^{\mu} = m^2\gamma^{\mu}+4im\sigma^{\mu\nu}\,k_{\nu}+k^2\gamma^{\mu}\end{aligned}$$

where $\doteqdot$ means that the spin states sandwich $\bar{u}(p)(\cdots)u(q)$, the Dirac equation and the mass-shell condition are tacitly understood. In the very last step we used of $\not{k}=\not{q}-\not{p}\doteqdot 0$. Next we find

$$\not{q}\gamma^{\mu} = [\,\not{q},\gamma^{\mu}\,]+\gamma^{\mu}\not{q} \doteqdot 4i\sigma^{\mu\nu}q_{\nu}+m\gamma^{\mu}$$
$$\gamma^{\mu}\not{p} = [\,\gamma^{\mu},\not{p}\,]+\not{p}\,\gamma^{\mu} \doteqdot m\gamma^{\mu}-4i\sigma^{\mu\nu}p_{\nu}$$

Then we can write

$$\begin{aligned}
&\gamma^{\lambda}\ [\not{p}(1-x)-\not{q}y+m\,]\ \gamma^{\mu}\ [\not{q}(1-y)-\not{p}x+m\,]\ \gamma_{\lambda}\\
=\ &\gamma^{\lambda}\ [\not{p}(1-x)-\not{q}y\,]\ \gamma^{\mu}\ [\not{q}(1-y)-\not{p}x\,]\ \gamma_{\lambda}+m^2\,\gamma^{\lambda}\,\gamma^{\mu}\,\gamma_{\lambda}\\
+\ &m\,\gamma^{\lambda}\ [\not{p}(1-x)-\not{q}y\,]\ \gamma^{\mu}\,\gamma_{\lambda}+m\,\gamma^{\lambda}\,\gamma^{\mu}\ [\not{q}(1-y)-\not{p}x\,]\ \gamma_{\lambda}\\
=\ &-2\,[\not{q}(1-y)-\not{p}x\,]\ \gamma^{\mu}\ [\not{p}(1-x)-\not{q}y\,]\\
+\ &4m\,(1-2x)\,p^{\mu}+4m\,(1-2y)\,q^{\mu}-2\,m^2\,\gamma^{\mu}\\
\doteqdot\ &-2\,[\not{q}(1-y)-mx\,]\ \gamma^{\mu}\ [\not{p}(1-x)-my\,]-2\,m^2\,\gamma^{\mu}\\
+\ &4m[\,(1-2x)(m\gamma^{\mu}-2i\sigma^{\mu\nu}\,p_{\nu})+(1-2y)(m\gamma^{\mu}+2i\sigma^{\mu\nu}\,q_{\nu})\,]\\
\doteqdot\ &-2\,(1-x)(1-y)[\,m^2\gamma^{\mu}+4im\sigma^{\mu\nu}\,(q-p)_{\nu}+k^2\gamma^{\mu}\,]-2\,m^2\,(1+xy)\,\gamma^{\mu}\\
+\ &2my(1-y)(4i\sigma^{\mu\nu}q_{\nu}+m\gamma^{\mu})+2mx(1-x)(m\gamma^{\mu}-4i\sigma^{\mu\nu}p_{\nu})\\
+\ &4m[\,(1-2x)(m\gamma^{\mu}-2i\sigma^{\mu\nu}\,p_{\nu})+(1-2y)(m\gamma^{\mu}+2i\sigma^{\mu\nu}\,q_{\nu})\,]\\
=\ &2\,m^2\,\gamma^{\mu}\,[\,2(1-x-y)-(x+y)^2\,]-\ 2\,k^2\,\gamma^{\mu}(1-x)(1-y)\\
+\ &8im\sigma^{\mu\nu}\,\{q_{\nu}\,[\,x-y\,(x+y)\,]-p_{\nu}\,[\,y-x\,(x+y)\,]\}
\end{aligned}\tag{1.127}$$

This result enables us to show that the finite and definite part $\Gamma^{\mu}_{\rm fin}(p,q)$ of the vertex function can be split on the mass-shell into two contributions, one proportional to γ^{μ} and the other one proportional to $\sigma^{\mu\nu}$, *viz.*,

$$\Gamma^{\mu}_{\rm fin}(p,q) \doteqdot \gamma^{\mu}\,\Gamma_1+i\,\sigma^{\mu\nu}\,\frac{k_{\nu}}{m}\,\Gamma_2$$

We find for $p^2=q^2=m^2$ but $k^2/m^2\equiv\varrho\neq 0$

$$\Gamma_1(\varrho)\ =\ \frac{\alpha}{\pi}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\ \frac{1-x-y-\frac{1}{2}(x+y)^2-\frac{1}{2}(1-x)(1-y)\,\varrho}{xy\,\varrho-(x+y)^2}$$

Now it is apparent that in the limit $\varrho \longrightarrow 0$, that corresponds to the mass-less photon, we would be faced to the infrared divergent integral

$$\Gamma_1(0) = \frac{\alpha}{\pi}\left[\frac{1}{4} - \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\,\frac{1-x-y}{(x+y)^2}\right] = \infty$$

To remedy this we could set, for instance,

$$\overline{\Gamma}_1(\eta) = \frac{\alpha}{\pi}\left[\frac{1}{4} + \int_0^1 \mathrm{d}x \int_\eta^{1-x} \mathrm{d}y\,\frac{y}{(x+y)^2} - \int_0^1 \mathrm{d}x\,(1-x)\int_\eta^{1-x} \mathrm{d}y\,\frac{1}{(x+y)^2}\right]$$

and since we have

$$\begin{aligned}
&\int_0^1 \mathrm{d}x \int_\eta^{1-x} \mathrm{d}y\,\frac{y}{(x+y)^2} = \int_0^1 \mathrm{d}x\;(1 + x\,\partial_x)\int_\eta^{1-x}\frac{\mathrm{d}y}{x+y} \\
= \;& -\int_0^1 \mathrm{d}x\;(1 + x\,\partial_x)\ln(\eta + x) \stackrel{\eta\to 0}{\longrightarrow} -\int_0^1 \mathrm{d}x\;(1 + x\,\partial_x)\ln x = 0 \\
&\int_0^1 \mathrm{d}x\,(1-x)\int_\eta^{1-x} \mathrm{d}y\,\frac{-1}{(x+y)^2} \\
= \;& \int_0^1 \mathrm{d}x\,(1-x)\left(1 - \frac{1}{\eta + x}\right) \\
= \;& \frac{1}{2} - \int_0^1 \mathrm{d}x\,\frac{1-x}{\eta + x} = \ln\eta + \frac{3}{2} + O(\eta)
\end{aligned}$$

we eventually come to the infrared regularized on the mass shell result

$$\overline{\Gamma}_1(\eta) = \frac{\alpha}{4\pi}\,(4\ln\eta + 7)$$

In general, it is customary in the Literature to define the on shell 3-point proper vertex in QED with $(q-p)^2 = k^2 \neq 0$: namely,

$$\Gamma^\mu(p,q) \doteqdot e\gamma^\mu\,F_1(k^2) - ie\,\sigma^{\mu\nu}\,\frac{k_\nu}{m}\,F_2(k^2) \qquad \left[\,p^2 = q^2 = m^2 \vee (q-p)^2 = k^2 \neq 0\,\right]$$

where the dimensionless quantities $F_1(k^2)$ and $F_2(k^2)$ are called the form factors. In perturbation theory the form factor $F_1(k^2)$ at one loop is ultraviolet divergent in four space-time dimensions but infrared finite as long as $k^2 \neq 0$. In the mass-less photon limit we find instead at the one loop approximation

$$\lim_{k^2\to 0} F_1(k^2) \approx 1 + \frac{\alpha}{4\pi}\left\{\frac{1}{\varepsilon_{\mathrm{UV}}} - \mathbf{C} + 6 + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + 4\ln\varepsilon_{\mathrm{IR}}\right\} \tag{1.128}$$

where $\varepsilon_{\mathrm{UV}} = \epsilon = 2 - \omega$ is the ultraviolet cutoff within dimensional regularization while $\varepsilon_{\mathrm{IR}} = \eta$ denotes the infrared cutoff. It is worthwhile to grasp that the finite part of the above regularized quantity, *i.e.* the fully on shell form factor $F_1(0)$ at one loop, does depend upon both the ultraviolet and infrared regulators.

Conversely, let us finally calculate the one loop value of the fully on shell second form factor $F_2(0)$, which turns out to be both ultraviolet and infrared finite. From the previous expressions (1.122) and (1.127) we see that the pair of basic integrals which are relevant for the second form factor $F_2(0)$ are

$$\begin{aligned}
& \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{y}{(x+y)^2} - \int_0^1 \mathrm{d}x \, x \int_0^{1-x} \mathrm{d}y \, \frac{1}{x+y} \\
= & \; -\int_0^1 \mathrm{d}x \, (1 - x + \ln x - x \ln x) = \frac{1}{4} \\
& \int_0^1 \mathrm{d}x \, x \int_0^{1-x} \mathrm{d}y \, \frac{1}{(x+y)^2} - \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{y}{x+y} \\
= & \int_0^1 \mathrm{d}x \, (-x \ln x) = \frac{1}{4}
\end{aligned}$$

and thereby

$$\begin{aligned}
& 2i\sigma^{\mu\nu} \frac{\alpha}{\pi} \left[\frac{p_\nu}{m} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{y - x(x+y)}{(x+y)^2} - \frac{q_\nu}{m} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{x - y(x+y)}{(x+y)^2} \right] \\
= & \; -\frac{\alpha}{\pi} \, i\sigma^{\mu\nu} \frac{k_\nu}{2m} \quad \Longleftrightarrow \quad F_2(0) \approx \frac{\alpha}{2\pi} = \Gamma_2
\end{aligned}$$

The magnetic coupling in the Dirac theory is provided by

$$\begin{aligned}
& -g\mu_{\mathrm{B}} \int \mathrm{d}^4x \, \bar{\psi}(x) \sigma^{\mu\nu} \psi(x) \, \mathrm{F}_{\mu\nu}(x) \\
= & \; -\frac{e\hbar}{m_e c} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \mathrm{d}^4 k \, \delta(k + p - q) \\
\times & \; \widetilde{\psi}^\dagger(q) \gamma^0 \sigma^{\mu\nu} \widetilde{\psi}(p) \left[ik_\mu \widetilde{A}_\nu(k) - ik_\nu \widetilde{A}_\mu(k) \right]
\end{aligned} \tag{1.129}$$

where μ_{B} is the Bohr magneton while g is the Landé factor which is equal to 2 in the Dirac relativistic theory, the overall minus sign being due to the negative electron charge. This shows that the effective magnetic coupling in momentum space is given by $2ig\mu_{\mathrm{B}} \, \sigma^{\mu\nu} k_\nu$ where

$$g = 2 \left(1 + \frac{\alpha}{2\pi} + \cdots \right)$$

Thus the quantity $2\mu_{\mathrm{B}}\Gamma_2$ is nothing but the lowest order term of the anomalous magnetic moment, which is entirely due to the radiative corrections. Actually we get the first correction to the Landé factor of the electron

$$a_e = \frac{g-2}{2} \approx \frac{\alpha}{2\pi} \simeq 1161.40 \times 10^{-6}$$

This celebrated value was first obtained by

Julian Seymour Schwinger [New York City (1918) - Los Angeles (1994)]
On Quantum Electrodynamics and the Magnetic Moment of the Electron
Phys. Rev. **73** (4) 416-417 (1948)

Present days experiments yield

$$a_e = (1159.65218073 \pm 0.00000028) \times 10^{-6}$$

Chapter 2

Renormalization

2.1 Divergences of Feynman Diagrams

2.1.1 Power Counting Criterion

Consider a Feynman diagram with V vertexes, E external lines, *i.e.* carrying incoming or outgoing external momenta, and I internal lines, *i.e.* those ones with both endpoints attached to some vertexes. To warm up, let me assume for a start that only scalar particles are involved. The number of independent internal momenta is the number of loops L of the diagram. The I internal momenta do satisfy $V - 1$ relations among themselves, because there is one constraint just owing to overall energy momentum conservation, so that

$$L = I - V + 1 \tag{2.1}$$

This equality allows to compute the naïve power counting in momentum space for the integrating functions of any diagram, a simple calculation which provides the so called superficial degree of divergence $\omega(G)$ of the Feynman graph G – superficial standing for apparent. To determine $\omega(G)$ we note that there are

- L independent loop integrals, providing D powers of the momenta for each loop in a $D-$dimensional space-time
- I internal momenta, each one providing a scalar Feynman propagator with two inverse powers of the momenta. Hence

$$\omega(G) = DL - 2I$$

We need one more relation among V, E and I. Let me denote by V_N the number of vertexes with N legs, *i.e.* N concurring entering momenta. In a graph G with V_N such vertexes we have NV_N lines which are either internal or external. It turns out that any internal line counts twice, for it originates and terminates at some vertex, in such a manner that

$$NV_N = E + 2I$$

The above relationships allows us to express the superficial degree of divergence in terms of the number of external lines, the number of vertexes and the number of space-time dimensions

$$\omega\,(G) = D - \tfrac{1}{2}\,(\,D-2\,)\,E + V_N\left[\,\tfrac{1}{2}\,N\,(D-2) - D\,\right] \tag{2.2}$$

In four dimensions $D = 4$ we find

$$\omega\,(G) = 4 - E + (\,N-4\,)\,V_N \qquad [\,\text{four dimensions}\,]$$

and in the case of the $\lambda\phi_4^4$ scalar self-interaction

$$\omega\,(G) = 4 - E \qquad [\,\lambda\,\phi^4 \text{ in } D = 4\,]$$

The key result here is that the superficial degree of divergence $\omega\,(G)$ does not depend upon the number of vertexes but solely on the number of external legs. Thus we have only two candidates with $\omega\,(G) \geq 0$

- $\widetilde{G}_c^{(2)}\,(k)$ with $\omega\,(G) = 2$ (quadratic divergence)
- $\widetilde{G}_c^{(4)}\,(k_1, k_2, k_3, k_4)$ with $\omega\,(G) = 0$ (logarithmic divergence)

Note that these 2− and 4−point Green's functions are directly related to the kinetic and interaction terms of the classical Lagrange density, a feature that we be proved to be crucial for a successful renormalization program to all orders in perturbation theory.

For example we have already seen in [22] §**4.1.1** that the lowest order contribution to $\widetilde{G}_2(k)$ is given by

$$\begin{aligned}
\widetilde{G}^{\,(2)}(k) \;&=\; \frac{i}{k^2 - m^2 + i\varepsilon} \\
&\times\; \left\{1 + \frac{i\lambda}{16\pi^2}\left(K^2 - m^2\left[\ln\frac{K}{m} - \frac{1}{2} + \ln 2 + O\left(\frac{m}{K}\right)^2\right]\right)\right. \\
&\times\; \left.\frac{i}{k^2 - m^2 + i\varepsilon}\right\} \;+\; O(\lambda^2)
\end{aligned} \tag{2.3}$$

where $K \sim M_{\rm P}$ is a large ultraviolet cut−off. On the other side, using dimensional regularization, the very same Green's function reads

$$\begin{aligned}\widetilde{G}^{(2)}(k) &= \frac{i}{k^2-m^2+i\varepsilon}\\ &\times \left\{1+\frac{i\lambda m^2}{32\pi^2}\left[\frac{1}{\epsilon}+\psi(2)-\ln\frac{m^2}{4\pi\mu^2}\right]\frac{i}{k^2-m^2+i\varepsilon}\right\}\\ &+ O(\lambda^2)\end{aligned} \tag{2.4}$$

while the expression of the *sweatmeat* diagram [22] yields

$$\widetilde{G}^{(4)}(k_1,k_2,k_3,k_4) = (-i\lambda)\prod_{j=1}^{4}\frac{i}{k_j^2-m^2+i\varepsilon}\ \times$$
$$\left\{1-\frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon}+\psi(2)-\ln\frac{m^2}{4\pi\mu^2}+1-\frac{2}{3}A(s,t,u)\right]+O(\lambda^2)\right\}$$

with $2\epsilon = 4-D$ while

$$\begin{aligned}A(s,t,u) &= \sum_{z=s,t,u}\left(\frac{4m^2}{z}-1\right)^{1/2}\operatorname{arcctg}\sqrt{\frac{4m^2}{z}-1}\quad [\,0\le z\le 4m^2\,]\\ &= \sum_{z=s,t,u}\left(1-\frac{4m^2}{z}\right)^{1/2}\operatorname{Arcth}\sqrt{1-\frac{4m^2}{z}}\quad [\,z<0\vee z>4m^2\,]\end{aligned}$$

where s, t and u are the Mandelstam variables

$$s=(k_1+k_2)^2 \qquad t=(k_2+k_3)^2 \qquad u=(k_3+k_1)^2$$

while $\operatorname{Arcth}(z) = -\,i\operatorname{arcctg}(-iz)$. Note that from the series representation

$$\begin{aligned}\operatorname{arcctg} x &= \frac{\pi}{2}-\operatorname{arctg} x = \frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\,x^{2n+1}} \qquad [\,x^2\ge 1\,]\end{aligned} \tag{2.5}$$

we readily get for example

$$\lim_{z\downarrow 0}\left(\frac{4m^2}{z}-1\right)^{\frac{1}{2}}\operatorname{arcctg}\sqrt{\frac{4m^2}{z}-1} = 1 \tag{2.6}$$

$$\lim_{z\uparrow 4m^2}\left(\frac{4m^2}{z}-1\right)^{\frac{1}{2}}\operatorname{arcctg}\sqrt{\frac{4m^2}{z}-1} = 0 \tag{2.7}$$

On the other hand, from the equality

$$\operatorname{Arcth}\xi = \frac{1}{2}\,\ln\frac{\xi+1}{\xi-1}$$

for $\xi = 1+\epsilon \quad (\epsilon \to 0\,)$ we find

$$\operatorname{Arcth}(1+\epsilon)\ \sim\ \ln\sqrt{2} - \ln\sqrt{\epsilon} \qquad\qquad [\,\epsilon\ \to\ 0\,]$$

in such a manner that we obtain the leading asymptotic behavior

$$\left(1-\frac{4m^2}{z}\right)^{\frac{1}{2}}\operatorname{Arcth}\sqrt{1-\frac{4m^2}{z}}\ \sim\ \tfrac{1}{2}\,\ln\left(-\,\frac{z}{m^2}\right) \qquad\qquad [\,z\ \to\ -\infty\,] \quad (2.8)$$

The above analysis does not actually prove by any means that all the remaining connected Green's functions

$$\widetilde{G}_c^{(n)}\,(k_1,\ldots,k_n) \qquad\qquad [\,n\in\mathbb{N},\ n>4\,]$$

which do exhibit a negative superficial degree of divergence $\omega\,(G) < 0$ are finite in four dimensions, because they generally contain sub-divergences. To this concern, the necessary and sufficient condition for the convergence of a Feynman graph G is provided by the so called Bogoliubov theorem.[1] In order to state this convergence theorem, let us first recall that the Green's functions and corresponding Feynman diagrams can be divided into disconnected and connected ones, the latter being just characterized by the property that all vertices of the corresponding Feynman graph are connected by at least one internal line. In general, as we have seen, a connected graph G is $1-$particle reducible (1PR) in the sense that it can be separated into two disconnected sub-graphs by cutting an internal line. Conversely, we shall call strongly connected or $1-$particle irreducible (1PI) any Feynman graph G that can not be separated into two disconnected sub-graphs by cutting one of its internal lines. As we have already discussed, the 1PI or strongly connected Green's functions or even $n-$point proper vertices in momentum space are commonly denoted by $\widetilde{\Gamma}^{\,(n)}(k_1\,,\,\ldots\,,\,k_n)$, which precisely correspond to the momentum space expansion coefficients (1.20) of the effective Action.

We shall use here a restrictive definition of a sub-diagram $g\subset G$ of a diagram G : this is a subset of vertexes of G and of all internal lines joining them in G. Then, to each strongly connected, *i.e.* $1-$particle irreducible,

[1] See *e.g.* N.N. Bogolyubov and D.M. Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience Publishers, New York, 1959 ; C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980, §**8–1–4** pp. 382 – 385.

graph G we associate the set $\mathcal{F}$ of all its strongly connected sub-graphs $g \in G$. Of course, $\mathcal{F}$ contains G itself.

Theorem If $\omega(g) < 0$, $\forall g \in \mathcal{F}$ then the Feynman integral which corresponds to G is absolutely convergent in the Euclidean formulation.

This means that in the $\lambda\phi_4^4$ model in four space-time dimensions, the generic sources of divergences are the 2-point and the 4-point proper vertexes and nothing else. Thus, if we will be able to remove the divergences, order by order in perturbation theory, from $\widetilde{\Gamma}^{(2)}(k_1)$ and $\widetilde{\Gamma}^{(4)}(k_1, \ldots, k_4)$, then all other Green's functions $\widetilde{G}^{(n)}(k_1, \ldots, k_n)$ of the theory will be divergence free. To the lowest order we have

$$\begin{aligned} \mathtt{reg}\,\widetilde{\Gamma}^{(2)}(k;\lambda,m,\mu,\epsilon) &= i\,[\,\widetilde{G}^{(2)}(k;\lambda,m,\mu,\epsilon)\,]^{-1} \\ = \quad k^2 - m^2 &\left\{1 - \frac{\lambda}{32\pi^2}\left[\frac{1}{\epsilon} + \psi(2) - \ln\frac{m^2}{4\pi\mu^2}\right]\right\} + O(\lambda^2) \end{aligned} \tag{2.9}$$

which corresponds to the divergent mass term of the classical Lagrangian

$$-\tfrac{1}{2}\,m^2\,\phi^2(x)\left(1 - \frac{\lambda}{32\pi^2}\cdot\frac{1}{\epsilon}\right) + O(\lambda^2) \tag{2.10}$$

Furthermore

$$\begin{aligned} &\mathtt{reg}\,\widetilde{\Gamma}^{(4)}(s,t,u;\lambda,m,\mu,\epsilon) = \\ &(-\lambda)\left\{1 - \frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon} - \mathbf{C} - \ln\frac{m^2}{4\pi\mu^2} + 2 - \tfrac{2}{3}\,A(s,t,u)\right]\right\} \\ + \quad &O(\lambda^3) \end{aligned} \tag{2.11}$$

the divergent part of which does correspond to the interaction potential

$$\frac{\lambda}{4!}\,\phi^4(x)\left(1 - \frac{3\lambda}{32\pi^2}\cdot\frac{1}{\epsilon}\right) + O(\lambda^2)$$

It is also worthwhile to observe that we have for $k = s = t = u = 0$

$$\begin{aligned} \mathtt{reg}\,\widetilde{\Gamma}^{(2)}(0) &= -m^2\left\{1 - \frac{\lambda}{32\pi^2}\left[\frac{1}{\epsilon} + \psi(2) + \ln\frac{4\pi\mu^2}{m^2}\right]\right\} + O(\lambda^2) \\ \mathtt{reg}\,\widetilde{\Gamma}^{(4)}(0) &= -\lambda\left\{1 - \frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon} + \psi(1) + \ln\frac{4\pi\mu^2}{m^2}\right]\right\} + O(\lambda^3) \end{aligned}$$

The graphs which contain the generic superficial divergences are named to be primitively divergent. The fact that in the $\lambda\phi_4^4$ scalar field theory the number of types of primitively divergent graphs is finite, *viz.,* 2- and 4-point proper vertexes, and correspond precisely to the type of terms appearing in the classical Lagrangian, is the NECESSARY CONDITION for the successful removal of all the ultraviolet divergences to all orders in perturbation theory. A field theory model for which this is possible is said power counting renormalizable. Actually, we can easily realize from the expression (2.2) for the superficial degree of divergence that quite a few number of scalar theories does fulfill this key requirement.

- In four space-time dimensions, *i.e.* $D = 4$, we see that $\omega\,(G)$ grows with the number of vertexes V_N with $N > 4$. This means that the scalar self-interactions of higher powers $g_N\,\phi^N(x) \quad (N > 4)$, although perfectly reasonable classically, necessarily lead at the quantum level to an infinite number of primitively divergent graphs. In such a nasty case, the situation gets quickly out of control and the possibility to remove the divergences to all orders in perturbation theory indeed disappears and thereby renormalizability is lost.

- In $D = 4$ and with $N > 4$ the coupling g_N have the canonical mass dimensions $[\,g_N\,] = \mathrm{eV}^{4-N}$. This fact strongly suggests that the very criterion of the power counting renormalizability is deeply connected, in four space-time dimensions, with the dimensionless nature of the coupling parameter for the interaction.

  This simple but crucial observation immediately led Werner Heisenberg to realize that the transition to the quantum theory of the Einstein General Relativity appears to be a formidable task. As a matter of fact, the Newton's constant G_N is nothing but, in natural units, the square of the Planck length $G_N = \ell_{\mathrm{P}}^2\,c^3/\hbar$ or the inverse square of the Planck mass $G_N = \hbar c/M_{\mathrm{P}}^2$. It follows that any probability amplitude involving quantized gravity will exhibit the $n-$th order radiative correction

$$\mathcal{A}^{(n)} \;\propto\; G_N^n \int_0^K \ell^{2n-1}\,d\ell \;\sim\; G_N^n\,K^{2n}$$

  where $\hbar K$ is the ultraviolet cutoff in momentum space, which evidently implies an infinite variety of infinities, their species being increasing *ad libitum* with the order of the perturbative expansion [2]. This means that the conventional quantum version of Einstein General Relativity can not give rise to a perturbative power counting renormalizable quantum field theory.

- When $D = 2$, *i.e.* one space and one time dimensions, the situation is completely reversed. There we have

$$\omega\,(G) = 2 - 2V_N \qquad (\text{two space} - \text{time dimensions})$$

[2] This simple dimensional argument is reported by Steven Weinberg, *Gravitation and cosmology : principles and applications of the general theory of relativity*, John Wiley & Sons, New York, 1972, chapter X § **8** p. 289.

in such a manner that the superficial degree of divergence does not depend upon N which labels the type of interaction. It depends only on the number of vertexes and the more vertexes are there the more convergent is the Feynman integral. The degree $\omega(G)$ does not depend on the number E of the external legs, the only primitively divergent diagrams having one or zero legs. Since divergences occur owing to loop integrals, this means that the latter occur only when a propagator from a vertex is closed on the very same vertex. But this is precisely the tadpole graph, the divergence of which can be removed by the normal ordering prescription. In other words, if we start from the free field quantum Action

$$S_0 = \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \ \tfrac{1}{2} : \partial_\mu \phi\,(t,x)\, \partial^\mu \phi\,(t,x) - m^2 \phi^2\,(t,x) :$$

then no divergences will appear in perturbation theory for the Green's functions. Incidentally, this is the ultimate simple reason why the method of dimensional regularization in the Euclidean formulation does render all the Feynman integral absolutely convergent.

Now we are ready to argue about the interacting field theories involving spinor fields, such as the Yukawa meson theory or quantum electrodynamics. Once again, it turns out that the number of possible fermion interactions is drastically reduced by the very strong requirement of the power counting renormalizability, which demands as a necessary condition that the number of species of primitively divergent graphs were finite. Let us therefore compute the superficial degree of divergence $\omega\,(G)$ on an arbitrary Feynman diagram involving scalar, spinor and vector fields.

Consider therefore a generic Feynman graph G with L loops, I_{b} boson internal lines, I_{f} fermion internal lines, V vertexes with N_{b} boson and N_{f} fermion concurring lines, E_{b} external boson lines and E_{f} external fermion lines. As already repeatedly remarked the numbers N_{f} and E_{f} must be even. The number of loops is given by

$$L = I - V + 1 = I_{\mathrm{b}} + I_{\mathrm{f}} - V + 1$$

The superficial degree of divergence in D space-time dimensions is

$$\omega\,(G) = D \cdot L - I_{\mathrm{f}} - 2\, I_{\mathrm{b}}$$

since each spinor propagator contributes only one power of momentum. In addition, the total number of fermionic spinor lines is given by

$$V N_{\mathrm{f}} = E_{\mathrm{f}} + 2 I_{\mathrm{f}}$$

and similarly for bosonic scalar and vector lines

$$V N_{\mathrm{b}} = E_{\mathrm{b}} + 2 I_{\mathrm{b}}$$

The above relations enable us to express the superficial degree of divergence in the form

$$\begin{aligned} \omega\,(G) &= D - \tfrac{1}{2}\,(D-1)\,E_{\mathrm{f}} - \tfrac{1}{2}\,(D-2)\,E_{\mathrm{b}} \\ &- V\,\left[\,D - \tfrac{1}{2}\,(D-1)\,N_{\mathrm{f}} - \tfrac{1}{2}\,(D-2)\,N_{\mathrm{b}}\,\right] \end{aligned} \tag{2.12}$$

which reduces to the previously obtained expression (2.2) for $N_{\mathrm{f}} = 0 = E_{\mathrm{f}}$ when only bosonic lines are present. In two space-time dimensions we find

$$\omega\,(G) = 2 - \tfrac{1}{2}\,E_{\mathrm{f}} - V\left(2 - \tfrac{1}{2}\,N_{\mathrm{f}}\right) \qquad [\,\text{two dimensions}\,]$$

which shows that one has to require $N_{\mathrm{f}} \leq 4$ for $D = 2$, otherwise the divergence would be growing with the number of vertexes. It turns out that there is a restriction on the type of allowed fermion interactions at the quantum level even in one space and one time dimensions: it must be of a degree not higher than $\bar{\psi}_{\alpha}\,(x)\,\psi_{\beta}\,(y)\,\bar{\psi}_{\delta}\,(z)\,\psi_{\eta}\,(w)$. We can understand this fact from another point of view : unlike boson fields, which are dimensionless in two space-time dimensions, the spinor fields in $D = 2$ have canonical dimensions $\mathrm{eV}^{1/2}$, so that $(\bar{\psi}\psi)^2$ is the local terms of the highest power that does not necessitate the introduction of a coupling parameter with an inverse mass power engineering dimensions.

In four space-time dimensions we have instead

$$\omega\,(G) = 4 - \tfrac{3}{2}\,E_{\mathrm{f}} - E_{\mathrm{b}} - V\left(4 - \tfrac{3}{2}\,N_{\mathrm{f}} - N_{\mathrm{b}}\right) \qquad [\,\text{four dimensions}\,]$$

Now the necessary condition that prevents the number of the species of the primitively divergent graphs to grow up with the number of vertexes yields

$$4 - \tfrac{3}{2}\,N_{\mathrm{f}} - N_{\mathrm{b}} \geq 0 \qquad (\,D = 4\,)$$

where N_{f} is even. The possible solutions are

- $N_{\mathrm{f}} = 2 \vee N_{\mathrm{b}} = 0$ which corresponds to a spinor mass term and not to an interaction vertex
- $N_{\mathrm{f}} = 0 \vee N_{\mathrm{b}} = 2\,,\,3\,,\,4$ which corresponds to a scalar or vector mass term as well as to the cubic and quartic bosonic field interactions, the cubic scalar one being ruled out even classically by the stability condition, *i.e.* by the request that the energy operator must be bounded

from below. This means that we are left with the cubic scalar-vector interaction vertex

$$\phi^*(x)\, A^\mu(x)\, \partial_\mu \phi(x) + \text{h.c.} \tag{2.13}$$

as well as the quartic interaction vertexes

$$\lambda\, [\, \phi(x)\, \phi^*(x)\,]^2 \qquad\qquad e^2 \phi(x)\, \phi^*(x)\, A^\mu(x)\, A^*_\mu(x) \tag{2.14}$$

in the case of *e.g.* charged scalar and Abelian vector fields.

- $N_\mathrm{f} = 2 \,\vee\, N_\mathrm{b} = 1$, the only nontrivial new solution that gives

  $$\omega(G) = 4 - \tfrac{3}{2}\, E_\mathrm{f} - E_\mathrm{b}$$

  This new solution describes the only boson−fermion interaction allowed by the requirement of power counting renormalizability and turns out to be incredibly restrictive : it must involve one fermion−antifermion pair and one scalar or vector real field, which means that in four space-time dimensions spinor fields must appear only as quadratic in the classical Lagrange density. Once again, this can be gathered because of the fact that in $D = 4$ the spinor field amplitudes have canonical dimensions $[\,\psi\,] = \mathrm{eV}^{3/2}$, while bosonic field amplitudes have $[\,\phi\,] = [\,A_\mu\,] = \mathrm{eV}$. Thus, the only nontrivial interaction of dimension four is the one with two spinor and one boson fields. If the discrete parity symmetry is there we come to the only two possible parity−even interactions

  $$g\, \phi(x)\, \bar\psi(x)\, \psi(x) \qquad\qquad e\, A_\mu(x)\, \bar\psi(x)\, \gamma^\mu\, \psi(x)$$

  that is, Yukawa meson field theory and quantum electrodynamics. This remarkable selection enormously simplifies the analysis about all the admissible quantum field theory models involving spinor fields.

In conclusion we can classify all the interacting quantum field theory models according to three categories which are characterized by the canonical engineering dimension of the coupling parameter:

1. super-renormalizable :

   only a **finite number** of Feynman diagrams G superficially diverge *i.e.* $\omega(G) \geq 0 \Leftrightarrow$ coupling has positive mass dimensions

2. renormalizable :

   only a **finite number** of proper vertexes $\widetilde{\Gamma}^{(n)}$ superficially diverge, and the divergences occur at all orders of perturbation theory $\Leftrightarrow$ coupling is dimensionless

3. non-renormalizable :

   all proper vertexes $\widetilde{\Gamma}^{(n)}$ superficially diverge at a sufficiently high order in perturbation theory $\Leftrightarrow$ coupling has negative mass dimensions

2.1.2 Renormalization

Let me remind the one loop structure of the primitively divergent proper vertexes of the self-interacting real scalar field theory in $D = 4$ space-time dimensions : using *e.g.* dimensional regularization with $D = 2\omega\,,\ \epsilon = 2 - \omega$ we find in momentum space

$$\begin{aligned}\mathtt{reg}\,\widetilde{\Gamma}^{(2)}(\,k\,;\lambda,m,\mu,\epsilon\,) &= k^2 - m^2\left\{1 - \frac{\lambda}{32\,\pi^2}\,\left[\frac{1}{\epsilon} + \psi\,(2) - \ln\frac{m^2}{4\pi\mu^2}\right]\right\} \\ &+ \; O\,(\,\lambda^2\,)\end{aligned}$$

$$\begin{aligned}&\mathtt{reg}\,\widetilde{\Gamma}^{(4)}(\,s,t,u\,;\lambda,m,\mu,\epsilon\,) \; = \\ -\;&\lambda\left\{1 - \frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon} - \mathbf{C} - \ln\frac{m^2}{4\pi\mu^2} + 2 - \tfrac{2}{3}\,A\,(\,s,t,u\,)\right]\right\} \\ +\;&O\,(\,\lambda^3\,)\end{aligned}$$

where

$$s = (k_1 + k_2)^2 \qquad t = (k_2 + k_3)^2 \qquad u = (k_3 + k_1)^2$$

are the Mandelstam variables, while

$$\begin{aligned}A\,(\,s,t,u\,) \;=\; \sum_{z\,=\,s\,,\,t\,,\,u}&\left\{\theta(z)\,\theta(4m^2 - z)\left(\frac{4m^2}{z} - 1\right)^{\frac{1}{2}}\operatorname{arcctg}\sqrt{\frac{4m^2}{z} - 1}\right. \\ &+\;\left.[\,\theta(-z) + \theta(z - 4m^2)\,]\left(1 - \frac{4m^2}{z}\right)^{\frac{1}{2}}\operatorname{Arcth}\sqrt{1 - \frac{4m^2}{z}}\;\right\}\end{aligned}$$

Notice that the finite part of the above expressions is arbitrary, depending upon the free mass scale μ . The idea for the removal of the poles in $\epsilon = 2 - \omega$ order-by-order in perturbation theory is very simple: modify the Feynman rules at each order in such a manner to obtain a finite result for $\epsilon \to 0$. To start with consider the divergence of the kinetic term. Its infinity can be cast away by inserting a new Feynman rule indicated by

$$-\!\!\times\!\!- \;\stackrel{\mathrm{def}}{=}\; -\,\frac{\lambda m^2}{32\,\pi^2}\,\left[\frac{1}{\epsilon} + \mathrm{F}_1\left(\epsilon\,,\,\frac{m^2}{4\pi\mu^2}\right)\right] \tag{2.15}$$

where F_1 is an arbitrary dimensionless function analytic when $\epsilon \to 0$, the presence of which just endorses the arbitrariness inside this procedure. As a matter of fact, after a subtraction of an infinity the remaining finite part can be anything. Thus, if we now calculate the new kinetic term we find

$$\widetilde{\Gamma}_R^{(2)}(k\,;\lambda,m,\mu) = k^2 - m^2 - \lambda\,\Sigma_1(m,\mu) + O(\lambda^2) \tag{2.16}$$

$$\Sigma_1(m,\mu) = \frac{m^2}{32\pi^2}\left[\mathrm{F}_1 + \ln\frac{m^2}{4\pi\mu^2} - \psi(2)\right] \tag{2.17}$$

which is finite up to the order $O(\lambda)$ although arbitrary. The extra-term (2.15) is named a mass counter-term. Actually, it is crucial to realize that its field dependence in configuration space is the same as the classical mass term in the Lagrangian, *viz.*

$$-\frac{1}{2}m^2\phi^2(x)\left\{1 + \frac{\lambda}{32\pi^2}\left[\frac{1}{\epsilon} + \mathrm{F}_1\left(\epsilon\,,\,\frac{m^2}{4\pi\mu^2}\right)\right]\right\} \tag{2.18}$$

Let us now turn to the 4−point proper vertex. Again, to the aim of removing its simple pole $1/\epsilon$ let us introduce the new Feynman rule

$$\bullet \;\stackrel{\text{def}}{=}\; (-i\lambda)\cdot\frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon} + \mathrm{G}_1\left(\epsilon\,,\,\frac{m^2}{4\pi\mu^2}\right)\right] \tag{2.19}$$

where G_1 is another arbitrary dimensionless function analytic when $\epsilon \to 0$. This extra $O(\lambda^2)$ vertex counterterm precisely corresponds to some new self-interaction term

$$-\frac{\lambda}{4!}\int \mathrm{d}x\;\phi^4(x)\left\{1 + \frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon} + \mathrm{G}_1\left(\epsilon\,,\,\frac{m^2}{4\pi\mu^2}\right)\right]\right\} \tag{2.20}$$

with the very same dependence upon the fields of the classical potential. Moreover, its addition drives to a new, finite albeit arbitrary 4−point proper vertex

$$\begin{aligned}
&\widetilde{\Gamma}_R^{(4)}(s,t,u\,;\lambda,m,\mu) \;=\; -\lambda \\
+\;& \frac{3\lambda^2}{32\pi^2}\left[2 - \mathrm{G}_1 - \mathbf{C} - \ln\frac{m^2}{4\pi\mu^2} - \tfrac{2}{3}A(s,t,u)\right] \\
+\;& O(\lambda^3)
\end{aligned} \tag{2.21}$$

To proceed further on requires a quite harder effort in the calculations. In particular, taking suitably into account the new 1-loop Feynman rules (2.15) and (2.19), the 2-loop $O(\lambda^2)$ calculation[3] of the 2-point proper vertex or inverse propagator yields

$$
\begin{aligned}
\texttt{reg}\,\widetilde{\Gamma}^{(2)}(k\,;\lambda,m,\mu,\epsilon) \;&=\; k^2 - m^2 - \lambda\,\Sigma_1(m,\mu) \\
+\;\; \frac{\hat{\lambda}^2}{4!}\cdot\frac{1}{\epsilon}\,k^2 &+ \tfrac{1}{2}\,\hat{\lambda}^2\,m^2\left[\frac{1}{\epsilon^2}+\frac{1}{2\,\epsilon}\,(\mathrm{F}_1+3\,\mathrm{G}_1-1)\right] \\
-\;\; \hat{\lambda}^2\,\widehat{\Sigma}_2(k\,,m) &+\; O(\lambda^3)
\end{aligned}
\tag{2.22}
$$

in which I have set

$$
\hat{\lambda} = \frac{\lambda}{16\,\pi^2} \qquad\qquad \widehat{m}^{\,2} = \frac{m^2}{4\pi\mu^2}
\tag{2.23}
$$

while $\widehat{\Sigma}_2\,(k\,,m)$ stands for the complicated coefficient of the $O(\lambda^2)$ 2-loop finite part for $\epsilon\to 0$, that I have not shown explicitly, which turns out to be regular for $k^2 = m^2$ and can be obtained from the subtraction of the first two terms of the Taylor's expansion from the 2-point proper regularized vertex

$$
\begin{aligned}
-\,\hat{\lambda}^2\,\widehat{\Sigma}_2(k\,,m) \;&\equiv\; \lim_{\epsilon\to 0}\Big[\texttt{reg}\,\widetilde{\Gamma}^{(2)}(k\,;\lambda,m,\mu,\epsilon) - \texttt{reg}\,\widetilde{\Gamma}^{(2)}(0\,;\lambda,m,\mu,\epsilon) \\
&-\;\; k^2\,\lim_{k\to 0}\frac{\partial}{\partial k^2}\,\texttt{reg}\,\widetilde{\Gamma}^{(2)}(k\,;\lambda,m,\mu,\epsilon)\Big]
\end{aligned}
\tag{2.24}
$$

in such a manner that we evidently have

$$
\widehat{\Sigma}_2(0\,,m) = \widehat{\Sigma}_2'(0\,,m) = 0
\tag{2.25}
$$

where

$$
\widehat{\Sigma}_2'(0\,,m) \equiv \lim_{k\to 0}\frac{\partial}{\partial k^2}\,\widehat{\Sigma}_2(k\,,m)
$$

Notice that the finite part $\widehat{\Sigma}_2\,(k\,,m)$ does exist for $\epsilon\to 0$ and consequently it does not depend upon μ. Hence the 2-loop counter-term reads

$$
\begin{aligned}
-\!\!\times\!\!- \;&=\; -\,k^2\,\hat{\lambda}^2\left(\frac{1}{24\,\epsilon}\;+\;\mathrm{H}_2\right) \\
&-\;\; m^2\,\hat{\lambda}^2\left\{\frac{1}{2\epsilon^2}+\frac{1}{4\epsilon}\,(\mathrm{F}_1+3\,\mathrm{G}_1-1)+\mathrm{F}_2\right\}
\end{aligned}
\tag{2.26}
$$

where $\mathrm{F}_2\,(\epsilon,\widehat{m}^2)$ and $\mathrm{H}_2\,(\epsilon,\widehat{m}^2)$ are some new arbitrary functions that are finite for $\epsilon\to 0$. Thus we eventually get the finite $O(\lambda^2)$ inverse propagator

$$
\begin{aligned}
\widetilde{\Gamma}_R^{(2)}(k\,;\lambda,m,\mu) \;&=\; k^2 - m^2 - \lambda\,\Sigma_1(m,\mu) - \lambda^2\,\Sigma_2(k\,;m,\mu) \\
&+\;\; O(\lambda^3)
\end{aligned}
\tag{2.27}
$$

[3]See here below and Pierre Ramond, *Field Theory: A Modern Primer*, Benjamin, Reading, Massachusetts, 1981, chapter IV §4 pp. 159–164

where

$$\lambda^2\,\Sigma_2\,(k\,;m,\mu) = \hat{\lambda}^2\,\left[m^2\;\mathrm{F}_2(\widehat{m}^2) + k^2\,\mathrm{H}_2(\widehat{m}^2) + \widehat{\Sigma}_2\,(k\,,m)\right] \tag{2.28}$$

which exhibits the two new arbitrary functions F_2, H_2 analytic for $\epsilon \to 0$.

Hence, putting altogether, we find that the $O(\lambda^2)$ modified Lagrangian in $2\omega = 4 - 2\epsilon$ space-time dimensions is provided by

$$\mathcal{L}_R = \mathcal{L} + \mathcal{L}_{\,\mathrm{c.t.}} \tag{2.29}$$

in which I have set

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi\ -\frac{1}{2}\,m^2\,\phi^2\ -\frac{\lambda}{4!}\,\phi^4$$

$$\mathcal{L}_{\,\mathrm{c.t.}}\ =\ \frac{1}{2}\,\delta_Z\,\partial_\mu\phi\,\partial_\mu\phi\ -\ \frac{1}{2}\,\delta m^2\,\phi^2\ -\ \frac{1}{4!}\,\delta_\lambda\,\phi^4 \tag{2.30}$$

and the corresponding new Feynman rules for the counter-terms

$$-\!\!\times\!\!-\ \stackrel{\mathrm{def}}{=}\ k^2\,\delta_Z - \delta m^2 \qquad\qquad \otimes\ \stackrel{\mathrm{def}}{=}\ -\,i\,\delta_\lambda \tag{2.31}$$

where

$$\delta_Z = -\,\hat{\lambda}^2\left(\frac{1}{24\,\epsilon} + \mathrm{H}_2\right) + O(\lambda^3) \tag{2.32}$$

in which, again, $\mathrm{H}_2\,(\epsilon, \widehat{m}^2)$ is a further arbitrary quantity, analytic for $\epsilon\ \to\ 0$, while

$$\begin{aligned}\delta m^2\ &=\ \tfrac{1}{2}\,m^2\left\{\hat{\lambda}\left(\frac{1}{\epsilon} + \mathrm{F}_1\right) + \hat{\lambda}^2\left[\frac{1}{\epsilon^2} + \frac{1}{2\epsilon}\,(\,\mathrm{F}_1 + 3\,\mathrm{G}_1 - 1\,) + 2\mathrm{F}_2\right]\right\}\\ &+\ O\left(\lambda^3\right)\end{aligned} \tag{2.33}$$

$$\delta_\lambda = \lambda\cdot\frac{3\hat{\lambda}}{2}\left(\frac{1}{\epsilon} + \mathrm{G}_1\right) + O\left(\lambda^3\right) \tag{2.34}$$

It is apparent that the $O(\lambda^2)$ modified Lagrangian $\mathcal{L}_R$ does exactly share the very same structure of the classical Lagrangian $\mathcal{L}$, up to the especially tuned divergent coefficients δ_Z, δm^2, δ_λ in such a manner that the related 2-point and 4-point proper vertexes are just finite though arbitrary when the regularization is removed, *i.e.* for $\epsilon\ \to\ 0$. Hence, after a redefinition of field amplitude, mass and self-coupling, we can recast the renormalized Lagrangian in the form

$$\mathcal{L}_R = \frac{1}{2}\,\partial_\mu \phi_0\,\partial^\mu \phi_0 \;-\; \frac{1}{2}\,m_0^2\,\phi_0^2 - \frac{\lambda_0}{4!}\,\phi_0^4 \tag{2.35}$$

with the customary standard definitions

$$\phi_0\,(x) = \phi\,(x)\,\sqrt{1+\delta_Z}\;\equiv\;\phi\,(x)\;Z_\phi^{1/2} \tag{2.36}$$

$$m_0^2\;\equiv\;(\,m^2 + \delta m^2\,)\;Z_\phi^{-\,1} \tag{2.37}$$

$$\lambda_0\;\equiv\;(\,\lambda + \delta_\lambda\,)\;Z_\phi^{-\,2} \tag{2.38}$$

or equivalently

$$\delta_Z \equiv Z_\phi - 1 \qquad\qquad \delta m^2 \equiv m_0^2\,Z_\phi - m^2 \qquad\qquad \delta_\lambda \equiv \lambda_0\,Z_\phi^2 - \lambda$$

that yields

$$\mathcal{L}_R = \frac{1}{2}\,Z_\phi\,\partial_\mu\phi\,\partial^\mu\phi \;-\; \frac{1}{2}\,\left(m^2 + \delta m^2\right)\phi^2 - \frac{1}{4!}\,(\,\lambda + \delta_\lambda\,)\,\phi^4 \tag{2.39}$$

The quantities ϕ_0 , m_0 , λ_0 , which are divergent for $\epsilon\;\to\;0$, are called the bare field, mass and coupling parameter respectively. It turns out that the bare quantities ϕ_0 , m_0 , λ_0 are evidently non-physical, because of their divergent nature when the regulator is removed. Moreover, it is worthwhile to remark that the bare non-physical parameters ϕ_0 , m_0 , λ_0 can be defined if and only if some regulator has been introduced.

`Two Loops Diagrams`. Here I present a detailed calculation of the so called Setting Sun or Saturn 2-loops diagram in the dimensional regularization. Call it $\mathtt{reg}\,\widetilde{\Gamma}_{\rm SS}(p)$, where p is the external momentum. Its analytic expression, once the two external free propagators have been amputated and after multiplication bi $(-i)$, is provided by

$$\begin{aligned}
&\mathtt{reg}\,\widetilde{\Gamma}_{\rm SS}(p) \;=\\
&\frac{1}{6}\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\int\frac{d^{2\omega}l}{(2\pi)^{2\omega}}\;\frac{\lambda^2\,\mu^{4\epsilon}}{[q^2-m^2+i\epsilon]\,[l^2-m^2+i\epsilon]\,[(p+q+l)^2-m^2+i\epsilon]}
\end{aligned}$$

By making use of the well known Feynman's parametric formula, after exchange of the parametric and loops integrals we come to the multiple integral

$$\mathtt{reg}\,\widetilde{\Gamma}_{\rm SS}(p) = \frac{1}{6}\,\lambda^2\,\mu^{4\epsilon}\,\Gamma(3)\int_0^1 \mathrm{d}\alpha\int_0^1 \mathrm{d}\beta\int_0^1 \mathrm{d}\gamma\;\delta(\alpha+\beta+\gamma-1)\,I(\alpha,\beta,\gamma,p^2)$$

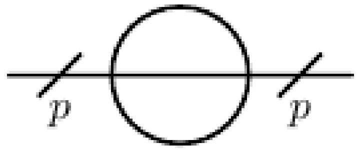

Figure 2.1: the truncated setting sun or Saturn diagram

where

$$\delta(\alpha+\beta+\gamma-1)\, I(\alpha,\beta,\gamma,p^2) = \delta(\alpha+\beta+\gamma-1) \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}}$$
$$\times \quad \left[(\alpha+\gamma)q^2 + (\beta+\gamma)l^2 + p^2\gamma + 2l\cdot q\gamma + 2l\cdot p\gamma + 2q\cdot p\gamma - m^2 + i\varepsilon\right]^{-3}$$

Now it is a clever expedient to introduce the notation $(k_\mu)_\imath = (q_\mu, l_\mu)$, $(z_\mu)_\imath = (\gamma p_\mu, \gamma p_\mu)$ with $\imath = 1,2$ together with the matrix

$$M_{\imath\jmath} = \begin{pmatrix} \alpha+\gamma & \gamma \\ \gamma & \beta+\gamma \end{pmatrix}$$

in such a manner that the denominator of the very last integrand can be recast in the form

$$\begin{aligned} \mathbb{D} &\equiv (\alpha+\gamma)q^2 + (\beta+\gamma)l^2 + p^2\gamma + 2l\cdot q\gamma + 2l\cdot p\gamma + 2q\cdot p\gamma - m^2(\alpha+\beta+\gamma) + i\epsilon \\ &= \sum_{\imath=1,2}\sum_{\jmath=1,2} M_{\imath\jmath}\, k_\imath\cdot k_\jmath + 2\sum_{\imath=1,2} k_\imath\cdot z_\imath + p^2\gamma - m^2 + i\varepsilon \end{aligned}$$

where $\gamma = 1-\alpha-\beta$ is understood, once the support of the Dirac $\delta-$distribution has been properly taken into account. The new pair of variables $\left(k'_\mu\right)_\imath = (q'_\mu, l'_\mu)$, which allow to put the denominator $\mathbb{D}$ into diagonal form, are defined by the expression

$$k_\imath = \sum_{\jmath=1,2}\left\{\left(M^{-1/2}\right)_{\imath\jmath} k'_\jmath - \left(M^{-1}\right)_{\imath\jmath} z_\jmath\right\}$$

in terms of which the denominator becomes

$$\begin{aligned} \mathbb{D} &= \sum_{\imath=1,2} k'_\imath k'_\imath - \sum_{\imath=1,2}\sum_{\jmath=1,2} z_\imath \left(M^{-1}\right)_{\imath\jmath} z_\jmath + p^2\gamma - m^2(\alpha+\beta+\gamma) + i\varepsilon \\ &= q'^2 + l'^2 - \left(m^2 - \frac{\alpha\beta\gamma\, p^2}{\det M}\right) + i\varepsilon \\ &= q'^2 + l'^2 - \left(m^2 - p^2\,\frac{\alpha\beta - \alpha^2\beta - \alpha\beta^2}{\alpha+\beta-\alpha\beta-\alpha^2-\beta^2}\right) - i\varepsilon \end{aligned}$$

where

$$\det M = \alpha\beta + \beta\gamma + \gamma\alpha \qquad \left(M^{-1}\right)_{\imath\jmath} = \begin{pmatrix} \beta+\gamma & -\gamma \\ -\gamma & \alpha+\gamma \end{pmatrix} (\det M)^{-1}$$

$$\gamma = 1 - \alpha - \beta \qquad 0 \le \alpha \le 1 \qquad 0 \le \beta \le 1$$

It turns out that the quantity

$$m^2 - p^2\, f(\alpha,\beta) \equiv m^2 - p^2\, \frac{\alpha\beta - \alpha^2\beta - \alpha\beta^2}{\alpha + \beta - \alpha\beta - \alpha^2 - \beta^2}$$

is always positive in the square domain $\mathfrak{D} = \{(\alpha,\beta) \in \mathbb{R}^2 \mid 0 \le \alpha \le 1\,,\, 0 \le \beta \le 1\}$ of the two independent Feynman's parameters space for $p^2 < 9m^2$, *i.e.* below the threshold for three physical particles production. Now we can perform the change of variables in the two loops integral $I(\alpha,\beta,\gamma,p^2)$

$$q_0' = \pm i q_4 \qquad l_0' = \pm i l_4$$

and set $p_0 \equiv \pm i p_4$ in the external four-momentum, in such a manner to turn to the Euclidean formulation. Then we get

$$I(\alpha,\beta,\gamma,p_E^2) = (\det M)^{-\omega} \int_{q_E} \int_{l_E} \left[q_E^2 + l_E^2 + f(\alpha,\beta)\, p_E^2 + m^2 \right]^{-3}$$

where we employ the usual Euclidean integration variables

$$q_{E\mu} \equiv (\mathbf{q}, q_4)\ ,\ \mathbf{q} \equiv (q_1, q_2, q_3, \ldots, q_{2\omega-1}) \qquad l_{E\mu} \equiv (\mathbf{l}, l_4)\ ,\ \mathbf{l} \equiv (l_1, l_2, l_3, \ldots, l_{2\omega-1})$$

and the shorthand notation

$$\int_{q_E} = \int \frac{d^{2\omega} q_E}{(2\pi)^{2\omega}} \qquad \int_{l_E} = \int \frac{d^{2\omega} l_E}{(2\pi)^{2\omega}}$$

in the non-physical $2\omega-$dimensional Euclidean space, while we set $p_{E\mu} = (p_1, p_2, p_3, p_4)$. After the change of variables the Feynman causal prescription is no longer necessary owing to the positive definiteness of the integrand. By making use of the Mellin transform of the exponential function we obtain for $\epsilon = 2 - \omega$ and $p^2 < 9m^2$

$$\begin{aligned} I(\alpha,\beta,p_E^2) &= (\det M)^{-\omega}\, \frac{1}{\Gamma(3)} \int_0^\infty t^2\, \exp\left\{ - \left[p_E^2\, f(\alpha,\beta) + m^2 \right] t \right\} \mathrm{d}t \\ &\times \int_{q_E} \int_{\ell_E} \exp\left\{ -q_E^2 t - \ell_E^2 t \right\} \\ &= (\det M)^{-\omega}\, \frac{(4\pi)^{2\epsilon - 4}}{\Gamma(3)} \left[p_E^2\, f(\alpha,\beta) + m^2 \right]^{2\omega - 3} \int_0^\infty t^{2 - 2\omega} \mathrm{e}^{-t} \mathrm{d}\, t \\ &= \tfrac{1}{2}\, (\det M)^{-\omega}\, (4\pi)^{2\epsilon - 4} \left[m^2 - f(\alpha,\beta) p^2 \right]^{2\omega - 3} \Gamma(2\epsilon - 1) \end{aligned}$$

where, in the very last equality, the analytic continuation towards the four-dimensional Minkowski space has been performed for the external momentum. Hence, the two loop integral corresponding to the amputated Saturn diagram takes the form

$$\begin{aligned} \mathtt{reg}\, \widetilde{\Gamma}_{\mathrm{SS}}(p) &= \tfrac{1}{6}\, \hat{\lambda}^2 (4\pi\mu^2)^{2\epsilon}\, \Gamma(2\epsilon - 1)\, \sigma_\epsilon(p) \\ \sigma_\epsilon(p) &\equiv \int_0^1 \mathrm{d}\alpha \int_0^1 \mathrm{d}\beta\, (\det M)^{\epsilon - 2} \left[m^2 - f(\alpha,\beta) p^2 \right]^{1 - 2\epsilon} \end{aligned}$$

where $\hat{\lambda} = \lambda / 16\pi$. The last step entails the analysis of the divergences[4] contained in the function $\sigma_\epsilon(p)$. It turns out that, in the limit $\epsilon \to 0$, there are singularities for $\alpha + \beta = 0$,

[4] See John C. Collins (1974) *Phys. Rev. D* **10**, 1213.

$\alpha + \gamma = 0$ and $\beta + \gamma = 0$. To the aim of exhibit the behavior at *e.g.* $\alpha + \beta = 0$ it is expedient to set

$$\alpha = \rho x \qquad \beta = \rho\,(1-x) \qquad \gamma = 1-\alpha-\beta = 1-\rho$$

$$\left|\frac{\partial(\alpha,\beta)}{\partial(\rho,x)}\right| = \rho$$

In so doing the singularity appears for $\alpha + \beta = \rho \to 0$. As a matter of fact, from the leading behaviors

$$\det M = \rho\,\{1-\rho\,[\,1-x(1-x)\,]\} = O(\rho) \qquad \text{for} \quad \rho \to 0^+$$

$$f(\alpha,\beta) = \varphi(\rho,x) = \frac{\rho\,(1-\rho)\,x(1-x)}{1-\rho\,[\,1-x(1-x)\,]} = O(\rho) \qquad \text{for} \quad \rho \to 0^+$$

$$m^2 - \varphi(\rho,x)\,p^2 = O(m^2) \qquad \text{for} \quad \rho \to 0^+$$

it is easy to extract the divergent part of $G_\epsilon\left(p^2, m^2\right)$: namely,

$$\mathtt{div}\,\sigma_\epsilon(p)\rfloor_{\epsilon\to 0^+} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}\rho\; m^{2-2\epsilon}\rho^{\epsilon-1} = \frac{1}{\epsilon}\,m^{2-4\epsilon}$$

The remaining two singularities for $\beta+\gamma = 0 \Leftrightarrow \alpha = \rho x = 1$ and $\alpha+\gamma = 0 \Leftrightarrow \beta = \rho(1-x) = 1$ yield the very same result, owing to the complete symmetry of the parametric integrating function

$$\mathfrak{I}(\alpha,\beta,\gamma) = \delta(\alpha+\beta+\gamma-1)\,(\alpha\beta+\beta\gamma+\gamma\alpha)^{\epsilon-2}\,\left[m^2 - \frac{\alpha\beta\gamma}{\alpha\beta+\beta\gamma+\gamma\alpha}\,p^2\right]^{1-2\epsilon}$$

in such a manner that we can eventually write

$$\mathtt{div}\,\sigma_\epsilon(p)\rfloor_{\epsilon\to 0^+} = \frac{3}{\epsilon}\,m^{2-4\epsilon}$$

Now one can readily obtain the finite part of $\sigma_\epsilon(p)$ that amounts to be an elementary polynomial, *viz.*,

$$\begin{aligned}
\widehat{\sigma}(p) \;&=\; \lim_{\epsilon\,\to\,0^+} \int_0^1 \mathrm{d}x \int_0^1 \rho\,\mathrm{d}\rho\,\Big\{(\det M)^{\epsilon-2}\left[\,m^2 - \varphi(\rho,x)\,p^2\,\right]^{1-2\epsilon} \\
&-\; m^{2-4\epsilon}\left[\,\rho^{\,\epsilon-2} - (1-x\rho)^{\,\epsilon-2} - (1-\rho-x\rho)^{\,\epsilon-2}\,\right]\,\Big\} \\
&=\; \int_0^1 \mathrm{d}x \int_0^1 \rho\,\mathrm{d}\rho\,\Big\{(\det M)^{-2}\left[\,m^2 - \varphi(\rho,x)\,p^2\,\right] \\
&-\; m^2\left[\frac{1}{\rho^2} - \frac{1}{(1-x\rho)^2} - \frac{1}{(1-\rho+x\rho)^2}\right]\Big\} \\
&=\; p^2 \int_0^1 \mathrm{d}x\,x(1-x) \int_0^1 \mathrm{d}\rho\;(\rho-1)\{1+\rho\,[\,x(1-x)-1\,]\}^{-3} \\
&+\; m^2 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}\rho\,\left\{\frac{1}{\rho\,\{1+\rho\,[\,x(1-x)-1\,]\}^2}\right. \\
&-\; \left.\frac{1}{\rho} - \frac{\rho}{(1-x\rho)^2} - \frac{\rho}{(1-\rho+x\rho)^2}\right\} \;=\; 3m^2 - \tfrac{1}{2}\,p^2
\end{aligned}$$

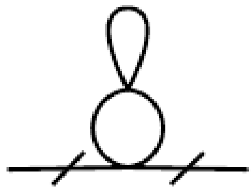

Figure 2.2: the truncated Eight Shaped or Double Scoop diagram

all the above double integrals being elementary. Putting everything together, after setting

$$\sigma'(p) \equiv \lim_{\epsilon\to 0^+} \frac{\partial}{\partial\epsilon}\,\sigma_\epsilon(p)$$

we can write

$$\begin{aligned}
\mathtt{reg}\,\widetilde{\Gamma}_{\rm SS}(p) &= \tfrac{1}{6}\,\hat{\lambda}^2(4\pi\mu^2)^{2\epsilon}\,\Gamma(2\epsilon-1)\,\left[\mathtt{div}\,\sigma_\epsilon(p)+\widehat{\sigma}(p)+\epsilon\,\sigma'(p)\right]\\
&= -\tfrac{1}{6}\,\hat{\lambda}^2\left[1+2\epsilon\ln(4\pi\mu^2)+2\epsilon^2\ln^2(4\pi\mu^2)+O(\epsilon^3)\right]\\
&\times\ \left\{\frac{1}{2\epsilon}+\psi(2)+\epsilon\left[\frac{\pi^2}{3}+\psi^2(2)-\psi'(2)\right]+O(\epsilon^2)\right\}\\
&\times\ \left[\frac{3m^2}{\epsilon}-6\ln m^2+3m^2-\tfrac{1}{2}\,p^2+\epsilon\,\sigma'(p)+O(\epsilon^2)\right]\\
&= -\tfrac{1}{6}\,\hat{\lambda}^2\left[\frac{3m^2}{2\epsilon^2}-\frac{p^2}{4\epsilon}+\frac{3m^2}{\epsilon}\left(\frac{3}{2}+\mathbf{C}-\ln\frac{m^2}{4\pi^2}\right)\right]+\widetilde{\Gamma}^{\rm fin}_{\rm SS}(p)+O(\epsilon)
\end{aligned}$$

The finite part $\widetilde{\Gamma}^{\rm fin}_{\rm SS}(p)$ of the Saturn diagram is hard to obtain, owing to the mass term, because the function $\sigma'(p)$ is not elementary and must be expressed in terms of the Spence or dilogarithm special functions defined by

$$L_{i2}(x) = -\int_0^1 \frac{\mathrm{d}t}{t}\,\ln(1-tx)$$

The other 2-loops diagram contributing to the 2-point proper vertex $\widetilde{\Gamma}^{(2)}(p)$ at the order λ^2 can be readily calculated in dimensional regularization. I will name it the amputated Eight Shaped or Double Scoop diagram and denote it by $\mathtt{reg}\,\Gamma_{\rm DS}(m)$. We find

$$i\,\mathtt{reg}\,\Gamma_{\rm DS}(m) = \tfrac{1}{4}(-i\lambda)^2\,\mu^{4\epsilon}\int_\ell \frac{i}{\ell^2-m^2+i\varepsilon}\int_q\left(\frac{i}{q^2-m^2+i\varepsilon}\right)^2$$

This factorized double integral can be reduced to well known 1-loop integrals *viz.*

$$\mathtt{reg}\,\Gamma_{\rm DS}(m) = -\tfrac{1}{4}\,\hat{\lambda}^2\,m^2\left\{\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left[\psi(2)-\mathbf{C}-2\ln\widehat{m}^2\right]\right\}+\Gamma^{\rm fin}_{\rm DS}(m)+O(\epsilon)$$

where $\Gamma^{\rm fin}_{\rm DS}(m)$ stands for the rather cumbersome expression of the arbitrary finite part of the regularized Double Scoop diagram. Finally, we have to calculate the $O(\lambda^2)$ two

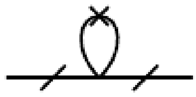

Figure 2.3: the truncated 1-loop counter-term diagram $i\,\Gamma_{\mathrm{c.t.}}(m)$

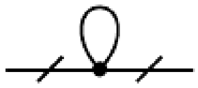

Figure 2.4: the truncated 1-loop counter-term diagram $i\,\overline{\Gamma}_{\mathrm{c.t.}}(m)$

contributions to the 2-point proper vertex involving the 1-loop counter-terms diagrams: namely,

$$\begin{aligned}
\Gamma_{\mathrm{c.t.}}(m) &= -\tfrac{1}{2}\,\hat{\lambda}\,m^2\left[\frac{1}{\epsilon}+\mathrm{F}_1(\epsilon,\widehat{m})\right]\tfrac{1}{2}\,(-i\lambda)\,\mu^{2\epsilon}\int_\ell\left(\frac{i}{\ell^2-m^2+i\varepsilon}\right)^2\\
&= \tfrac{1}{4}\,\hat{\lambda}^2\,m^2\left\{\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left[\mathrm{F}_1(\epsilon,\widehat{m})-\mathbf{C}-\ln\widehat{m}^{\,2}\right]\right\}+\Gamma^{\mathrm{fin}}_{\mathrm{c.t.}}(m)+O(\epsilon)\\
\overline{\Gamma}_{\mathrm{c.t.}}(m) &= -\,\lambda\cdot\tfrac{3}{2}\,\hat{\lambda}\left[\frac{1}{\epsilon}+\mathrm{G}_1(\epsilon,\widehat{m})\right]\tfrac{1}{2}\,\mu^{2\epsilon}\int_\ell\frac{i}{\ell^2-m^2+i\varepsilon}\\
&= \tfrac{3}{4}\,\hat{\lambda}^2\,m^2\left\{\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left[\mathrm{G}_1(\epsilon,\widehat{m})+\psi(2)-\ln\widehat{m}^{\,2}\right]\right\}+\overline{\Gamma}^{\mathrm{fin}}_{\mathrm{c.t.}}(m)+O(\epsilon)
\end{aligned}$$

Summing up, it follows therefrom that the $O(\lambda^2)$ superficially divergent expression of the 2-point proper vertex in dimensional regularization takes the form

$$\begin{aligned}
\mathtt{reg}\,\widetilde{\Gamma}^{\,(2)}\,(k\,;\lambda,m,\mu,\epsilon)\Big]_{2-\ell\mathrm{oop}} &= k^2-m^2-\lambda\,\Sigma_1(m^2)\\
&+ \frac{\hat{\lambda}^2}{24\epsilon}\,k^2+\tfrac{1}{2}\,\hat{\lambda}^2\,m^2\left\{\frac{1}{\epsilon^2}+\frac{1}{2\epsilon}\left[\mathrm{F}_1+3\mathrm{G}_1-1\right]\right\}\\
&- \lambda^2\,\widehat{\Sigma}_2(k\,;\lambda,m)\;+\;O\left(\lambda^3,\epsilon\right)
\end{aligned}$$

the right-hand side graphically corresponding to the Figure 2.5.

2.1.3 The Physical Renormalization Condition

It is very important to gather that the renormalized Lagrangian $\mathcal{L}_R$ looks exactly the same as the classical Lagrangian $\mathcal{L}$ but for parameters and fields.

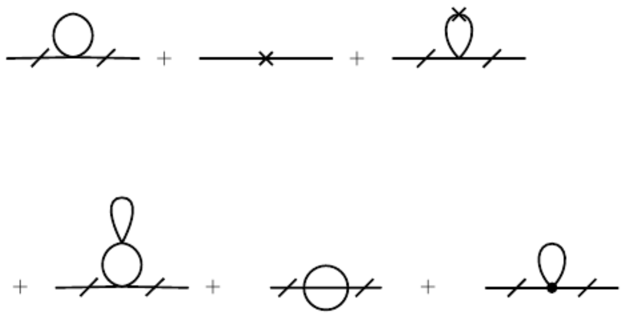

Figure 2.5: diagrams contributing to $\mathtt{reg}\,\widetilde{\Gamma}^{(2)}(k)$ up to the order λ^2

Moreover, the renormalized Lagrangian $\mathcal{L}_R$ leads to a finite theory while the classical Lagrangian $\mathcal{L}$ does not. This fact indicates that we can always put all the infinities of perturbation theory inside ϕ_0 , m_0 , λ_0 , which means that, thanks to the renormalization procedure, all those infinities are then removed order-by-order from the Green's functions and in turn from the $S-$matrix elements. The bare quantities do diverge for $\epsilon \to 0$, while the renormalized quantities ϕ, m, λ give finite although arbitrary values when the ultraviolet regulators are removed, *i.e.* , for $\epsilon \to 0$. The latter have to be connected with the physical parameters and fields of the theory, by means of the so called PHYSICAL RENORMALIZATION CONDITION OR PRESCRIPTION.

Suppose that the renormalized mass m does correspond to the physical mass of the neutral scalar particle as experimentally measured, *e.g.* the mass of a heavy neutral Higgs boson recently discovered at LHC

$$m = m_{\text{Higgs}} = 126 \text{ GeV}$$

Contextually, let me assume that the coupling λ just corresponds to the expected Higgs particle self-interaction, as measured in an elastic two Higgs bosons scattering at energies close to the threshold $z \gtrsim 4m^2$. In such a circumstance we come to the following defining relations, which constitute the physical renormalization prescription: namely,

$$1. \qquad \lim_{k^2 \to m^2} \widetilde{\Gamma}_R^{(2)}(k\,;\lambda, m) = 0 \tag{2.40}$$

$$2. \qquad \lim_{k^2 \to m^2} (\partial/\partial k^2)\, \widetilde{\Gamma}_R^{(2)}(k\,;\lambda, m) = 1 \tag{2.41}$$

$$3. \qquad \lim_{z \to 4m^2} \widetilde{\Gamma}_R^{(4)}(s, t, u\,;\lambda, m) \;=\; -\,\lambda \tag{2.42}$$

with

$$\widetilde{\Gamma}^{(2)}_{R}(\,k\,;\lambda,m\,) = k^{2} - m^{2} - \Sigma(\,k\,;\lambda,m\,) \tag{2.43}$$

$$\Sigma(\,k\,;\lambda,m\,) = \sum_{n=0}^{\infty} \lambda^{n}\,\Sigma_{n}\,(\,k\,;m\,) \tag{2.44}$$

$$\lim_{k^{2}\to m^{2}} \ \Sigma(\,k\,;\lambda,m\,) = 0 \tag{2.45}$$

$$\lim_{k^{2}\to m^{2}} \ \frac{\partial}{\partial k^{2}}\ \Sigma(\,k\,;\lambda,m\,) = 0 \tag{2.46}$$

Hence, from the previous 1-loop expressions (2.17), (2.21) and the limits (2.6), (2.7) we definitely obtain

$$\mathrm{F}_{1} = 1 - \mathbf{C} - \ln \widehat{m}^{2} \qquad\qquad \mathrm{G}_{1} = 2 - \ \mathbf{C} - \ln \widehat{m}^{2} \tag{2.47}$$

$$-\!\!\times\!\!- \ = \ -\ \frac{\lambda m^{2}}{32\,\pi^{2}}\ \left[\,\frac{1}{\epsilon} + 1 - \ \mathbf{C} - \ln \widehat{m}^{2}\,\right] \tag{2.48}$$

$$\bullet \ = \ (\,-\,i\lambda\,)\ \cdot\ \frac{3\lambda}{32\pi^{2}}\ \left[\,\frac{1}{\epsilon} + 2 - \ \mathbf{C} - \ln \widehat{m}^{2}\,\right] \tag{2.49}$$

while from the 2-loop expression (2.27) we get

$$\lim_{k^{2}\to m^{2}} \ \Sigma_{2}\,(\,k\,;m\,) \ = \ \lim_{k^{2}\to m^{2}} \ \frac{\partial}{\partial k^{2}}\ \Sigma_{2}\,(\,k\,;m\,) \ = \ 0 \tag{2.50}$$

together with

$$\begin{aligned} -\!\!\times\!\!\!\times\!\!- \ &= \ -\,\frac{\hat{\lambda}^{2}}{24}\ \cdot\ \frac{1}{\epsilon}\,k^{2} \ - \ \hat{\lambda}^{2}\,k^{2}\,\mathrm{H}_{2}\,(\widehat{m}^{2}) \\ - \ \tfrac{1}{2}\,m^{2}&\left\{\frac{\hat{\lambda}^{2}}{\epsilon^{2}} + \frac{1}{\epsilon}\left[\,\hat{\lambda} - 2\hat{\lambda}^{2}\left(\mathbf{C} + \ln \widehat{m}^{2}\right)\right] + 2\hat{\lambda}^{2}\,\mathrm{F}_{2}\right\} \end{aligned} \tag{2.51}$$

The renormalized proper vertexes then become

$$\widetilde{\Gamma}^{(2)}_{R}(k\,;\lambda,m) = k^{\,2} - m^{2} - \lambda^{2}\,\Sigma_{\,2}(k\,;m) \ + \ O(\lambda^{3})$$

$$\widetilde{\Gamma}^{(4)}_{R}(s,t,u\,;\lambda,m) = -\,\lambda\left[\,1 - \hat{\lambda}\,A(s,t,u) + O(\lambda^{2})\,\right]$$

It is very important to gather that for very large values of the Mandelstam variables in the so called deep Euclidean region $(-\,z) \ = \ \mathbf{k}^{2} \ \gg \ 4m^{2}$, where z is any of the three Mandelstam variables s,t,u , taking the leading behavior (2.8) into account we find the effective running coupling

$$\begin{aligned} -\,\widetilde{\Gamma}^{(4)}_{R}\,(s,t,u\,;\lambda,m) \ &= \ \lambda_{\,\mathrm{eff}}(\mathbf{k}^{2}) \approx \lambda\left\{1 + \frac{3\lambda}{32\pi^{2}}\ \ln\frac{\mathbf{k}^{2}}{m^{2}}\right\} \\ &\simeq \ \lambda\cdot\frac{1}{1 - \dfrac{3\lambda}{16\pi^{\,2}}\ \ln\dfrac{|\mathbf{k}|}{m}} \end{aligned} \tag{2.52}$$

that yields for $\lambda = 1$, *i.e.* at the border of validity of perturbation theory, a Landau-Pomerančuk like pole at the huge scale

$$\hbar c \ |\, \mathbf{k} \,| \simeq \ mc^2\, \mathrm{e}^{52} \simeq mc^2 \times 10^{23}$$

which is of the order 10^{25} GeV for a Higgs boson mass of 125 GeV/c^2, far beyond the quantum gravity scale $M_{\mathrm{Planck}} = 1.22093(7) \times 10^{19}$ GeV. Note that also the ϕ_4^4 scalar self-interaction coupling is increasing with the scale, just like the fine structure coupling (see [22] §**4.2.2**).

2.2 The Renormalization Group

The concepts of counter-terms, bare Lagrangian, fields, masses and couplings can be introduced only after some regularization method has been employed. For example, within the framework of dimensional regularization, one has to deal with a D-dimensional Minkowski space with $\mathrm{D} = \Re\mathrm{e}\, 2\omega = 4 - 2\epsilon$ ($\epsilon > 0$) so that the renormalized Action in terms of the bare and dressed scalar fields, masses and couplings reads

$$\begin{aligned} \mathcal{A}_R \ &= \ \hbar \int \mathrm{d}^{\mathrm{D}} x \ \left(\frac{1}{2}\, \partial_\mu \phi_0\, \partial^\mu \phi_0 \ - \frac{1}{2}\, m_0^2\, \phi_0^{\,2} - \frac{\lambda_0}{4!}\, \phi_0^{\,4} \right) \\ &= \ \frac{\hbar}{2} \int \mathrm{d}^{\mathrm{D}} x \ \left(Z_3\, \partial_\mu \phi\, \partial^\mu \phi \ - m^2\, Z_2\, \phi^{\,2} - \frac{\lambda}{12}\, Z_1\, \mu^{\,2\epsilon}\, \phi^{\,4} \right) \end{aligned} \qquad (2.53)$$

with the slightly different though quite customary definitions

$$\phi_0\,(x) = \phi\,(x)\, \sqrt{Z_\phi} \ \equiv \ \phi\,(x)\ Z_3^{1/2} \qquad (2.54)$$

$$m_0^2 \ \equiv \ m^2\, Z_2\, Z_3^{-\,1} \qquad (2.55)$$

$$\lambda_0 \ \equiv \ \mu^{2\epsilon}\, \lambda\, Z_1\, Z_3^{-\,2} \qquad (2.56)$$

It is worthwhile to remark once more that the renormalized Lagrangian as well as the bare, *i.e.* divergent, parameters λ_0 and m_0 do exist only in the presence of some regularization: for example, if dimensional regularization is employed, then they exist only in the D$-$dimensional Minkowski space with $\mathrm{D} < 4$ and $\epsilon > 0$. On the contrary, the free[5] dressed parameters λ

[5]The dressed finite parameters are called *free* in the sense that they can not be *a priori* specified in any model of quantum field theory, albeit they can be extracted only by experimental investigation.

and m are well defined and finite in the four dimensional Minkowski space. In so doing the classical and counter-term Lagrangian in a D-dimensional Minkowski space take the form

$$\mathcal{L}_R = \mathcal{L}^{(\mathrm{D})} + \mathcal{L}_{\mathrm{c.t.}}$$

$$\mathcal{L}^{(\mathrm{D})} = \tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi\, - \tfrac{1}{2}\,m^2\,\phi^2 - \frac{\lambda}{4!}\,\mu^{2\epsilon}\,\phi^4 \tag{2.57}$$

$$\mathcal{L}_{\mathrm{c.t.}} = \tfrac{1}{2}\,\delta_Z\,\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}\,\delta m^2\,\phi^2 - \tfrac{1}{4!}\,\delta_\lambda\,\phi^4 \tag{2.58}$$

where

$$\delta_Z \equiv Z_3 - 1 \qquad \delta m^2 \equiv m^2\,(\,Z_2 - 1\,) \qquad \delta_\lambda \equiv \mu^{2\epsilon}\,\lambda\,(\,Z_1 - 1\,)$$

Note that in D space-time dimensions the bare and the dressed scalar fields both have canonical engineering dimensions

$$[\,\phi_0\,] = [\,\phi\,] = \mathrm{cm}^{1-\mathrm{D}/2} = \mathrm{eV}^{\,\frac{\mathrm{D}}{2}-1}$$

in natural units, while only the dressed coupling λ keeps dimensionless. It is also worthwhile to observe that $\mathcal{L}_R$ and $\mathcal{L}_{\mathrm{c.t.}}$ solely exist in D space-time dimensions with $\epsilon < 0$, because both diverge for $\epsilon \to 0$.

In the functional integral approach, one integrates over the field variable. Thus, its rescaling by Z_3 can be always reabsorbed provided one redefines the classical source accordingly, by introducing the bare source

$$J_0\,(x) = Z_3^{-1/2}\,J\,(x) \tag{2.59}$$

and the bare classical fields

$$\phi_{\,c\ell\,,0}(x) = \sqrt{Z_3}\,\phi_{\,c\ell}(x) \tag{2.60}$$

Then, on the one hand, starting from the renormalized Lagrangian (2.35) and taking functional derivatives with respect to the bare source $J_0\,(x)$ or the bare classical fields $\phi_{c\ell\,,\,0}\,(x)$, we obtain the Green's functions or the proper vertexes of perturbation theory, in which the parameters m and λ are replaced by the bare ones m_0 and λ_0 . On the other hand, had we started instead from the renormalized Lagrangian (2.39) and taken functional derivatives with respect to the classical source $J\,(x)$, then we end up with the finite Green's functions and proper vertexes. For the 1PI Green's functions this equality reads

$$\widetilde{\Gamma}_0^{(n)}\,(k_1,\ldots,k_n\,;\,\lambda_0, m_0, \epsilon) = Z_3^{-\,n/2}\,\widetilde{\Gamma}_R^{(n)}\,(k_1,\ldots,k_n\,;\,\lambda, m, \mu, \epsilon) \tag{2.61}$$

where $\epsilon = 2 - \mathrm{D}/2$, whereas $\widetilde{\Gamma}_R^{(n)}$ are finite as $\epsilon \to 0$, because the theory is renormalizable by power counting and the free dressed parameters λ and m are always finite in the four dimensional Minkowski space[6]. In eq. (2.61) one can understand the bare parameters (m_0, λ_0) as functions of the renormalized or dressed parameters (m, λ) and of the regularisation tools (μ, ϵ), or suppose the bare parameters to be truly independent ones.

In the latter case, the dressed parameters are then functions of the bare ones. Hence, in so doing, it turns out that the left hand side of (2.61) becomes absolute and does not depend at all on the arbitrary mass scale μ, which is nothing but an *ad hoc* tool of the regularisation scheme, whilst the right hand side will depend upon μ both explicitly as well as implicitly through $m(\mu), \lambda(\mu)$ and $Z_3(\epsilon, \mu)$. Therefore, by differentiating the relationship (2.61) with respect to μ we eventually obtain a differential equation, wich is called the renormalization group equation, that summarizes the deep content of the renormalization procedure: namely,

$$\left(\mu\,\frac{\partial}{\partial\mu} + \mu\,\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\cdot\frac{\partial}{\partial\lambda} + \mu\,\frac{\mathrm{d}m}{\mathrm{d}\mu}\cdot\frac{\partial}{\partial m} - \frac{n}{2}\,\mu\,\frac{\partial}{\partial\mu}\ln Z_3\right)\widetilde{\Gamma}_R^{(n)} = 0 \qquad (2.62)$$

The conceptual content of the above equation is that the physical quantities on the Minkowski space, such as the scattering amplitude and cross-section, can never and by no means depend upon the arbitrary and unphysical scale which is necessarily involved in any regularisation procedure. Hence, the beauty of this renormalization group equation is that it merely involves the renormalized 1PI Green's functions which are finite as long as $\epsilon \to 0$, *i.e.* in the physical Minkowski space. It is very important to realize that the above relationships (2.61,2.62), together with their related features, hold true **only** for renormalizable quantum field theories, which meand that they normally fail for non-renormalizable, super-renormalizable, or any other effective field theory model valid up to some fixed energy scale[7].

The various derivatives in (2.62) come from the implicit dependence of the renormalized proper vertexes upon μ through m, λ and Z_3. Define the coefficients

$$\beta(\lambda, m/\mu, \epsilon) \equiv \mu\,\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \qquad (2.63)$$

$$\gamma_m(\lambda, m/\mu, \epsilon) \equiv \tfrac{1}{2}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\ln m^2 \qquad (2.64)$$

[6]Of course, the very same relationship (2.61) holds true *mutatis mutandis* for any other regularisation such as UV cut-off or Pauli-Villars.

[7]It is worthwhile to gather that the range in which a renormalizable perturbative QFT is valid and predictive is merely bounded by the requirement that the coupling parameters stay less than one.

$$\gamma_d(\lambda, m/\mu, \epsilon) \equiv \tfrac{1}{2}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_3 \tag{2.65}$$

They are analytic for $\epsilon \to 0$ and dimensionless for they depend only upon λ and m/μ. On the other side, the renormalized proper vertexes $\widetilde{\Gamma}_R^{(n)}$ have canonical engineering mass dimensions equal to $\mathrm{D} + n\,(1 - \mathrm{D}/2)$ in the D dimensional Minkowski space. Thus, if we set

$$p = (\,p_1, p_2, \ldots, p_n\,) \qquad p \quad\longmapsto\quad \mathrm{e}^{\,\varrho}\,p \qquad \varrho \in \mathbb{R}$$

where ϱ denotes the energy momentum rescaling parameter, we can write in accordance with Euler theorem on homogeneous functions for $\mathrm{D} = 4 - 2\epsilon$

$$\left(\frac{\partial}{\partial\varrho} + \mu\,\frac{\partial}{\partial\mu} + m\,\frac{\partial}{\partial m} + \epsilon\,(2-n) + n - 4\right)\widetilde{\Gamma}_R^{(n)}\,(\mathrm{e}^{\,\varrho}\,p;\lambda, m, \mu\,) = 0 \tag{2.66}$$

It follows that after eliminating the term $\mu\,\partial/\partial\mu$ between (2.62) and (2.66) we eventually obtain

$$\left(\frac{\partial}{\partial\varrho} - \beta\,\frac{\partial}{\partial\lambda} + (1-\gamma_m)\,m\,\frac{\partial}{\partial m} - n\,(1-\gamma_d) + 4\right)\widetilde{\Gamma}_R^{(n)}\,(\mathrm{e}^{\,\varrho}\,p;\lambda, m/\mu\,) = 0 \tag{2.67}$$

$$\beta\,(\lambda, m/\mu) = \lim_{\epsilon\to 0}\ \mu\,\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \qquad \gamma_m\,(\lambda, m/\mu) = \lim_{\epsilon\to 0}\ \mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\ \ln m$$

$$\gamma_d\,(\lambda, m/\mu) = \lim_{\epsilon\to 0}\ \mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\ \ln\sqrt{Z_3}$$

This equation encodes the behavior of all the renormalized 1PI Green's functions of the theory when the energy momentum variables are rescaled. Its solution will unravel how the the Green's functions as well as the collision matrix elements at some four momentum p are related to the corresponding quantities at the energy momentum scale $p' = \mathrm{e}^{\,\varrho}\,p$.

The difficulty in solving the partial differential equations (2.67) of the Monge type lies in the fact that the coefficient functions $\beta, \gamma_m, \gamma_d$ depend upon two dimensionless variables λ and m/μ. In turn, the latter ones can be computed order by order in perturbation theory, provided the finite part of the counter-terms has been selected. This can be done quite generally in terms of the so called RENORMALIZATION PRESCRIPTIONS that will be discussed in the next paragraph. Hence the solution of the scaling equation (2.67) is unique, provided some renormalization prescription has been definitely selected. Historically, equations of the kind (2.67) has been obtained within different field theoretic developments by

1. Murray Gell-Mann & F.E. Low (1954) *Quantum Electrodynamics at Small Distances*, Phys. Rev. **95**, 1300-1312

2. Kurt Symanzik (1970) *Small Distance Behavior in Field Theory and Power Counting*, Commun. Math. Phys. **18**, 227

3. Curtis G. Callan Jr. (1970) *Broken Scale Invariance in Scalar Field Theory*, Phys. Rev. D **2**, 1541-1547

It is always possible to express the bare parameters as Laurent series, the coefficients of which are functions of λ and m/μ, *viz.,*

$$\lambda_0 = \mu^{2\epsilon}\left[a_0(\lambda, m/\mu, \epsilon) + \sum_{n=1}^{\infty} a_n(\lambda, m/\mu)\,\epsilon^{-n}\right] \tag{2.68}$$

$$m_0^2 = m^2\left[b_0(\lambda, m/\mu, \epsilon) + \sum_{n=1}^{\infty} b_n(\lambda, m/\mu)\,\epsilon^{-n}\right] \tag{2.69}$$

$$Z_3 = Z_\phi = c_0(\lambda, m/\mu, \epsilon) + \sum_{n=1}^{\infty} c_n(\lambda, m/\mu)\,\epsilon^{-n} \tag{2.70}$$

where the coefficients a_0, b_0, c_0 are analytic for $\epsilon \to 0$. A comparison with the already obtained lowest order counter-terms (2.15), (2.19) and (2.26) yield for $\epsilon \to 0$ and with $\widehat{\lambda} = \lambda/16\pi^2$

$$\begin{aligned}
a_0 &= \lambda\left[1 + \tfrac{3}{2}\widehat{\lambda}\,G_1 + \cdots\right] \\
b_0 &= 1 + \tfrac{1}{2}\widehat{\lambda}\,F_1 + \widehat{\lambda}^2 F_2 + \cdots \\
c_0 &= 1 - \widehat{\lambda}^2 H_2 + \cdots \\
a_1 &= \tfrac{3}{2}\lambda\widehat{\lambda} + \cdots \\
b_1 &= \tfrac{1}{2}\widehat{\lambda} + \tfrac{1}{4}\widehat{\lambda}^2\left(F_1 + 3G_1 - \tfrac{5}{6}\right) + \cdots \\
c_1 &= -\tfrac{1}{24}\widehat{\lambda}^2 + \cdots \\
b_2 &= \tfrac{1}{2}\widehat{\lambda}^2 + \cdots \\
&\vdots
\end{aligned} \tag{2.71}$$

2.2.1 Renormalization Prescriptions

In the previous paragraph we have seen that, besides the introduction of the arbitrary scale μ within dimensional regularization, the removal of all the divergences order by order in perturbation theory can be achieved at the price of a further large source of arbitrariness which is encoded in the choice of the finite part of the counter-term functions $\mathrm{F}_1, \mathrm{G}_1, \mathrm{F}_2, \mathrm{H}_2, \ldots$, and so on.

Since the structure of the renormalized Lagrangian for *e.g.* the real scalar self-interacting field theory in D space-time dimensions is provided by

$$\mathcal{L}_R = \mathcal{L}^{(\mathrm{D})} + \mathcal{L}_{\mathrm{c.t.}}$$

where owing to (2.58) $\mathcal{L}^{(\mathrm{D})}$ and $\mathcal{L}_{\mathrm{c.t.}}$ share the very same form, it is clear that a different choice of the finite parts of the counter-terms $\delta Z, \delta m^2$ and $\delta\lambda$ which enter in $\mathcal{L}^{(\mathrm{D})}$ can be always traded with a corresponding suitable redefinition of the dressed parameters ϕ, m^2, λ appearing into $\mathcal{L}^{(\mathrm{D})}$. This means that the finite part of the counter-terms is fixed iff the finite dressed parameters have been specified. Any such definite choice is called a renormalization prescription or condition. The latter is dictated, for example, either by the simplicity of the physical interpretation, *i.e.* the correspondence with the measurable quantities, or by the convergence properties of the perturbative series, or else for any other sake of convenience, like the possibility to solve explicitly eq. (2.67) in order to find the asymptotic behavior of the Green's functions and scattering amplitudes.

BPHZ renormalization prescription

The renormalization condition universally adopted in QED is fixed by the requirement of a 1:1 correspondence between the dressed parameters to the measured quantities like the electron mass and charge. Actually, it turns out that in QED, which is power counting renormalizable and gauge invariant, the only primitively divergent proper vertexes are the electron proper vertex $\widetilde{\Gamma}(\not{p})$, the transverse photon vertex $\widetilde{\Gamma}^{\mu\nu}(k)$, which satisfies $k_\mu\, \widetilde{\Gamma}^{\mu\nu}(k) = 0$ because of gauge invariance, and the $e^-e^+\gamma$ proper vertex $\widetilde{\Gamma}^{\mu}(p, p')$. Then the conventional renormalization condition is the so called on the mass shell or Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization condition

$$\begin{cases} \widetilde{\Gamma}^{\mu\nu}_R(k) = (\,k^\mu k^\nu - \,k^2 g^{\mu\nu}\,)\,[\,1 - \Pi_R(k^2)\,]\,, \\ \qquad \widetilde{\Gamma}^{\mu}_R(p, p'\,) \equiv ie\,\gamma^\mu\,, \qquad\qquad p = p' \\ \quad \widetilde{\Gamma}_R(\not{p}\,) = \not{p} - \,m_e - \Sigma_R(\not{p}\,)\,, \end{cases} \tag{2.72}$$

$$\lim_{k\to 0} \Pi_R(k^2) = 0 \qquad\qquad \lim_{p^2\to m_e^2} \Sigma_R(\not{p}) = 0 = \lim_{p^2\to m_e^2} \Sigma'_R(\not{p}) \tag{2.73}$$

where prime means derivative with respect to the argument. In the above expressions the renormalized self-energies are subtracted at zero momenta, while the $e^-e^+\gamma$ vertex is subtracted at $k^\mu = 0$ *i.e.* in the Thomson limit. In so doing the finite dressed parameters are identified with the measured electron mass m_e and charge $-\,e$, while the residues at the physical poles for the full renormalized photon and electron propagators are equal to i, since

$$S_R(\not p) = \frac{i}{\not p - m_e - \Sigma_R(\not p)} \tag{2.74}$$

$$\widetilde{G}_R^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon} \left\{ \frac{1}{1 - \Pi_R(k^2)} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + \frac{k^\mu k^\nu}{k^2} \right\} \tag{2.75}$$

the full photon propagator being understood in the Feynman gauge.

BPHZ. Renormalization of quantum electrodynamics was first obtained by Freeman J. Dyson (1949) *The S-matrix in Quantum Electrodynamics*, Phys. Rev. vol. 75 p. 1736. The first complete proof of the renormalizability of QED to all orders has been given by Nicolai N. **Bogoliubov** & O.S. **Parasiuk** (1957) *On the Multiplication of Propagators in Quantum Field Theory*, Acta Math. vol. 97 p. 227 and is thoroughly discussed in the classic textbook N.N. Bogoliubov & D.V. Shirkov (1959) *An Introduction to the Theory of Quantized Fields*, Interscience, New York. The convergence of the renormalized Feynman diagrams has been proved in full detail by Klaus **Hepp** (1966) *Proof of the Bogoliubov-Parasiuk Theorem on Renormalization*, Commun. Math. Phys. vol. 2 pp. 301-326 and developed in detail in his book *Théorie de la Renormalisation* (1969) Springer-Verlag, Berlin. Finally, Wolfhart **Zimmermann** gave a comprehensive rigorous account on the renormalization, including the famous proof based upon the so called Forest Formula and his method of zero momentum subtraction of the divergent integrals, in the classic *Lectures on Elementary Particles and Quantum Field Theory* (1970) Brandais Summer Institute, S. Deser, M. Grisaru and H. Pendleton Eds., Cambridge, Massachusetts MIT Press.

The BPHZ renormalization prescription is such that the physical electric charge is identified with the renormalized proper vertex $e^+e^-\gamma$ at zero photon four momentum, while the electron mass and null photon mass are identified with the related 2-point 1PI vertexes always at zero energies and momenta. Hence we shall say that the zero four momentum is the subtraction point for the relevant Green's functions and/or the scale at which the latter ones are equated to the renormalized dressed parameters.

In general, the specification of the scale or subtraction point at which the finite dressed parameters are identified with the corresponding renormalized 1PI Green's functions is *a priori* largely arbitrary. There are nonetheless some important constraints, which are due to the unavoidable occurrence of the physical singularities. In the case of the field theories involving mass-less particles, like QCD in the perturbative regime of very high energies momenta and negligible quark masses, it is not possible to subtract the renormalized 1PI vertexes at zero energy momentum owing to the appearance of infrared divergences due to masslessness of the particles (*e.g.* quarks and gluons) and possible parallelism of the related momenta. Moreover, as we have already encountered and discussed, some scattering amplitudes develop branch points and cuts in the energy plane beyond some physical thresholds, like *e.g.* for $s = (k_1 + k_2)^2 \geq 4m^2$ in the case of elastic scattering of neutral spinless particles. Thus, it would be unwise to subtract the ultraviolet divergences at some large time-like values of the four momenta. Actually, it is worth while to keep in mind that, in order to deal with well defined absolutely convergent regularized Feynman integrals, before the subtraction procedure, it is mandatory to turn to the Euclidean formulation, that means to perform the Wick rotation. But as I have repeatedly emphasized, this is generally always possible and safe only in the deep Euclidean region, *i.e.* for suitably selected large space-like values of energies momenta. The transition from the Minkowskian to Euclidean formulation for the Green's function is provided by the following recipe.

Renormalization Prescriptions in the Euclidean

Before proceeding further on, let me turn to the Euclidean formulation in $D = 2\omega$ space dimensions, in such a manner to always deal with absolutely convergent dimensionally regularized integrals. The Feynman rules have been obtained in [22] §**2.6** and the transition from the Euclidean space to the Minkowski space for a generic Feynman integral is quite simple. Let me consider in fact a Feynman diagram G with L loops, V vertexes and I internal lines. In order to perform the continuation from Minkowskian to Euclidean spaces we have to multiply each propagator and each vertex by a factor $+i$ and a further factor $+i$ arises for each loop integration owing to Wick rotation $\mathrm{d}^{2\omega}\ell = i\,\mathrm{d}^{2\omega}\ell_E$. In fact we can write symbolically

$$\begin{aligned} G_{\mathrm{M}}^{(n)}(m^2 - i\varepsilon) &= \left(\int \mathrm{d}^{2\omega}\ell\right)^L \left(\frac{i}{k^2 - m^2 + i\varepsilon}\right)^I (-\,i\lambda)^V \\ &= i^L \left(\int \mathrm{d}^{2\omega}\ell_E\right)^L \left(\frac{1}{k_E^2 + m^2}\right)^I (-\,i)^I\,(-\,\lambda)^V\,i^V \\ &= i^{\,L-I+V}\;\; G_{\mathrm{E}}^{(n)}(\,m^2\,) \end{aligned} \tag{2.76}$$

and taking the topological relation $L - I + V = 1$ into account we eventually find

$$G_{\mathrm{M}}^{(n)}(m^2 - i0) = i\,G_{\mathrm{E}}^{(n)}(m^2)$$

which is valid for all $n > 2$, while for $n = 2$ we have to replace i by $-\,i$ as it's evident from the free propagators. This entails in turn the following relation between Minkowskian and Euclidean proper vertexes: namely,

$$\widetilde{\Gamma}_{\mathrm{M}}^{(n)}(\,k_1\,,\,\ldots\,,\,k_n\,;\,m^2 - i0\,) = i\,\widetilde{\Gamma}_{\mathrm{E}}^{(n)}(\,\bar{k}_1\,,\,\ldots\,,\,\bar{k}_n\,;\,m^2\,)$$

$$\bar{k}_{\mu\jmath} = (\,\mathbf{k}_\jmath\,,\, k_{4\jmath}\,) \qquad k^\mu_\jmath = (\,\mathbf{k}_\jmath\,,\, k^0_\jmath\,) \qquad k_{0\jmath} = i\, k_{4\jmath} \qquad (\,\jmath = 1,2,\,\ldots\, n\,)$$

It follows that in the Euclidean formulation we have

$$\begin{aligned} \mathtt{reg}\,\widetilde{\Gamma}^{(2)}(\bar{k}) \;&=\; \bar{k}^{\,2} + m^2 \left\{ 1 - \frac{\lambda}{32\,\pi^2} \;\left[\frac{1}{\epsilon} + \psi\,(2) - \ln\frac{m^2}{4\pi\mu^2} \right] \right\} \\ &+\;\; O\,(\,\lambda^2\,) \end{aligned}$$

$$\begin{aligned} &\mathtt{reg}\,\widetilde{\Gamma}^{(4)}(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) \;=\; \\ &i\lambda \left\{ 1 - \frac{3\lambda}{32\pi^2} \left[\frac{1}{\epsilon} - \mathbf{C} - \ln\frac{m^2}{4\pi\mu^2} + 2 - \tfrac{2}{3}\, A\,(\,s,t,u\,) \right] \right\} \\ &+ O\,(\,\lambda^3\,) \end{aligned}$$

$$\begin{aligned} A\,(\,s,t,u\,) \;&=\; \sum_{z=s\,,\,t\,,\,u} \left(\frac{4m^2}{z} + 1 \right)^{1/2} \\ &\times\;\; \left[\ln\left(1 + \sqrt{\frac{4m^2}{z} + 1} \right) - \ln\left(-1 + \sqrt{\frac{4m^2}{z} + 1} \right) \right] \end{aligned}$$

$$s = (\bar{k}_1 + \bar{k}_2)^2 \qquad t = (\bar{k}_1 + \bar{k}_3)^2 \qquad u = (\bar{k}_1 + \bar{k}_4)^2$$

where I have omitted the lower case suffix E for the sake of brevity because the Euclidean nature of the proper vertexes is apparent in the momentum dependence $\bar{k}_{\mu\jmath}$. Then the 1−loop Euclidean counter-terms become

$$-\!\!\times\!\!- \;\stackrel{\text{def}}{=}\; -\,\frac{\lambda\, m^2}{32\,\pi^2} \;\left[\frac{1}{\epsilon} + \mathrm{F}_1 \left(\epsilon\,,\, \frac{m^2}{4\pi\mu^2} \right) \right] \tag{2.77}$$

$$\otimes \;\stackrel{\text{def}}{=}\; (\,-\,\lambda\,) \cdot \frac{3\lambda}{32\pi^2} \left[\frac{1}{\epsilon} + \mathrm{G}_1 \left(\epsilon\,,\, \frac{m^2}{4\pi\mu^2} \right) \right] \tag{2.78}$$

and the corresponding renormalized finite proper vertexes

$$\begin{aligned} \widetilde{\Gamma}^{(2)}_R(\bar{k}) \;&=\; \bar{k}^{\,2} + m^2 \left[1 + \tfrac{1}{2}\hat{\lambda} \left(\mathrm{F}_1 - 1 + \mathbf{C} + \ln\frac{m^2}{4\pi\mu^2} \right) \right] \\ &+\;\; \hat{\lambda}^2 \left[\, \tfrac{1}{2}\, m^2\, \mathrm{F}_2 + \bar{k}^{\,2}\, \mathrm{H}_2 \,\right] + \widehat{\Sigma}_2\,(\bar{k}\,;\lambda\,, m) + O\,(\,\lambda^3\,) \end{aligned}$$

where $\widehat{\Sigma}_2(0;\lambda,m) = \widehat{\Sigma}'_2(0;\lambda,m) = 0$, together with

$$\begin{aligned} &\widetilde{\Gamma}^{(4)}_R\,(s,t,u\,;\lambda\,, m\,,\mu) \;=\; i\lambda \\ \times\;\; &\left\{ 1 + \tfrac{3}{2}\,\hat{\lambda} \left[\mathrm{G}_1 + \mathbf{C} + \ln\frac{m^2}{4\pi\mu^2} - 2 + \tfrac{2}{3}\, A\,(\,s,t,u\,) \right] + O\,(\,\lambda^2\,) \right\} \\ &s = (\bar{k}_1 + \bar{k}_2)^2 \qquad t = (\bar{k}_2 + \bar{k}_3)^2 \qquad u = (\bar{k}_3 + \bar{k}_1)^2 \end{aligned}$$

Hence we come to the following most popular renormalization prescriptions, using once more the simplest case of the real scalar self-interacting field theory in $\mathrm{D} = \Re\mathrm{e}\, 2\omega = 4 - \epsilon \;(\,\epsilon > 0\,)$ Euclidean dimensions as a paradigmatic example, within dimensional regularization.

1. BPHZ prescription

$$\widetilde{\Gamma}_R^{(2)}(0) = m^2 \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}k_E^2}\,\widetilde{\Gamma}_R^{(2)}(0) = 1$$
$$\widetilde{\Gamma}_R^{(4)}(0,0,0,0) \;=\; i\lambda\,\mu^{2\epsilon}$$

   where the dressed mass m just corresponds to the physical measured mass of the spin-less scalar particle. This renormalization condition fixes the finite part of the counter-terms to be

$$\mathrm{F}_1 = 1 - \mathbf{C} - \ln\widehat{m}^{\,2} \qquad \mathrm{G}_1 = -\,\mathbf{C} - \ln\widehat{m}^{\,2} \qquad \mathrm{F}_2 = \mathrm{H}_2 = 0\ \ldots$$

2. Subtraction at arbitrary Euclidean momentum scale $\bar{K} = \sqrt{(\,\bar{K}_\mu\,\bar{K}_\mu\,)}$

$$\widetilde{\Gamma}_R^{(2)}(\bar{K}^2) = \bar{K}^2 + m_K^2$$
$$\widetilde{\Gamma}_R^{(4)}(\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4) \;=\; i\lambda_K\,\mu^{2\epsilon}$$

   the latter point being symmetrically chosen so that

$$(\bar{k}_1 + \bar{k}_2)^2 = (\bar{k}_2 + \bar{k}_3)^2 = (\bar{k}_3 + \bar{k}_1)^2 = \bar{K}^2$$

   leaving understood, of course, that any other value for $\bar{k}_\imath \;(\,\imath = 1,2,3,4\,)$ at which $\widetilde{\Gamma}_R^{(2)}$ and $\widetilde{\Gamma}_R^{(4)}$ are normalized is equally well possible. In so doing the finite parts of the counter-terms become

$$\begin{aligned} \mathrm{F}_1 &= \psi(2) - \ln(m_K^2/4\pi\mu^2) \\ \mathrm{G}_1 &= \psi(1) - \ln\frac{m_K^2}{4\pi\mu^2} - \int_0^1 \mathrm{d}x\;\ln[\,1 + x(1-x)\,\bar{K}^2/m_K^2\,] \\ &\ \vdots \end{aligned}$$

   the relevant number in this prescription being clearly the ratio $\bar{K}/m_K$.

3. On the mass shell prescription at the physical subtraction point

$$\widetilde{\Gamma}_R^{(2)}(\bar{k}^2) = 0 \qquad (\,\mathrm{d}/\mathrm{d}\bar{k}^2\,)\,\widetilde{\Gamma}_R^{(2)}(\bar{k}^2) = 1 \qquad \text{for}\quad \bar{k}^2 \;=\; -\,m^2$$
$$\widetilde{\Gamma}_R^{(4)}(\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4) \;=\; i\lambda\,\mu^{2\epsilon} \quad \text{for}\quad \bar{k}_\imath\,\bar{k}_\jmath = -\,m^2\left(\,\delta_{\imath\jmath} - \tfrac{1}{4}\,\right)$$

where m denotes again the physical mass of the scalar particle, while λ coincides with the physical measured self-coupling at the physical threshold $z = -\,4m^2$ where $z = s, t, u$ as usual. The finite parts of the counter-terms can be readily obtained in close correspondence with the Minkowskian formulation of Section 2.1.3.

4. The **Minimal Subtraction Scheme** (MSS) is the renormalization prescription in which one simply sets equal to zero ALL the finite parts of the counter-terms , order by order in perturbation theory: namely,

$$\mathrm{F}_1 = \mathrm{G}_1 = \mathrm{F}_2 = \mathrm{H}_2 = \cdots \equiv 0$$

or equivalently with $\hat{\lambda} \equiv \lambda/16\pi^2$

$$Z_1 = 1 + \frac{3\hat{\lambda}}{2\epsilon} \; + \; O\left(\lambda^2\right) \tag{2.79}$$

$$Z_2 = 1 + \frac{1}{2\epsilon}\left(\hat{\lambda} - \tfrac{1}{2}\,\hat{\lambda}^2\right) + \frac{\hat{\lambda}^2}{2\epsilon^2} \; + \; O\left(\lambda^3\right) \tag{2.80}$$

$$Z_3 = Z_\phi = 1 - \frac{\hat{\lambda}^2}{24\epsilon} \; + \; O\left(\lambda^3\right) \tag{2.81}$$

The main virtue of this renormalization prescription invented by

Gerardus 't Hooft (1973)
Dimensional regularization and the renormalization group
Nuclear Physics vol. **B61**, pp. 455-468

Steven Weinberg (1973)
New Approach to the Renormalization Group
The Physical Review vol. D**8**, pp. 3497-3509

is the mass independence of all the coefficients (2.71), a key feature that opens the possibility to solve the renormalization group equations (2.62). This fact actually entails a great simplification, which allows to calculate the coefficients β, γ_d and γ_m in a straightforward way, order by order in perturbation theory.

As a matter of fact, from (2.71) we can write in the MSS

$$\lambda_0 = \mu^{2\epsilon}\,\lambda\, Z_1(\lambda, \epsilon) = \mu^{2\epsilon}\left[\lambda + \sum_{n=1}^{\infty}\frac{a_n(\lambda)}{\epsilon^n}\right] \tag{2.82}$$

and acting with the derivative operator to $\mu\,(\mathrm{d}/\mathrm{d}\mu)$ at fixed λ_0 we get

$$2\epsilon\left[\lambda+\sum_{n=1}^{\infty}\frac{a_n(\lambda)}{\epsilon^n}\right] \;=\; -\,\beta(\lambda,\epsilon)\left[1+\sum_{n=1}^{\infty}\frac{a_n'(\lambda)}{\epsilon^n}\right] \tag{2.83}$$

Taking into account the analytic behavior for $\epsilon\to 0$ of the renormalization group coefficients, after setting

$$\beta(\lambda,\epsilon)\equiv\beta(\lambda)-2\epsilon\lambda$$

by identifying the residua of the poles in ϵ we eventually find

$$2a_1(\lambda) \;=\; -\,\beta(\lambda)+2\lambda\,a_1'(\lambda) \tag{2.84}$$

$$2a_{n+1}(\lambda) \;=\; -\,a_n'(\lambda)\,\beta(\lambda)+2\lambda\,a_{n+1}'(\lambda) \qquad (\,n\in\mathbb{N}\,) \tag{2.85}$$

or even

$$\begin{aligned}\beta(\lambda)\equiv\lim_{\epsilon\to 0}\,\mu\,\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} &= 2\left(\lambda\,\frac{\mathrm{d}}{\mathrm{d}\lambda}-1\right)a_1(\lambda)\\ &= 2\lambda\,a_1'(\lambda)-2a_1(\lambda)\end{aligned} \tag{2.86}$$

together with the recursive relations

$$a_{n+1}(\lambda)-\lambda\,a_{n+1}'(\lambda)=a_n'(\lambda)\,[\,a_1(\lambda)-\lambda\,a_1'(\lambda)\,] \qquad (\,n\in\mathbb{N}\,) \tag{2.87}$$

which are equivalent to

$$\left(\lambda\,\frac{\mathrm{d}}{\mathrm{d}\lambda}-1\right)a_{n+1}(\lambda)=a_n'(\lambda)\left(\lambda\,\frac{\mathrm{d}}{\mathrm{d}\lambda}-1\right)a_1(\lambda) \qquad (\,n\in\mathbb{N}\,) \tag{2.88}$$

Let us now evaluate the $\beta-$function to the lowest order in perturbation theory: namely,

$$\begin{aligned}\beta(\lambda) &= 2\left(\lambda\,\frac{\mathrm{d}}{\mathrm{d}\lambda}-1\right)\left[\frac{3\lambda^2}{32\pi^2}\;+\;O(\lambda^3)\right]\\ \mu\,\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} &= \frac{3\lambda^2}{16\pi^2}\;+\;O(\lambda^3)\end{aligned} \tag{2.89}$$

that can be readily integrated to give

$$\lambda(\,\mu\,)\approx\lambda(\,\mu_o\,)\cdot\frac{1}{1-\dfrac{3\lambda(\,\mu_o\,)}{16\pi^{\,2}}\,\ln\dfrac{\mu}{\mu_o}} \tag{2.90}$$

which shows that λ is a monotonically increasing function of μ as long as we stay within the range of validity of the lowest order perturbative regime, *i.e.*

$$\frac{3\lambda(\mu_o)}{16\pi^2}\ln\frac{\mu}{\mu_o}\ll 1$$

It is important to remark that, if we take μ_o equal to the physical mass m and the larger scale μ equal to the high momentum magnitude $|\,\mathbf{k}\,|$, then the above equation (2.90) precisely matches the leading behavior (2.52) for the effective running coupling $\lambda_{\text{eff}}(\mathbf{k})$. The deep meaning of this occurrence lies in that the asymptotic behavior of a renormalizable model is universal, *i.e.* it does not depend upon the specifically chosen renormalization prescription, as I shall further discuss later on.

2.2.2 Fixed Points and Effective Couplings

The main application of the renormalization group invariance in quantum field theory is the evaluation of the large or small momentum behaviors of the Green's functions as well as the scale dependent running parameters such as couplings or masses. The behaviors (2.90) and (2.168) correspond to lowest order expressions for small values of the couplings and sufficiently small momentum scales. For instance, if we fix $\mu_o = 1\ \text{eV/c}^2$, which is the characteristic scale of atomic Physics, and $\lambda(\mu_o) = 1$, which is the bound of validity of perturbation thory, then we obtain the benchmark value

$$\mu \le \mathrm{e}^{16\pi^2/3}\ \text{eV/c}^2 \approx 10^{14}\ \text{GeV/c}^2$$

Of course, there is no reason to believe that the above 1-loop expressions for $\beta(\lambda)$ – or $\beta(e)$ – keep valid for very large values of the couplings and very high momentum scales. We do not know how to calculate the $\beta-$function for large couplings, *i.e.* outside perturbation theory, but let us merely **speculate** about some possible scenarios starting from $\beta(0) = 0$, the no interaction point.

1. $\beta(\lambda)$ keeps positive for large λ thus drawing a concave or convex curve depending on the sign of $\beta'(\lambda)$. For example, to the lowest order $\beta(\lambda)$ is a concave parabola. If $\beta(\lambda)$ blows up for some value of λ, then λ itself becomes infinite (Landau-Pomeranchŭck point). As already noticed, the no interaction point $\lambda = 0$ is a trivial or Gaussian fixed point at which $\beta'(0) > 0$. This means that above it $\mu\,\mathrm{d}\lambda/\mathrm{d}\mu$ is positive leading $\lambda(\mu)$ away from it as the distance decreases: such a fixed point is called INFRARED STABLE. For small couplings, *i.e.* within the perturbative

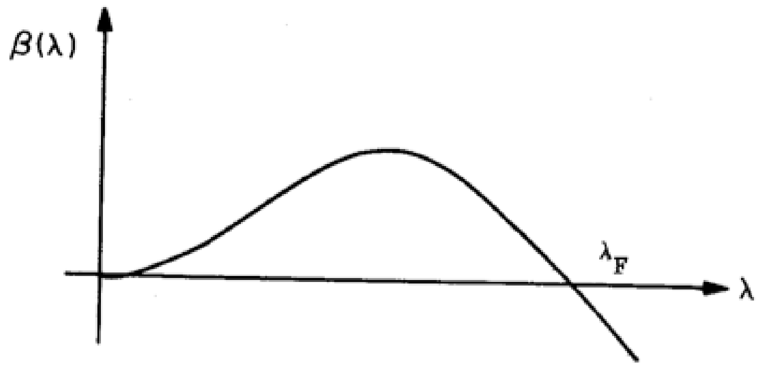

Figure 2.6: A possible behavior of the function $\beta(\lambda)$

regime, most field theories behave this way, with $\beta(\lambda)$ starting out positive.

2. $\beta(\lambda)$ starts out positive for small λ and then turns over and become negative, by crossing the positive $\lambda-$axis at $\lambda = \lambda_{\mathrm{F}}$, which is called a FIXED POINT for, if for some reason the coupling was originally at λ_{F} it woul remain there – see Fig. 2.6. As a matter of fact, we can analyze the behavior of $\lambda(\mu)$ by expanding the $\beta-$function around λ_{F} getting the approximation

$$\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \approx (\lambda - \lambda_{\mathrm{F}})\, \beta'(\lambda_{\mathrm{F}}) \tag{2.91}$$

If $\beta'(\lambda_{\mathrm{F}}) < 0$ as in Fig. 2.6 one readily realizes that $\lambda(\mu)$ will be driven towards λ_{F} as μ increases: such a fixed point is called ULTRAVIOLET STABLE, because

$$\lim_{\mu\to\infty} |\, \lambda(\mu) - \lambda_{\mathrm{F}} \,| = 0$$

from above and from below – see Fig. 2.7. If $\lambda_{\mathrm{F}} < 1$ and had we started from $\lambda(\mu) < \lambda_{\mathrm{F}}$ we never leave the perturbative regime. Alternatively, had we started from $\lambda(\mu) > \lambda_{\mathrm{F}}$ then as the distance decreases we will driven to the perturbative regime. These two occurrences are depicted in Fig. 2.7. No field theory in four space-time dimensions is known to exhibit such a perturbative behavior, but there are speculations that a non-perturbative treatment of Quantum Gravity could perhaps fall into this kind of behavior, a property which is called asymptotic safety

Steven Weinberg (1979)

Ultraviolet Divergences In Quantum Theories Of Gravitation

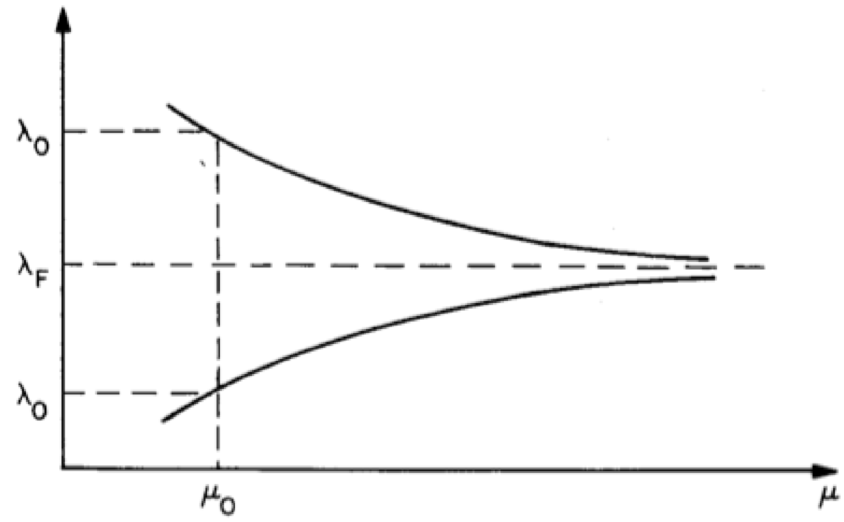

Figure 2.7: The running coupling $\lambda(\mu)$ in the neighborhood of an ultraviolet stable non-trivial non-perturbative fixed point λ_{F}

in General Relativity: An Einstein centenary survey

Eds. S. W. Hawking and W. Israel, Cambridge University Press, p.790

3. $\beta(\lambda)$ starts out negative for small values of $\lambda(\,\mu)$ that appears to be a monotonically decreasing function of μ. In such a circumstance the perturbative approximation is getting better and better as the distance decreases, in such a manner that $\lim_{\mu\to\infty}\lambda(\,\mu) = 0$ so that the no interaction point $\lambda = 0$ turns out to be an ultraviolet stable fixed point. This kind of perturbative behavior for small coupling is exhibited by the non-Abelian gauge theories, a phenomenon that is known as asymptotic freedom.

4. $\beta(\lambda)$ starts out negative from the no interaction point, then turns over and becomes positive crossing the $\lambda-$axis at λ_{F} , In such a case $\beta\,'(\lambda_{\mathrm{F}}) > 0$ so that λ_{F} is an infrared fixed point. This means that if at μ_0 one has $\lambda_0 < \lambda_{\mathrm{F}}$ then $\lim_{\mu\to\infty}\lambda(\,\mu) = 0$, but if $\lambda_0 > \lambda_{\mathrm{F}}$ it will be driven away from λ_{F} towards larger and larger values.

2.2.3 Asymptotic Behavior

It is not difficult to gather that one can repeat the very same steps which have driven to the perturbative evaluation of the $\beta-$function in the MSS

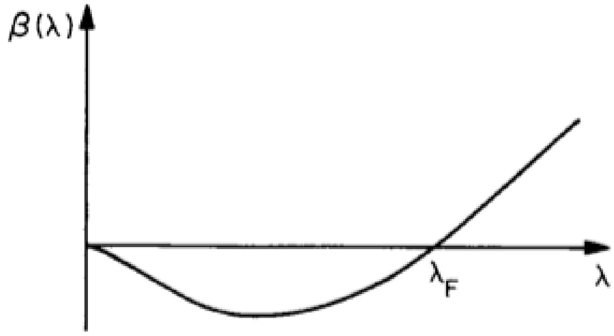

Figure 2.8: Another possible behavior of the function $\beta(\lambda)$

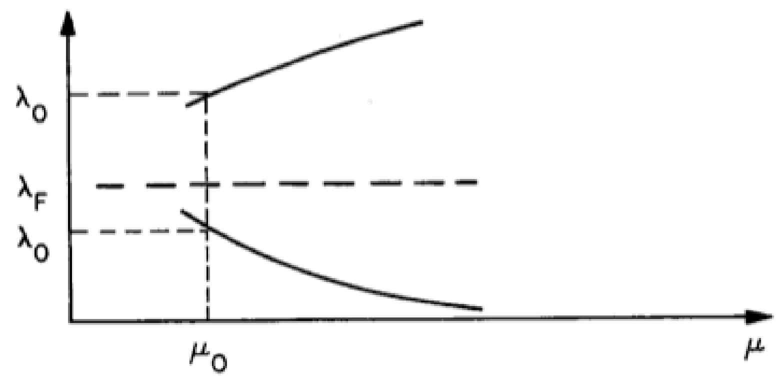

Figure 2.9: The running coupling $\lambda(\mu)$ in the neighborhood of an infrared non-trivial non-Gaussian fixed point λ_F

to obtain the remaining renormalization group coefficients γ_d and γ_m in a straightforward way. As a matter of fact, from the very definitions

$$
\begin{aligned}
\phi_0(x) &= \phi(x)\,\sqrt{Z_3(\lambda,\epsilon)} \\
Z_3(\lambda,\epsilon) &= 1 + \sum_{n=1}^{\infty} c_n(\lambda)\,\epsilon^{-n}
\end{aligned}
\tag{2.92}
$$

together with

$$
\gamma_d(\lambda,\epsilon) \equiv \gamma_d(\lambda) \qquad\qquad \beta(\lambda,\epsilon) \equiv \beta(\lambda) - 2\epsilon\lambda
$$

we readily obtain

$$
\begin{aligned}
Z_3(\lambda,\epsilon)\,\gamma_d(\lambda) \;&=\; \tfrac{1}{2}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,Z_3(\lambda,\epsilon) \\
&\Updownarrow \\
\gamma_d(\lambda) \;&+\; \sum_{n=1}^{\infty} \gamma_d(\lambda)\,c_n(\lambda)\,\frac{1}{\epsilon^n} \;= \\
-\,\lambda\,c_1'(\lambda) \;&+\; \sum_{n=1}^{\infty} \left[\,\tfrac{1}{2}\,\beta(\lambda)\,c_n'(\lambda) - \lambda\,c_{n+1}'(\lambda)\,\right]\,\frac{1}{\epsilon^n}
\end{aligned}
$$

that yields

$$
\gamma_d(\lambda) \;=\; -\,\lambda\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\,c_1(\lambda) \tag{2.93}
$$

$$
\left[\,\tfrac{1}{2}\,\beta(\lambda)\,\frac{\mathrm{d}}{\mathrm{d}\lambda} - \gamma_d(\lambda)\right] c_n(\lambda) \;=\; \lambda\,c_{n+1}'(\lambda) \qquad\qquad (\,n \in \mathbb{N}\,) \tag{2.94}
$$

and from the $\beta-$function defining equation (2.86) we derive the recursive relations

$$
c_{n+1}'(\lambda) = c_n(\lambda)\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\,c_1(\lambda) + c_n'(\lambda)\,\left(\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{1}{\lambda}\right)\,a_1(\lambda) \qquad (\,n \in \mathbb{N}\,) \tag{2.95}
$$

Note that to the lowest order we find

$$
c_1(\lambda) \;=\; -\,\frac{1}{4!}\left(\frac{\lambda}{16\pi^2}\right)^2 +\; O(\lambda^3) \tag{2.96}
$$

$$
\gamma_d(\lambda) \;=\; \frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 +\; O\left(\lambda^3\right) \tag{2.97}
$$

Finally, from the MSS mass counter-term

$$
\begin{aligned}
m_0^2 &= \frac{Z_2}{Z_3}\, m^2 = m^2 \left[1 + \sum_{n=1}^{\infty} b_n(\lambda)\, \epsilon^{-n} \right] \\
b_1(\lambda) &= \frac{\lambda}{32\pi^2} - \frac{5}{24}\left(\frac{\lambda}{16\pi^2}\right)^2 + \cdots \\
b_2(\lambda) &= \frac{1}{2}\left(\frac{\lambda}{16\pi^2}\right)^2 + \cdots
\end{aligned}
\tag{2.98}
$$

the application of the derivative operator $\mu\,\mathrm{d}/\mathrm{d}\mu$ at fixed bare mass entails

$$
\left[1 + \sum_{n=1}^{\infty} b_n(\lambda)\, \frac{1}{\epsilon^n} \right] \mu\, \frac{\mathrm{d}}{\mathrm{d}\mu}\, m^2(\lambda) = m^2(\lambda) \sum_{n=1}^{\infty} b_n'(\lambda)\, [\, 2\epsilon\lambda - \beta(\lambda)\,]\, \frac{1}{\epsilon^n}
$$

and consequently from the MSS definition

$$
\gamma_m(\lambda) \equiv \tfrac{1}{2}\, \mu\, \frac{\mathrm{d}}{\mathrm{d}\mu}\, \ln m^2(\lambda) \tag{2.99}
$$

we come to the equality

$$
\begin{aligned}
\gamma_m(\lambda) \;&+\; \sum_{n=1}^{\infty} \gamma_m(\lambda)\, b_n(\lambda)\, \frac{1}{\epsilon^n} = \lambda\, \frac{\mathrm{d}}{\mathrm{d}\lambda}\, b_1(\lambda) \\
&+\; \sum_{n=1}^{\infty} \left[\, b_{n+1}'(\lambda)\, \lambda - \tfrac{1}{2}\, \beta(\lambda)\, b_n'(\lambda) \,\right] \frac{1}{\epsilon^n}
\end{aligned}
\tag{2.100}
$$

and thereby

$$
\gamma_m(\lambda) \;=\; \lambda\, \frac{\mathrm{d}}{\mathrm{d}\lambda}\, b_1(\lambda) = \frac{\lambda}{32\pi^2} - \frac{5}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 + O\left(\lambda^3\right) \tag{2.101}
$$

together with the corresponding recursive relations

$$
b_{n+1}'(\lambda) = b_n(\lambda)\, \frac{\mathrm{d}}{\mathrm{d}\lambda}\, b_1(\lambda) + b_n'(\lambda) \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{1}{\lambda} \right) a_1(\lambda) \qquad (\, n \in \mathbb{N}\,) \tag{2.102}
$$

The above listed recursive relations for the renormalization group coefficients are especially useful to calculate, order by order in the dressed coupling $\lambda(\mu)$, the residues of all the higher order poles in $\epsilon = 2-\mathrm{D}/2$ from the knowledge of the residues $a_1(\lambda), b_1(\lambda)$ and $c_1(\lambda)$ at the simple poles. This occurrence is one of most profound meaning of renormalizability, the residues of the higher order poles of a higher loop diagram being indirectly calculable without explicit painful evaluation of a complicated Feynman integral.

It is possible to solve in perturbation theory eq. (2.67) in the minimal subtraction scheme. To this concern let us introduce the running coupling and running mass functions as solutions of the first order ordinary differential equations according to

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\,\lambda(\varrho) \equiv \lambda'(\varrho) = \beta\,(\lambda(\varrho)) \qquad\qquad \lambda(0) = \lambda \tag{2.103}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\,m(\varrho) \equiv m'(\varrho) = [\,\gamma_m\,(\lambda(\varrho)) - 1\,]\,m(\varrho) \qquad\qquad m(0) = m \tag{2.104}$$

Then, according to the classical Monge method[8] to solve partial differential equations, eq. (2.67) reduces to a first order differential equation in ϱ, *viz.*,

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varrho} - n\,(1-\gamma_d) + 4\right)\widetilde{\Gamma}_R^{(n)}(\mathrm{e}^{\,\varrho}\,p\,;\lambda(\varrho),m(\varrho)/\mu\,) = 0 \tag{2.105}$$

that yields

$$\begin{aligned}\widetilde{\Gamma}_R^{(n)}(\mathrm{e}^{\,\varrho}\,p;\lambda,m/\mu\,) \;&=\; \mathrm{e}^{\,(n-4)\varrho}\,\widetilde{\Gamma}_R^{(n)}\,(p\,;\lambda(\varrho),m(\varrho)/\mu\,)\\ &\times\;\exp\{\,-n\textstyle\int_0^\varrho \mathrm{d}\varrho'\,\gamma_d\,(\lambda(\varrho'\,))\end{aligned} \tag{2.106}$$

Now to the lowest order we have

$$\lambda'(\varrho) \approx \frac{3}{16\pi^2}\,\lambda^2(\varrho) \qquad\qquad \lambda(0) = \lambda$$

which can be readily integrated to give once again

$$\lambda(\varrho) \approx \frac{\lambda}{1 - \dfrac{3\lambda\varrho}{16\pi^2}} \qquad\qquad \left[\varrho \ll \frac{16\pi^2}{3\lambda}\right]$$

and thereby

$$\int_0^\varrho \mathrm{d}\varrho'\,\lambda\,(\varrho'\,) \;\approx\; -\,\frac{16\pi^2}{3}\,\ln\left(1 - \frac{3\lambda\varrho}{16\pi^2}\right)$$

$$\frac{5}{12(16\pi^2)^2}\int_0^\varrho \mathrm{d}\varrho'\,\lambda^2(\varrho'\,) \;\approx\; 5\varrho\,\frac{\hat{\lambda}^2}{12}\left(\frac{\lambda\,(\varrho)}{\lambda}\right)$$

On the other side, the lowest order equation for the running mass reads

$$m'(\varrho) \approx \left[\,\frac{\lambda(\varrho)}{32\pi^2} - \frac{5\lambda^2(\varrho)}{12(16\pi^2)^2} - 1\,\right]\,m(\varrho) \qquad\qquad m(0) = m \tag{2.107}$$

[8] Gaspard Monge, Compte de Péluse, Beaune 1746 - Paris 1818, *Application de l'analyse à la géométrie* (1785).

which can also be readily integrated using the above results and yields

$$
\begin{aligned}
m(\varrho) &\approx m \exp\left\{-\varrho + \frac{1}{32\pi^2}\int_0^{\varrho} \mathrm{d}\varrho'\, \lambda(\varrho')\left[1 - \frac{5\lambda(\varrho')}{6(16\pi^2)}\right]\right\} \\
&= m \exp\left\{-\varrho - \tfrac{1}{6}\ln\left(1 - 3\lambda\varrho/16\pi^2\right) - 5\varrho\,\frac{\hat{\lambda}^2}{12}\left(\frac{\lambda(\varrho)}{\lambda}\right)\right\} \\
&= m\left(\frac{\lambda(\varrho)}{\lambda}\right)^{\frac{1}{6}} \exp\left\{-\varrho - 5\varrho\,\frac{\hat{\lambda}^2}{12}\left(\frac{\lambda(\varrho)}{\lambda}\right)\right\}
\end{aligned}
\tag{2.108}
$$

Finally we have

$$
\int_0^{\varrho} \mathrm{d}\varrho'\, \gamma_d\left(\lambda(\varrho')\right) \approx \varrho\,\frac{\hat{\lambda}^2}{12}\left(\frac{\lambda(\varrho)}{\lambda}\right)
$$

that eventually leads to the leading asymptotic behavior of the renormalized proper vertexes to the lowest order in the dressed coupling constant λ *viz.*,

$$
\begin{aligned}
&\widetilde{\Gamma}_R^{(n)}(\mathrm{e}^{\varrho}p; \lambda, m/\mu) \sim \widetilde{\Gamma}_R^{(n)}(p; \lambda(\varrho), m(\varrho)/\mu) \\
&\times \exp\left\{-n\varrho\,\frac{\hat{\lambda}^2}{12}\left(\frac{\lambda(\varrho)}{\lambda}\right) + \varrho(n-4)\right\} \\
&\text{for} \quad 0 \le 3\lambda\varrho \lesssim 16\pi^2
\end{aligned}
\tag{2.109}
$$

2.3 1-Loop Renormalization of QED

Consider the primitively divergent graphs at one loop in QED. If we regulate the theory using dimensional regularization, then from [22] §**4.2.2** we get photon self-energy or vacuum polarization

$$
\begin{aligned}
\mathtt{reg}\,\Pi^{\mu\nu}(k, m_e, \mu) &= \left(k^2 g^{\mu\nu} - k^{\mu}k^{\nu}\right)\mathtt{reg}\,\Pi(k^2, m_e^2) \\
\mathtt{reg}\,\Pi(k^2, m_e^2) &\doteq \frac{\alpha}{3\pi}\left\{-\frac{1}{\epsilon} + \mathbf{C} - \ln\frac{2\pi\mu^2}{m_e^2}\right\} + \widehat{\Pi}(k^2, m_e^2) \\
\widehat{\Pi}(k^2, m_e^2) &= \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x\; 2x(1-x)\,\ln\left[1 - x(1-x)\,k^2/m_e^2\right]
\end{aligned}
$$

where we have definitely identified the fermion mass parameter with the physical electron mass $m_e = 0.510998910 \pm 0.000000013$ MeV, while the fine structure constant $\alpha = e^2/4\pi\hbar c$ has the value $1/137.035999679(94)$. In order to remove the ultraviolet divergence from the above expression for $2 - \omega = \epsilon \to 0$ a 1-loop counter-term is needed of the form

$$
-\,(Z_3 - 1)\,\tfrac{1}{4}\,\mathrm{F}^{\mu\nu}\,\mathrm{F}_{\mu\nu} \tag{2.110}
$$

where

$$(Z_3 - 1) = \frac{-\alpha}{3\pi}\left\{\frac{1}{\epsilon} + \mathrm{F}_3\left(\epsilon, \frac{2\pi\mu^2}{m_e^2}\right)\right\} + O(\alpha^2) \qquad (2.111)$$

whereas F_3 is arbitrary and analytic for $\epsilon \to 0$. The renormalized invariant polarization function, which is finite in four space-time dimensions, is then provided by

$$\Pi_R(k^2, m_e^2) = \frac{\alpha}{3\pi}\left[\mathbf{C} - \ln\frac{2\pi\mu^2}{m_e^2} - \mathrm{F}_3\left(\frac{2\pi\mu^2}{m_e^2}\right)\right] + \widehat{\Pi}(k^2, m_e^2) \qquad (2.112)$$

To fix the arbitrary part we use the so called physical, on the mass shell, or BPHZ renormalization prescription: namely,

$$\Pi_R(0, m_e^2) = 0 \quad \Longleftrightarrow \quad \mathrm{F}_3^{\mathrm{BPHZ}} = \ln\frac{m_e^2}{2\pi\mu^2} + \mathbf{C} \qquad (2.113)$$

$$Z_3 = 1 - \frac{\alpha}{3\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{2\pi\mu^2}{m_e^2}\right\} + O(\alpha^2) \qquad (2.114)$$

Turning to the electron self-energy, from the expression (1.103) we have

$$\mathtt{reg}\,\widetilde{\Gamma}(\not p)\Big|_{1-\mathrm{loop}} = \not p - m_e - \mathtt{reg}\,\Sigma_2(\not p) \qquad (2.115)$$

$$\mathtt{reg}\,\Sigma_2(\not p) = A(p, m_e\,; \mu, \epsilon)\, m_e - B(p, m_e\,; \mu, \epsilon)\,\not p \qquad (2.116)$$

where the following 1-loop expressions (1.116) and (1.117) hold true: namely,

$$A \doteq \frac{\alpha}{\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \frac{3}{2} + \ln\frac{4\pi\mu^2}{m_e^2} - \left(1 - \frac{m_e^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m_e^2}\right)\right\}$$

$$B \doteq \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + 1 + \ln\frac{4\pi\mu^2}{m_e^2} + \frac{m_e^2}{p^2} - \left[1 - \left(\frac{m_e^2}{p^2}\right)^2\right]\ln\left(1 - \frac{p^2}{m_e^2}\right)\right\}$$

Thus, in order to remove the divergence at one loop in the Dirac mass term, we add the electron mass counter-term

$$-\,\delta m\,\bar\psi\,\psi \qquad (2.117)$$

with

$$\frac{\delta m}{m_e} \;=\; -\,\frac{\alpha}{\pi}\left\{\frac{1}{\epsilon} + \mathrm{F}_{\mathrm{m}}\left(\epsilon, \frac{4\pi\mu^2}{m_e^2}\right)\right\} + O(\alpha^2) \qquad (2.118)$$

the arbitrary mass counter-term function F_{m} being analytic for $\epsilon \to 0$, as well as the electron wave function counter-term

$$(Z_2 - 1)\,\bar\psi\, i\,\not\partial\,\psi \qquad (2.119)$$

in which

$$Z_2 = 1 - \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} + \mathrm{F}_2\left(\epsilon, \frac{4\pi\mu^2}{m_e^2}\right)\right\} + O(\alpha^2) \tag{2.120}$$

where the electron wave function 1-loop counter-term F_2 is analytic for $\epsilon \to 0$, as usual. In order to fix those 1-loop counter-terms by means of the BPHZ renormalization prescription, we have to impose the so called on-mass-shell subtraction, which corresponds to the following **symbolic or formal limits**

$$\lim_{\not p \to m_e} \widetilde{\Gamma}_R(\not p) = 0 \quad \Longleftrightarrow \quad \lim_{\not p \to m_e} \Sigma_R(\not p) = 0 \tag{2.121}$$

$$\lim_{\not p \to m_e} \frac{\partial \widetilde{\Gamma}_R(\not p)}{\partial \not p} = 1 \quad \Longleftrightarrow \quad \lim_{\not p \to m_e} \frac{\partial \Sigma_R(\not p)}{\partial \not p} = 0 \tag{2.122}$$

where we understand the renormalized UV finite relationship $\Gamma_R(\not p) = \not p - m_e - \Sigma_R(\not p)$, whereas the derivative with respect to a matrix variable $\not p$ has to be understood as follows: namely,

$$\frac{\partial}{\partial \not p}\,\not p = \tfrac{1}{4}\,\gamma^{\mu}\,\frac{\partial}{\partial p^{\mu}}\,\gamma^{\nu} p_{\nu} = \tfrac{1}{4}\,\gamma^{\mu}\,\gamma^{\nu}\,g_{\mu\nu} = \tfrac{1}{8}\,\{\gamma^{\mu}, \gamma^{\nu}\}\,g_{\mu\nu} = \mathbb{I} \tag{2.123}$$

As a matter of fact we have for $\varkappa = p^2/m_e^2$

$$\begin{aligned} m_e\,\frac{\partial}{\partial \not p}\,A(\varkappa) &= \tfrac{1}{4}\,m_e\,\gamma^{\mu}\,\frac{\partial}{\partial p^{\mu}}\,A(\varkappa) = \frac{\not p}{2m_e}\,A'(\varkappa) \\ \frac{\partial}{\partial \not p}\,B &= \tfrac{1}{4}\,\gamma^{\mu}\,\frac{\partial}{\partial p^{\mu}}\,B(\varkappa) = \frac{\not p}{2m_e}\,B'(\varkappa) \end{aligned}$$

and thereby

$$\frac{\partial}{\partial \not p}\,\Sigma_2(\not p) = \frac{\not p}{2m_e}\,A'(\varkappa) - B(\varkappa) - \frac{\varkappa}{2}\,B'(\varkappa) \tag{2.124}$$

in which we have taken into account that the momentum dependence of the Poincaré invariant dimensionless coefficient functions A and B must be on the variable $\varkappa = p^2/m_e^2$ by dimensional reasons. Hence, after subtraction of the divergent part, we get the $O(\alpha)$ renormalized UV finite 1-loop coefficients

$$A_1 = \frac{\alpha}{\pi}\left\{\psi(2) + \ln\frac{4\pi\mu^2}{m_e^2} + \frac{1}{2} + \frac{1}{\varkappa}\,(1-\varkappa)\ln(1-\varkappa) - \mathrm{F_m}\right\}$$

$$B_1 = \frac{\alpha}{4\pi}\left\{\psi(2) + \ln\frac{4\pi\mu^2}{m_e^2} + \frac{1}{\varkappa} - \left(1 - \frac{1}{\varkappa^2}\right)\ln(1-\varkappa) - \mathrm{F}_2\right\}$$

that yield the 1-loop relationship

$$\widetilde{\Gamma}_{1\ell} = \not{p} - m_e - \widehat{\Sigma}_2(\not{p}) = \not{p} - m_e + \not{p}\, B_1(\varkappa) - A_1(\varkappa)\, m_e$$

together with

$$A_1'(\varkappa) = \frac{-\,\alpha}{\pi\varkappa}\left[1 + \frac{1}{\varkappa}\,\ln(1-\varkappa)\right] \tag{2.125}$$

$$B_1'(\varkappa) = \frac{-\,\alpha}{4\pi\varkappa}\left\{1 + \frac{2}{\varkappa}\left[1 + \frac{1}{\varkappa}\,\ln(1-\varkappa)\right]\right\} \tag{2.126}$$

Notice that the derivatives $A_R'(\varkappa)$ and $B_R'(\varkappa)$ do not depend upon the finite parts of the 1-loop counter-terms, but turn out to be infrared singular on the mass shell $\varkappa = 1$. Then it appears to be manifest that the on-mass-shell subtraction is not allowed unless a suitable INFRARED REGULATOR is provided. For example one could introduce a pretty small mass parameter $\mu_\gamma \lll m_e$ which can be eventually traded for a tiny photon mass [9]

$$\ln\left(1 - \frac{p^2}{m_e^2}\right) \qquad \longmapsto \qquad \ln\left(1 - \frac{p^2 - \mu_\gamma^2}{m_e^2}\right) \stackrel{\varkappa\downarrow 1}{\longrightarrow} \ln\frac{\mu_\gamma^2}{m_e^2}$$

in such a manner that we can definitely obtain

$$A_1'(1) = \frac{-\,\alpha}{\pi\varkappa}\left(1 + 2\ln\frac{\mu_\gamma}{m_e}\right) \simeq \frac{2\alpha}{\pi}\,\ln\frac{m_e}{\mu_\gamma} \tag{2.127}$$

$$B_1'(1) = \frac{-\,\alpha}{4\pi\varkappa}\left(3 + 4\ln\frac{\mu_\gamma}{m_e}\right) \simeq \frac{\alpha}{\pi}\,\ln\frac{m_e}{\mu_\gamma} \tag{2.128}$$

The BPHZ conditions (2.121) and (2.122) at 1-loop yield

$$A_1(1) = B_1(1) = \tfrac{1}{2}[\,A_1'(1) - B_1'(1)\,] \tag{2.129}$$

Then we can set up to the 1-loop approximation

$$\mathrm{F}_2 = 2 - \mathbf{C} + \ln\frac{4\pi\mu^2}{m_e^2} + 2\ln\frac{\mu_\gamma}{m_e} \tag{2.130}$$

$$\mathrm{F_m} = \frac{3}{2} - \mathbf{C} + \ln\frac{4\pi\mu^2}{m_e^2} + \frac{1}{2}\,\ln\frac{\mu_\gamma}{m_e} \tag{2.131}$$

Finally, the vertex counter-term and its renormalization constant become

$$(Z_1 - 1)\, e\,\bar{\psi}\,\not{A}\,\psi$$

[9] We recall that the present day experimental upper bound on the photon mass is $\mu_\gamma < 1 \times 10^{-18}$ eV.

$$Z_1 = 1 - \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} + \mathrm{G}_1\left(\epsilon, \frac{4\pi\mu^2}{m_e^2}\right)\right\} + O(\alpha^2)$$

where F_1 is analytic for $\epsilon \to 0$. The BPHZ renormalization prescription that fixes the otherwise arbitrary 1-loop counter-term function F_1 is provided by

$$\lim_{k \to 0} \Gamma_R^{\mu}(k,p,q) \doteqdot ie\,\gamma^{\mu} \qquad \left[p^2 = q^2 = m_e^2 \vee (q-p)^2 = k^2 \to 0\right] \quad (2.132)$$

Here $\doteqdot$ means that the 3-point proper vertex $e^- e^+ \gamma$ is understood to be averaged between spin states with the fermion momenta on the physical mass shell and in the so called Thomson limit of vanishing photon momentum, *i.e.* the deep infrared limit. Hence, if we recall the ultraviolet and infrared regularized expression (1.128) of the form factor F_1 we obtain

$$\lim_{k\to 0} \Gamma_R^{\mu}(p,q) \doteqdot ie\gamma^{\mu} \quad \Longleftrightarrow \quad F_1(0) = 1 \qquad (2.133)$$

$$\mathrm{G}_1 = 4\ln\varepsilon_{\mathrm{IR}} + 6 - \mathbf{C} + \ln\frac{4\pi\mu^2}{m_e^2} \qquad (2.134)$$

A comparison between the renormalization constants $Z_1 = Z_2$ in the BPHZ subtraction scheme shows that we can identify the renormalization constant, up to the 1-loop approximation, iff the infrared regulators μ_γ and $\varepsilon_{\mathrm{IR}}$ are related by

$$e\,\varepsilon_{\mathrm{IR}} = \sqrt{\frac{\mu_\gamma}{m_e}} \qquad (2.135)$$

In such a circumstance we get

$$Z_1 = 1 - \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + 2 + \ln\frac{4\pi\mu^2}{m_e^2} + 2\ln\frac{\mu_\gamma}{m_e}\right\} + O(\alpha^2) = Z_2 \qquad (2.136)$$

In conclusion we can write the renormalized Lagrangian of spinor quantum electrodynamics in $D = 2\omega$ space-time dimensions and in the Feynman gauge according to

$$\begin{aligned}\mathcal{L}_R \;&=\; -\tfrac{1}{4}\,\mathrm{F}^{\mu\nu}\,\mathrm{F}_{\mu\nu} + \bar{\psi}\,(\,i\,\partial\!\!\!/ - m_e + e\,A\!\!\!/\,)\,\psi + A^{\mu}\,\partial_{\mu}B + \tfrac{1}{2}\,B^2 \\ &-\; (Z_3 - 1)\,\tfrac{1}{4}\,\mathrm{F}^{\mu\nu}\,\mathrm{F}_{\mu\nu} + (Z_2 - 1)\,\bar{\psi}\,i\,\partial\!\!\!/\,\psi - \delta\,m\,\bar{\psi}\,\psi + (Z_1 - 1)\,e\,\bar{\psi}\,A\!\!\!/\,\psi\end{aligned}$$

where the renormalization constant in the physical on the mass shell BPHZ renormalization prescription read, up to the one loop approximation,

$$Z_1 \;=\; Z_2 \;\approx\; 1 - \frac{\alpha}{4\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + 2 + \ln\frac{4\pi\mu^2}{m_e^2} + \ln\frac{\mu_\gamma^2}{m_e^2}\right\} \qquad (2.137)$$

$$Z_3 \;\approx\; 1 - \frac{\alpha}{3\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{2\pi\mu^2}{m_e^2}\right\} \qquad (2.138)$$

$$\frac{\delta m}{m_e} \;\approx\; -\frac{\alpha}{\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \frac{3}{2} + \ln\frac{4\pi\mu^2}{m_e^2} + \ln\frac{\mu_\gamma^2}{m_e^2}\right\} \qquad (2.139)$$

It is worthwhile to remark that the BPHZ prescritions (2.121) and (2.122) are formal and symbolic conditions, the meaning of which is nothing but the order-by-order fixing of the finite part of the counterterms. Notice that the gauge fixing term does not renormalize, which means that the longitudinal part of the photon propagator is not affected by radiative corrections, which endorses in turn that gauge invariance is preserved by the renormalization procedure for QED order-by-order in perturbation theory.

2.3.1 The Ward's Identities

The renormalized Lagrangian of quantum electrodynamics is still based upon the minimal coupling *Ansatz*, which enforces gauge invariance. In the D dimensional Minkowski space $\mathcal{M}_{\mathrm{D}}$ with $\mathrm{D} = \Re\mathrm{e}\, 2\omega = 4 - 2\epsilon \quad (\epsilon > 0)$ and in terms of the bare and dressed fields, masses, and couplings it can be written in different suggestive forms: namely,

$$
\begin{aligned}
\mathcal{L}_R^{(\mathrm{D})} &= \mathcal{L}_{\mathrm{inv}} + \mathcal{L}_{\mathrm{c.t.}} + \mathcal{L}_{\mathrm{g.f.}} \\
&= -\tfrac{1}{4}\, g_{\mu\rho}\, g_{\nu\sigma}\, \mathrm{F}_0^{\rho\sigma}\, \mathrm{F}_0^{\mu\nu} + \bar{\psi}_0 \,(i\, \partial\!\!\!/ + e_0\, A\!\!\!/_0)\, \psi_0 - m_0 \bar{\psi}_0\, \psi_0 + \mathcal{L}_{\mathrm{g.f.}} \\
&= -\tfrac{1}{4}\, \mathrm{F}^{\mu\nu}\, \mathrm{F}_{\mu\nu} + \bar{\psi}\, (i\, \partial\!\!\!/ - m_e + e\, \mu^{\epsilon}\, A\!\!\!/)\, \psi + (Z_1 - 1)\, e\, \mu^{\epsilon}\, \bar{\psi}\, A\!\!\!/ \psi \\
&+ \ (Z_2 - 1)\, \bar{\psi}\, i\, \partial\!\!\!/ \psi - (Z_3 - 1)\, \tfrac{1}{4}\, \mathrm{F}^{\mu\nu}\, \mathrm{F}_{\mu\nu} - \delta m\, \bar{\psi}\psi + \mathcal{L}_{\mathrm{g.f.}} \\
\mathcal{L}_{\mathrm{inv}} &= -\tfrac{1}{4}\, \mathrm{F}^{\mu\nu}\, \mathrm{F}_{\mu\nu} + \bar{\psi}\, (i\, \partial\!\!\!/ - m_e + \mu^{\epsilon}\, e\, A\!\!\!/)\, \psi \\
\mathcal{L}_{\mathrm{c.t.}} &- \ \tfrac{1}{4}\, \delta Z_3\, \mathrm{F}^{\mu\nu}\, \mathrm{F}_{\mu\nu} + \delta Z_2\, \bar{\psi}\, i\, \partial\!\!\!/ \psi - \delta m\, \bar{\psi}\psi + \delta e\, \bar{\psi}\, A\!\!\!/ \psi \\
\mathcal{L}_{\mathrm{g.f.}} &= A^{\mu}\, \partial_{\mu} B + \tfrac{1}{2}\, \xi\, B^2 = A_0^{\mu}\, \partial_{\mu} B_0 + \tfrac{1}{2}\, \xi_0\, B_0^2
\end{aligned}
$$

the relations between bare and dressed quantities being as follows

$$
\begin{gathered}
A_0^{\mu}(x) = \sqrt{Z_3}\, A^{\mu}(x) \qquad \psi_0(x) = \sqrt{Z_2}\, \psi(x) \qquad \sqrt{Z_3}\, B_0(x) = B(x) \\
e_0 = \frac{Z_1\, \mu^{\epsilon}}{Z_2 \sqrt{Z_3}}\, e \qquad m_0 = (m_e + \delta m)/Z_2 \qquad \xi_0 = Z_3\, \xi \\
\delta Z_2 = (Z_2 - 1) \qquad \delta Z_3 = (Z_3 - 1) \qquad \delta e = e\, \mu^{\epsilon}\, (Z_1 - 1)
\end{gathered}
$$

where the canonical dimensions for both the bare and renormalized fields in D space-time dimensions and in natural units is provided by

$$
[\, A_0^{\mu}\,] = [\, A^{\mu}\,] = \mathrm{eV}^{\,\frac{\mathrm{D}}{2} - 1} \qquad\qquad [\, \psi_0\,] = [\, \psi\,] = \mathrm{eV}^{\,\frac{\mathrm{D}}{2} - \frac{1}{2}} \tag{2.140}
$$

Here the BPHZ on the mass shell renormalization scheme or prescription (2.72) is understood, in which m_e and $-\, e = -\sqrt{4\pi\alpha\hbar c}$ are respectively the physical mass and charge of the electron, as available from the Particle Data Group tables. Notice that the gauge fixing Lagrangian $\mathcal{L}_{\mathrm{g.f.}}$ does not renormalize. It turns out that the renormalized Action, up to the gauge fixing,

$$S_R = \int \mathrm{d}^{\mathrm{D}}x \left[-\tfrac{1}{4}\, g_{\mu\rho}\, g_{\nu\sigma}\, \mathrm{F}_0^{\rho\sigma}\, \mathrm{F}_0^{\mu\nu} + \bar{\psi}_0 \,(i\,\partial\!\!\!/ + e_0\, A\!\!\!/_0)\, \psi_0 - m_0 \bar{\psi}_0\, \psi_0 \right] \quad (2.141)$$

is invariant under the gauge transformations of the bare fields

$$\begin{aligned} A_0^{\mu}(x) &\quad\longmapsto\quad A_0^{\mu}(x) + \partial^{\mu} f_0(x) && (2.142)\\ \psi_0(x) &\quad\longmapsto\quad \exp\{ie_0 f_0(x)\}\ \psi_0(x) && (2.143)\end{aligned}$$

and of the renormalized fields

$$\begin{aligned} A_{\mu}(x) &\quad\longmapsto\quad A_{\mu}(x) + \partial_{\mu} f(x) && (2.144)\\ \psi(x) &\quad\longmapsto\quad \exp\{ie\mu^{\epsilon} f(x) Z_1/Z_2\}\ \psi(x) && (2.145)\\ f_0(x) &\ =\ \sqrt{Z_3}\, f(x) && (2.146)\end{aligned}$$

Consider now the generating functional for the disconnected renormalized Green's functions of quantum electrodynamics

$$\begin{aligned} Z_R[\,J, \bar{\zeta}, \zeta\,] \ &=\ \mathcal{N} \int \mathfrak{D}\, A \int \mathfrak{D}\, B \int \mathfrak{D}\, \psi \int \mathfrak{D}\, \bar{\psi} \\ &\qquad \exp\left\{ i \int \mathrm{d}^{\mathrm{D}}x \left[\mathcal{L}_R^{(\mathrm{D})} + A^{\nu}\, J_{\nu} + \bar{\zeta}\, \psi + \bar{\psi}\, \zeta \right] \right\} && (2.147)\end{aligned}$$

$$\int \mathfrak{D}\, A \ \equiv\ \prod_{x \in \mathcal{M}_{\mathrm{D}}} \int_{-\infty}^{\infty} \mathrm{d}A_x^0 \int \mathrm{d}\mathbf{A}_x$$

where the sources term can be written also in term of the bare sources

$$\begin{aligned} & A^{\mu}(x)\, J_{\mu}(x) + \bar{\zeta}(x)\, \psi(x) + \bar{\psi}(x)\, \zeta(x) \\ =\ & g_{\mu\nu}\, A_0^{\mu}(x)\, J_0^{\nu}(x) + \bar{\zeta}_0(x)\, \psi_0(x) + \bar{\psi}_0(x)\, \zeta_0(x) && (2.148)\\ & J^{\nu}(x) = \sqrt{Z_3}\, J_0^{\nu}(x) \qquad\qquad \zeta(x) = \sqrt{Z_2}\, \zeta_0(x) && (2.149)\end{aligned}$$

As it does not play any role, I have omitted the source term for the auxiliary ghost scalar field, so that one can readily perform the Gaussian functional integration with respect to $B(x)$ and turn the gauge fixing Lagrangian into

$$\mathcal{L}_{\mathrm{g.f.}} = -\,\frac{1}{2\xi}\, \partial_{\mu}\, A^{\mu}(x)\, \partial_{\nu}\, A^{\nu}(x) \qquad (2.150)$$

in such a manner that generating functional can be recast in the form

$$\begin{aligned} & Z_R[\,J, \bar{\zeta}, \zeta\,] = \mathcal{N}' \int \mathfrak{D}\, A \int \mathfrak{D}\, \psi \int \mathfrak{D}\, \bar{\psi}\ \mathrm{e}^{\,i\, S_{\mathrm{eff}}} \\ & S_{\mathrm{eff}} = \int \mathrm{d}^{\mathrm{D}}x \left[\mathcal{L}_{\mathrm{inv}} + \mathcal{L}_{\mathrm{c.t.}} + \mathcal{L}_{\mathrm{g.f.}} + A^{\nu}\, J_{\nu} + \bar{\zeta}\, \psi + \bar{\psi}\, \zeta \right] && (2.151)\end{aligned}$$

In the functional integral one can perform the change of variables on the renormalized field functions, real and Graßmann valued,

$$
\begin{aligned}
A'_\mu(x) &= A_\mu(x) + \partial_\mu f(x) && (2.152)\\
\psi'(x) &= \exp\{ie\mu^\epsilon f(x) Z_1/Z_2\}\ \psi(x) && (2.153)\\
& && (2.154)
\end{aligned}
$$

where $f(x)$ is an arbitrary real function on the D dimensional Minkowski space-time $\mathcal{M}_\mathrm{D}$. Notice that the change of variable on the vector potential is a translation while the Graßmann valued spinor field is rescaled by a unit phase factor. Hence the functional measure

$$
\mathcal{N} \prod_{x\in\mathcal{M}_\mathrm{D}} \int \mathrm{d}\bar{\psi}_x \int \mathrm{d}\psi_x \prod_{\mu=0}^{3} \int_{-\infty}^{\infty} \mathrm{d}A_x^\mu \qquad (2.155)
$$

is manifestly invariant. Owing to the gauge invariance of the renormalized Action (2.141) only the sources and gauge fixing terms are affected by the above change of functional variables. On the one hand, up to the first order in the gauge function $f(x)$, the variation in the exponent of the integrating functional takes the form

$$
\begin{aligned}
\Delta_f &= \int \mathrm{d}^\mathrm{D}x \,\Big\{ \xi^{-1}\,\partial_\mu A^\mu(x)\,\Box f(x) - J^\mu(x)\,\partial_\mu f(x)\\
&- \ ie\mu^\epsilon f(x)\,\frac{Z_1}{Z_2}\,\big[\,\bar{\zeta}(x)\,\psi(x) \,-\bar{\psi}(x)\,\zeta(x)\,\big]\Big\}
\end{aligned}
$$

On the other hand a change of integration variables cannot change the value of an integral. Then we obtain that

$$
\delta Z_R[\,J,\bar{\zeta},\zeta\,]/\delta f(x)\big]_{f=0} \,=\, 0 \qquad (2.156)
$$

and from the first order variation, up to an irrelevant boundary term which vanishes at infinity,

$$
\begin{aligned}
\Delta_f\, Z_R[\,J,\bar{\zeta},\zeta\,] &= \mathcal{N}' \int \mathfrak{D}A \int \mathfrak{D}\psi \int \mathfrak{D}\bar{\psi}\ \mathrm{e}^{i\,S_\mathrm{eff}}\\
\times \ &\int \mathrm{d}^\mathrm{D}x \,\Big\{\partial_\mu J^\mu(x) + \xi^{-1}\,\partial_\mu A^\mu(x)\,\Box\\
- \ &ie\mu^\epsilon\,\frac{Z_1}{Z_2}\,\big[\,\bar{\zeta}(x)\,\psi(x) \,-\bar{\psi}(x)\,\zeta(x)\,\big]\Big\}\, f(x) \qquad (2.157)
\end{aligned}
$$

we come to the Ward-Takahashi identity [10] for the generating functional of the renormalized Green's functions

$$\begin{aligned} 0 &= \partial_\mu J^\mu(x)\, Z_R[\, J, \bar{\zeta}, \zeta\,] - \frac{i}{\xi}\, \partial^\mu \Box\, [\, \delta Z_R/\delta J^\mu(x)\,] \\ &- e\mu^\epsilon\, \frac{Z_1}{Z_2}\, \big\{ \bar{\zeta}(x)\, [\, \delta Z_R/\delta \bar{\zeta}(x)\,] + [\, \delta Z_R/\delta \zeta(x)\,]\, \zeta(x) \big\} \end{aligned} \tag{2.158}$$

Turning to the generating functional for the renormalized connected Green's functions $W_R = -i \ln Z_R$ one gets

$$\begin{aligned} 0 &= \partial_\mu J^\mu_x - \xi^{-1}\, \partial^\mu \Box\, \frac{\delta W_R}{\delta J^\mu_x} \\ &- ie\mu^\epsilon\, \frac{Z_1}{Z_2}\, \big[\, \bar{\zeta}_x\, (\delta W_R/\delta \bar{\zeta}_x) + (\delta W_R/\delta \zeta_x)\, \zeta_x\, \big] \end{aligned} \tag{2.159}$$

This is known as the Ward-Takahashi identity for the renormalized connected Green's function in the class of linear covariant gauges. It is also very useful to obtain the corresponding relations for proper vertexes or 1PI renormalized Green's functions. First we define as it's customary

$$\frac{\delta W_R}{\delta J_\mu} = A^\mu_{c\ell} \qquad \delta W_R/\delta \bar{\zeta} = \psi_{c\ell} \qquad \frac{\delta W_R}{\delta \zeta} = -\, \bar{\psi}_{c\ell}$$

$$\Gamma_R \left[\, A_{c\ell}\,, \psi_{c\ell}\,, \bar{\psi}_{c\ell}\,\right] = W_R[\, J, \bar{\zeta}, \zeta\,] - \int \mathrm{d}^{\mathrm{D}} x\, \left[\, J_\mu\, A^\mu_{c\ell} + \bar{\zeta}\, \psi_{c\ell} + \bar{\psi}_{c\ell}\, \zeta\,\right]$$

$$\frac{\delta \Gamma_R}{\delta A^\mu_{c\ell}} = -\, J_\mu \qquad \zeta = -\, \delta \Gamma_R/\delta \bar{\psi}_{c\ell} \qquad \bar{\zeta} = \frac{\delta \Gamma_R}{\delta \psi_{c\ell}}$$

Then in terms of the generating functional for the renormalized 1PI Green's functions the Ward-Takahashi identities read

$$\begin{aligned} & ie\mu^\epsilon\, \frac{Z_1}{Z_2}\, \big\{ \bar{\psi}_{c\ell}(x)\, [\, \delta \Gamma_R/\delta \bar{\psi}_{c\ell}(x)\,] + [\, \delta \Gamma_R/\delta \psi_{c\ell}(x)\,]\, \psi_{c\ell}(x) \big\} \\ = \; & \xi^{-1}\, \partial^\mu \Box\, A^{c\ell}_\mu(x) - \partial_\mu\, [\, \delta \Gamma_R/\delta A^{c\ell}_\mu(x)\,] \end{aligned} \tag{2.160}$$

Examples and applications.

Taking the functional derivative of the Ward's identity (2.159) with respect to $J_\nu(y)$ and putting all sources to zero one gets

$$\xi\, \partial_\nu\, \delta(x-y) = \partial^\mu_x \Box\, \frac{\delta^{(2)} W_R}{\delta J^\mu_x\, \delta J^\nu_y} \bigg\rfloor_{J=\zeta=\bar{\zeta}=0} = i\partial^\mu_x \Box\, G^R_{\mu\nu}(x-y) \tag{2.161}$$

[10] John Clive Ward (1950) *An identity in quantum electrodynamics*, The Physical Review **78**, 182; Yasushi Takahashi (1957) *On the generalized Ward identity*, Il Nuovo Cimento, Serie 10, **6**, 370.

where $G^R_{\mu\nu}$ stands for the dressed renormalized photon propagator. Going to the momentum space we obtain

$$\xi\, k_\nu = ik^\mu\, k^2\, \widetilde{G}^R_{\mu\nu}(k) \tag{2.162}$$

the general causal solution of which is

$$\widetilde{G}^R_{\mu\nu}(k) = \frac{-\,i}{k^2+i\varepsilon}\left[\left(k^2\, g_{\mu\nu} - k^\mu k^\nu\right)\, \frac{1}{1-\Pi_R(k)} + \xi\, \frac{k^\mu k^\nu}{k^2+i\varepsilon}\right] \tag{2.163}$$

where $\Pi_R(k)$ is the renormalized invariant vacuum polarization function, which tells us that the longitudinal part of the dressed photon propagator is kinematic and does not require any counter-term. Hence the gauge fixing Lagrangian does not renormalize, as previously claimed.

Taking instead the functional derivative of eq. (2.160) with respect to the spinor classical field $\bar\psi_{c\ell}(y)$ one obtains

$$\begin{aligned} &\partial^x_\mu\, \{\delta^{(2)}\,\Gamma_R/\delta\,\bar\psi_{c\ell}(y)\,\delta\, A^{c\ell}_\mu(x)\} + ie\mu^\epsilon\, \frac{Z_1}{Z_2}\, \delta(x-y)\, \{\delta\,\Gamma_R/\delta\,\bar\psi_{c\ell}(x)\} \\ +\;\; & ie\mu^\epsilon\, \frac{Z_1}{Z_2}\, \{\delta^{(2)}\,\Gamma_R/\delta\,\bar\psi_{c\ell}(y)\,\delta\,\psi_{c\ell}(x)\}\, \psi_{c\ell}(x) = 0 \end{aligned}$$

One more functional derivative of eq. (2.160) with respect to the spinor classical field $\psi_{c\ell}(z)$ yields

$$\begin{aligned} 0 \;&=\; \partial^x_\mu\, \{\delta^{(3)}\,\Gamma_R/\delta\,\psi_{c\ell}(z)\,\delta\,\bar\psi_{c\ell}(y)\,\delta\, A^{c\ell}_\mu(x)\} \\ &+\;\; ie\mu^\epsilon\, \frac{Z_1}{Z_2}\, \delta(x-y)\, \{\delta^{(2)}\,\Gamma_R/\delta\,\psi_{c\ell}(z)\,\delta\,\bar\psi_{c\ell}(x)\} \\ &+\;\; ie\mu^\epsilon\, \frac{Z_1}{Z_2}\, \{\delta^{(2)}\,\Gamma_R/\delta\,\bar\psi_{c\ell}(y)\,\delta\,\psi_{c\ell}(x)\}\, \delta(x-z) \end{aligned}$$

and after setting all the classical fields to zero in the Minkowski space $\epsilon \downarrow 0$

$$\partial^x_\mu\, \Gamma^\mu_R(x,y,z) = ie\, \frac{Z_1}{Z_2}\, \left[\Gamma^{(2)}_R(x-y)\,\delta(x-z) - \delta(x-y)\,\Gamma^{(2)}_R(x-z)\right]$$

Taking the Fourier transform we get

$$Z_2\, k_\mu\, \widetilde{\Gamma}^\mu_R(q,p) = e\, Z_1 \left[\widetilde{\Gamma}^{(2)}_R(\not q) - \widetilde{\Gamma}^{(2)}_R(\not p)\right] \qquad (\, k+p=q\,) \tag{2.164}$$

where $\widetilde{\Gamma}^{(2)}_R(\not p) = i\hbar/\widetilde{S}^{(2)}_R(\not p)$ denotes the renormalized dressed or full spinor propagator in momentum space. To the lowest order in e and in 4 space-time dimensions one finds $Z_1 = Z_2 = 1 \vee \epsilon \to 0$ so that

$$k_\mu\, e\gamma^\mu = e(-\not p + m_e) + e(\not q - m_e) \quad\Leftrightarrow\quad \not q = \not k + \not p$$

as it does. A very interesting consequence of the above Ward's identity (2.164) is as follows. Since the renormalized proper vertexes are finite in D = 4 to all orders in perturbation theory, it turns out that even the ratio Z_1/Z_2 must be so in a renormalizable theory. Actually this implies that the divergent parts of Z_1 and Z_2 must be equal order-by-order in perturbation theory and for any regularization method employed. Hence it is quite convenient, although not strictly necessary, to select a renormalization prescription in which $Z_1 = Z_2$ including the finite parts. This is what happens in the physical BPHZ renormalization scheme – see eq. (2.137) – and in the minimal subtraction scheme, as we shall see in the sequel. It is worthwhile to keep in mind, however, that only in the physical on shell renormalization scheme the value of the renormalized charge is such that $e^2/4\pi = 1/137$. Moreover, if $Z_1 = Z_2$ the counterterms do constitute a gauge invariant set, including their finite parts, and the Ward-Takahashi identities share the very same form for the renormalized Green's functions and the bare Green's functions, for any gauge invariant regularization method like dimensional regularization or Pauli-Villars regularization.

A further application of the Ward identities can be done to obtain the $\beta-$function in QED. As a matter of fact, from the previously introduced general relationship

$$e_0 = \frac{Z_1}{Z_2}\,\mu^\epsilon\, e(\mu,\epsilon)\, Z_3^{-\frac{1}{2}} \qquad [\,\mathrm{D} = \Re\mathrm{e}\, 2\omega = 4 - 2\epsilon\,] \tag{2.165}$$

keeping in mind the MSS lowest order value

$$Z_3 = 1 - \frac{e^2}{12\pi^2}\cdot\frac{1}{\epsilon} + \cdots \tag{2.166}$$

and the Ward identity $Z_1 = Z_2$, then we get

$$e_0 = \mu^\epsilon \left[e(\mu,\epsilon) + \frac{e^3(\mu,\epsilon)}{24\pi^2\,\epsilon} + \cdots \right]$$

Application of the differential operator $\mu\,\mathrm{d}/\mathrm{d}\mu$ to the above relation yields

$$\begin{aligned} 0 &= \epsilon\, e(\mu,\epsilon) \left[1 + \frac{e^2(\mu,\epsilon)}{24\pi^2\,\epsilon} + \cdots \right] \\ &+ \beta(e,\epsilon) \left[1 + \frac{e^2(\mu,\epsilon)}{8\pi^2\,\epsilon} + \cdots \right] \end{aligned}$$

and thereby

$$\beta(e,\epsilon) \;=\; -\,\epsilon\, e(\mu,\epsilon)\, \left[1 + \frac{e^2(\mu,\epsilon)}{24\pi^2\,\epsilon} + \cdots \right] \left[1 + \frac{e^2(\mu,\epsilon)}{8\pi^2\,\epsilon} + \cdots \right]^{-1}$$

$$= \quad -\,\epsilon\, e(\mu,\epsilon)\ \left[1+\frac{e^2(\mu,\epsilon)}{24\pi^2\,\epsilon}-\frac{e^2(\mu,\epsilon)}{8\pi^2\,\epsilon}+\cdots\right]$$

Taking eventually the limit $\epsilon\to 0$ we obtain the running electric charge

$$\beta(e)\equiv\lim_{\epsilon\to 0}\,\mu\,\frac{\mathrm{d}e}{\mathrm{d}\mu}=\frac{e^3}{12\pi^2}\ +\ O(e^5) \tag{2.167}$$

$$e(\mu)\approx\ e(\mu_o)\cdot\frac{1}{1-\dfrac{e^2(\mu_o)}{6\pi^2}\,\ln\dfrac{\mu}{\mu_o}} \tag{2.168}$$

or equivalently

$$\alpha(\mu)\approx\ \alpha(\mu_o)\cdot\frac{1}{1-\dfrac{2\alpha(\mu_o)}{3\pi}\,\ln\dfrac{\mu}{\mu_o}} \tag{2.169}$$

which shows that $\alpha(\mu)$ is a monotonically increasing function of μ – just like $\lambda(\mu)$ in the self-interacting $\lambda\phi_4^4$ scalar field theory – as long as we stay within the range of validity of the lowest order perturbative regime

$$\frac{2\alpha(\mu_o)}{3\pi}\,\ln\frac{\mu}{\mu_o}\ll 1$$

This is the reason why the no interaction point for those models, *i.e.*

$$\lim_{\mu\to 0}\lambda(\mu)=0=\lim_{\mu\to 0}\alpha(\mu)$$

is called an INFRARED STABLE FIXED POINT, or GAUSSIAN or even TRIVIAL FIXED POINT, which corresponds to the 0-th order of perturbation theory.

2.4 Decay Rates of Unstable Particles

In the former treatment of collision theory we have considered incoming and outgoing particles as stable asymptotic states before their interactions and until the reaching of detectors. Actually, in the real world only electrons and protons are stable long-living particles on their mass shells, protons being not even elementary but quarks bound states. Nonetheless, even unstable particles with a finite and short lifetime can be perfectly well treated and included within the framework of the standard relativistic quantum collision theory. The only requirement is that the time and distance of the particle propagation from the source to the target and from the target to the detector to be very large in respect to the time and range of the particle interactions. Hence, even the interactions of short-living particles such as $\pi-$mesons, the mean lifetime of which is of the order of 10^{-8} s, can be readily described in the above terms as long as they undergo interaction and detection before decaying. The latter is usually the case, because the mean lifetime is normally lengthened by relativistic velocities: as a matter of fact, beams of relativistic pions can travel hundred of meters in actual laboratory experiments, while they interact in targets which have the dimensions of a few millimeters. These distances fully satisfy the above requirements, so that the Golden Formula can be safely employed to calculate the differential cross-sections for those processes and reactions.

However, it is neither convenient nor sufficient to restrict ourselves only to the scattering of unstable particles, for one has to further analyze and understand their decay properties, in order to compare theoretical predictions and experimental findings. In order to unravel the dynamical mechanism at the root of the decay processes it is essential to turn to the quantum effects due to radiative corrections, as dramatically encoded, for example, in the celebrated Lamb shift effect. In Section 1.2.4 we have investigated the Schwinger-Dyson equations and the structure of the complete of full propagator. For a scalar field of physical mass m we have the specific form (1.99) in momentum space, once the UV divergences have been subtracted and some definite renormalization prescription has been adopted: namely,

$$\widetilde{G}_R^{(2)}(k) = \frac{i\hbar c}{k^2 - (mc/\hbar)^2 - \hbar c\,\Sigma_R(k^2) + i\varepsilon} \tag{2.170}$$

We recall that in the physical C.G.S. system of units we have $[\,\widetilde{G}^{(2)}\,] =$ eV cm^3, while $[\,\Sigma\,] =$ eV^{-1} cm^{-3}. Moreover, to lighten notations, from now onward the suffix R will be omitted, since we understand all the quantities to be properly renormalized. For example, within the $\lambda\phi_4^4$ self-interacting model the self-energy can be expressed and calculated order by order in perturbation theory

$$\Sigma_R(k^2/m^2, \lambda) = \sum_{n=1}^{\infty} \lambda^n\, \Sigma_R^{(n)}(k^2/m^2)$$

and turns out to be a regular function when $k^2 \to m^2$. For example, had we chosen the physical renormalization prescription - see the condition (2.45) - then we set

$$\lim_{k^2 \to m^2} \Sigma_R^{(n)}(k^2/m^2) = 0 \qquad\qquad (\,\forall\, n \in \mathbb{N}\,)$$

that fixes the form of the renormalized UV finite self-energy scalar function.

Chapter 3

Quantum Gauge Theories

3.1 The Yang-Mills Quantum Theories

The quantization and renormalization of the Yang-Mills theories in covariant gauges have been widely studied from many points of view since a long time and may be considered as firmly established. In this section I want to briefly recall, for the sake of completeness, the quantization procedure for the non-Abelian Yang-Mills theories in the presence of a Lorentz covariant subsidiary condition, highlighting the main features involved: the introduction of the Faddeev-Popov ghosts, the need of a Fock space of states with an indefinite metric and thereby the definition of a physical Hilbert subspace in which a unitary $S-$matrix restriction is suitably recovered. All the above mentioned fundamental concepts will be discussed in the framework of perturbation theory, as usual, and with special emphasis to the functional integration technique, according to the historical development.

3.1.1 Gauge Invariance and Generating Functional

The construction of a gauge invariant Lagrangian under a Non-Abelian group of local transformations has been already obtained in [22] §**2.5**; according to the insofar introduced notations and conventions it reads

$$\mathcal{L}_{\mathrm{YM}} \equiv -\tfrac{1}{2}\,\mathrm{tr}\,[\,F_{\mu\nu}(x)F^{\mu\nu}(x)\,] + \bar{\psi}(x)\,(i\not{D} - M)\,\psi(x)$$

where

$$\not{D} \equiv \gamma^{\mu}D_{\mu} = \not{\partial} - ig\,\gamma^{\mu}A_{\mu}(x) = \not{\partial} - ig\,\gamma^{\mu}A_{\mu}^{a}(x)\,\boldsymbol{\tau}_{F}^{a}$$

$$\mathrm{tr}\left(\boldsymbol{\tau}_{F}^{a}\,\boldsymbol{\tau}_{F}^{b}\right) = \tfrac{1}{2}\,\delta^{ab}$$

The gauge transformations , which leave the Action invariant, may be written either in finite form

$$\psi(x) \mapsto \psi_\omega(x) = \exp\{\, ig\,\omega^a(x)\boldsymbol{\tau}_F^a\} \equiv U_\omega(x)\,\psi(x)$$

$$U_\omega^{-1}(x) = \exp\{-\, ig\,\omega^a(x)\boldsymbol{\tau}_F^a\} = U_\omega^\dagger(x) = U_{-\omega}(x) \in SU(N)$$

$$A_\mu(x) \mapsto A_\mu^\omega(x) = U_\omega(x)\,A_\mu(x)\,U_\omega^\dagger(x) - \frac{i}{g}\,[\,\partial_\mu U_\omega(x)\,]\,U_\omega^\dagger(x)$$

$$F_{\mu\nu}^\omega(x) = \frac{i}{g}\left[\,D_\mu^\omega\,,\,D_\nu^\omega\,\right] = U_\omega(x)\,F_{\mu\nu}(x)\,U_\omega^\dagger(x)$$

or in infinitesimal form as

$$\begin{aligned}
\delta\psi(x) &= ig\,\boldsymbol{\tau}_F^a\,\psi(x)\,\delta\,\omega^a(x)\\
\delta\overline{\psi}(x) &= -\,ig\,\overline{\psi}(x)\,\boldsymbol{\tau}_F^a\,\delta\,\omega^a(x)\\
\delta A_\mu^a(x) &= \partial_\mu\,\delta\,\omega^a(x) - gf^{abc}\,A_\mu^c(x)\,\delta\,\omega^b(x) \equiv \nabla_\mu^{ab}\delta\,\omega^b(x)\\
\delta F_{\mu\nu}^a(x) &= gf^{abc}\,F_{\mu\nu}^b(x)\,\delta\,\omega^c(x)
\end{aligned}$$

It is worthwhile to remark that another quadratic and gauge invariant term in the gauge potentials does actually exist, namely,

$$\tfrac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,[\,F_{\mu\nu}(x)F_{\rho\sigma}(x)\,]$$

However, it is easy to recognize that such a term can be written as a total four-divergence and hence does not modify the equations of motion for the gauge potentials, in particular it is inessential in perturbative calculations.

Gauge theories are examples of constrained systems described by singular Lagrangian, namely of systems in which it is impossible to express all the velocities in terms of the canonical variables, due to the presence of constraint relationships that appear in the definitions of the momenta. As a matter of fact we find the generalized, functional, conjugate momenta

$$\frac{\delta\mathcal{L}_{\mathrm{YM}}}{\delta\,\partial_0 A^0(x)} \equiv \Pi_0(x) = 0 \tag{3.1}$$

$$\frac{\delta\mathcal{L}_{\mathrm{YM}}}{\delta\,\partial_0 A^k(x)} \equiv \Pi_k(x) = F_{k0}(x) = E_k(t,\mathbf{x}) \tag{3.2}$$

Thus, owing to the presence of the constraint

$$\Pi_0^a(t,\mathbf{x}) = 0 \qquad\qquad a = 1,2,\ldots,N^2-1$$

it is not possible to obtain the velocities $\dot{A}_\lambda(t,\mathbf{x})$ in terms of the functional canonical variables $A^\nu(t,\mathbf{x})$, $\Pi_\mu(t,\mathbf{x})$. Moreover, the temporal components

$A_0^a(t, \mathbf{x})$ of the gauge potential do appear in the Lagrangian, as well as in the Hamiltonian, as Lagrange multiplier functions, because the corresponding conjugate momenta identically vanish. For such a kind of singular mechanical systems, the main difficulty one has to overcome is to keep compatible, within the dynamics, the presence of the constraints on the one side and the algebra of the canonical Poisson brackets (or canonical commutation relations at the quantum level) on the other side. This problem can be solved, in general, by means of the so called Dirac procedure

Paul Adrian Maurice Dirac
Lectures on Quantum Mechanics
Yeshiva University Press, 1964

i.e. by replacing the Poisson brackets by the Dirac brackets, which turn out to be fully compatible with the presence of the whole set of constraints. In the development of the Dirac formalism for the constrained systems, which lies outside the scope of the present short review, the gauge choices, or auxiliary, or subsidiary conditions, become a subset of the whole set of constraints, made compatible with the dynamics thanks to the definition of the Dirac brackets. Notwithstanding the general setting of the Dirac procedure, still it is not tailored to any arbitrary kind of gauge choices, but only to a particular subclass called Dirac-compatible.

`Remark.` There are excellent reviews and presentations of the Dirac theory of constrained systems available in the Literature: for example, A. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems*, Acc. Naz. dei Lincei, Roma, 1976; E.C.G. Sudarshan and M. Mukunda, *Classical Dynamics. A Modern Perspective*, Wiley Interscience, New York, 1974; K. Sundermeyer, *Constrained Dynamics: with Applications to Yang-Mills Theories, General Relativity, Classical Spin, Dual String Model*, Springer Verlag, Berlin, 1982. However, for a better comprehension of these notes, I remind the reader that the PRIMARY CONSTRAINTS are those arising from the definition of the conjugate momenta of the system, just like $\Pi_0^a(t, \mathbf{x}) = 0 \quad (a = 1, 2, \ldots, N^2 - 1)$ in the previous example. The SECONDARY CONSTRAINTS are obtained by imposing the temporal consistency for all the primary constraints, i.e., by requiring that the time derivative of the primary constraints does vanish. A set of constraints $\psi^\imath$ $(\imath = 1, 2, \ldots)$ is said to be SECOND CLASS if for each constraint $\psi^\jmath$ there exists at least one constraint $\psi^\varkappa$ such that the Poisson bracket $\{\psi^\jmath, \psi^\varkappa\}$ does not vanish on the constraint manifold $\psi^\imath = 0$. Otherwise the set of constraints is said to be FIRST CLASS.

A necessary conditions for a gauge choice to be Dirac compatible is that it must be a proper functional of the canonical variables, that means, it must not involve the Lagrange multipliers which are associated to the presence of primary constraints. For example, the temporal gauge choice $A_0(t, \mathbf{x}) = 0$

is not Dirac-compatible. Even the Lorentz invariant non-Abelian Lorenz condition

$$\partial^{\mu} A^{a}_{\mu}(x)\, \boldsymbol{\tau}^{a}_{F} = 0$$

is not Dirac-compatible, while the axial gauge $A_3(t,\mathbf{x}) = 0$, or the Coulomb, or radiation gauge $\nabla \cdot \mathbf{A}(t,\mathbf{x}) = 0$ are truly Dirac compatible subsidiary conditions. Thus, some of the most frequently used gauge conditions in the quantization of the Yang-Mills theories do not turn out to be Dirac compatible and, thereby, one has to resort to alternative procedures in those cases. This is what the so called Faddeev-Popov trick has been invented for.

Let me briefly review the transition to the quantum theory of the Yang-Mills fields in the functional integral approach and let me disregard for a moment the spinor field multiplet, because its quantization can be readily obtained in a straightforward and usual way. The Yang-Mills Lagrangian can be decomposed as a sum of the free ($g = 0$) and an interacting part

$$\begin{aligned}
\mathcal{L}_{\mathrm{YM}} &= \mathcal{L}_0 + \mathcal{L}_{\,\mathrm{int}} && (3.3)\\
\mathcal{L}_0 &= -\tfrac{1}{2}\, g^{\mu\rho}\, g^{\nu\sigma} \Big[\, \partial_{\mu} A^{a}_{\nu}(x) - \partial_{\nu} A^{a}_{\mu}(x) \Big]\, \partial_{\rho} A^{a}_{\sigma}(x) && (3.4)\\
\mathcal{L}_{\,\mathrm{int}} &= -\, g\, f^{abc}\, g^{\mu\rho}\, g^{\nu\sigma}\, A^{b}_{\rho}(x)\, A^{c}_{\sigma}(x)\, \partial_{\mu} A^{a}_{\nu}(x)\\
&\quad - \tfrac{1}{4}\, g^{2}\, f^{abc}\, f^{ade}\, g^{\mu\rho}\, g^{\nu\sigma}\, A^{b}_{\mu}(x)\, A^{c}_{\nu}(x)\, A^{d}_{\rho}(x)\, A^{e}_{\sigma}(x) && (3.5)
\end{aligned}$$

and the generating functional of the Green's functions $Z[\,J\,]$ can be written as a functional differential operator acting on the free part $Z_0[\,J\,]$: namely,

$$\begin{aligned}
Z[J] &= \mathcal{N} \int \mathfrak{D}A\, \exp\left\{ i \int \mathrm{d}x\, \left[\mathcal{L}_{\mathrm{YM}} + J^{a,\mu} A^{a}_{\mu} \right] \right\} && (3.6)\\
&= \exp\left\{ i \int \mathrm{d}x\, \mathcal{L}_{\,\mathrm{int}} \left[\delta / i\delta J^{a}_{\mu}(x) \right] \right\} Z_0[J] && (3.7)\\
Z_0[J] &= \mathcal{N} \int \mathfrak{D}A\, \exp\left\{ i \int \mathrm{d}x\, \Big[\, \tfrac{1}{2}\, A^{a}_{\mu}(x)\, \mathcal{K}^{\mu\nu}_{x} A^{a}_{\nu}(x) \right.\\
&\quad \left. +\; J^{a,\mu}(x)\, A^{a}_{\mu}(x) \Big] \right\} && (3.8)
\end{aligned}$$

where

$$\mathcal{N}^{-1} = Z_0[\,0\,] \qquad \mathcal{K}^{\mu\nu}_{x} = g^{\mu\nu}\, \Box_{x} - \partial^{\mu}_{x} \partial^{\nu}_{x} \tag{3.9}$$

$$\int \mathfrak{D}A = \prod_{a=1}^{N^2-1} \prod_{\mu=0}^{3} \prod_{x \in \mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}A^{a}_{\mu\,,\,x} \tag{3.10}$$

This functional integral, however, is meaningless as the inverse operator $\mathcal{K}^{-1}$ does not exist because $g_{\mu\nu}\, \mathcal{K}^{\mu\nu}_{x} = 0$. As we already know in the Abelian case, this difficulty is peculiar to gauge theories, *i.e.* theories involving a local symmetry. Actually, in equation (3.6) one should take the integral over all the configurations of the gauge potentials $\{A_\mu^a(x)\}$, including those ones which are related by gauge transformations. By the way, an ORBIT generated by any given vector potential $A_\mu(x)$ is the set of all configurations $\{A_\mu^\omega(x)\}$ which are connected to the selected $A_\mu(x)$ by gauge transformations. Since the Yang-Mills Lagrangian is gauge invariant it turns out to be constant along the orbits, so that there is no damping factor in equation (3.6) when the integration variable $\{A_\mu^a(x)\}$ runs over the its orbit.

3.1.2 Faddeev-Popov Trick

In order to solve this problem it will be necessary and sufficient to integrate only over gauge nonequivalent configurations of the vector potential, namely to integrate over one and only one representative of each gauge orbit. In so doing it will be possible to factorize in equation (3.6) the measure of the volume of the gauge group; hence, to integrate over representatives of the gauge orbits is equivalent to integrate over the quotient space $\{A_\mu\}/\{A_\mu^\omega\}$. As a consequence, in a correct quantum functional procedure one has to integrate on a manifold

$$\Phi[\,A_\mu(x)\,] = 0$$

which has to intersect each orbit only once: namely, the system of equations

$$\begin{cases} \Phi[\,A_\mu^\omega(x)\,] = 0 \\ \Phi[\,A_\mu(x)\,] = 0 \end{cases} \tag{3.11}$$

must have one and only one trivial solution $\omega^a(x) = 0$ $(a = 1, \ldots, N^2-1)$ for any given potential $A_\mu(x)$. Thus the gauge choice, or auxiliary or subsidiary condition, must be such that to faithfully select one representative in each gauge orbit.

`The Gribov problem.` Consider for example the non-Abelian linear and Lorentz invariant Lorenz condition $\partial^\mu A_\mu^a(x)\,\boldsymbol{\tau}_F^a = 0$ then we get

$$\begin{aligned} & [\,\Box\, U_\omega(x)\,]\, U_\omega^\dagger(x) + [\,\partial_\mu U_\omega(x)\,]\partial^\mu U_\omega^\dagger(x) \\ + \quad & ig\,[\,\partial^\mu U_\omega(x)\,]\, A_\mu(x) U_\omega^\dagger(x) - \, ig\, U_\omega(x)\, A_\mu(x)\, \partial^\mu U_\omega^\dagger(x) = 0 \end{aligned}$$

This is a complicated system of coupled differential equations in the unknown functions $\omega^a(x)$ $(a = 1, 2, \ldots, N^2-1)$ for some given potential $A_\mu(x)$. In general the above system of equations does possess many solutions, which means that there are many representatives for each gauge orbit even in the presence of the subsidiary conditions. This phenomenon is called the Gribov problem or Gribov ambiguity:

Vladimir Naumovich Gribov (Leningrad 1930 - Budapest 1997)

Quantization Of Non-Abelian Gauge Theories
Nuclear Physics **B139** (1978) pp. 1-19.

Actually, in the case of the non-Abelian Coulomb gauge

$$\Phi\,[\,A_\mu(x)\,] = \partial_\imath A^a_\imath(t,\mathbf{x}) = 0$$

with $A_\imath(t,\mathbf{x}) \in \mathrm{SU}(2)$ Gribov has found explicit nontrivial solutions for the system (3.11) which, however, turn out to be topologically nontrivial in the sense that the corresponding gauge invariant quantity called topological charge or Pontryagin index

$$\nu \equiv \frac{1}{24\pi^2}\int \mathrm{d}\mathbf{x}\,\varepsilon_{\imath\jmath k}\,\mathrm{tr}\,\left(A_\imath\,A_\jmath\,A_k\right)$$

is a non-vanishing integer or half-integer for all the Gribov copies. Therefore it appears to be plausible to expect – although there is no general proof of this statement – that all the eventual nontrivial solutions of the system (3.11) should exhibit a nontrivial topological structure, *i.e.* a non-vanishing topological charge ν. On the one hand, if we restrict the functional space integration, at least in perturbation theory, to all the topologically trivial deformations with $\nu = 0$ around the null potential, which is certainly the case of the perturbation theory, then we could definitely ignore the Gribov problem and ambiguities. On the other hand, there are algebraic non-covariant gauges such as the axial gauge

$$n^\mu A_\mu(x) = 0 \qquad\qquad n^\mu = (0,1,0,0)$$

or the planar gauge

$$n^\mu A_\mu(x) = \phi(x) \qquad \Box\phi(x) = 0 \qquad\qquad n^\mu = (0,1,0,0)$$

for which the Gribov copies do not exist at all [A. Bassetto, I. Lazzizzera and R. Soldati (1981) Phys. Lett. **107B**, p. 278]. Hence, if we believe that gauge invariance has to be definitely endorsed at the quantum level, then the Gribov ambiguities appear to be a formal problem, related to some specific choices of the subsidiary conditions, that can and has to be suitably eventually settled within a correct mathematical framework.

In order to factorize out the volume of the local gauge group, *i.e.* the volume of the orbit space, we shall employ the formal Faddeev-Popov procedure by defining

$$\triangle^{-1}_{\Phi}[\,A_\mu\,] = \int \mathfrak{D}\,\mathrm{g}_\omega\,\delta\left[\,\Phi\left(A^\omega_\mu\right)\,\right] \tag{3.12}$$

where I have introduced the functional INVARIANT MEASURE over the unitary gauge group $SU(N)$

$$\int \mathfrak{D}\,\mathrm{g}_\omega \equiv \prod_{x\,\in\,\mathcal{M}}\int_{\mathrm{D}}\mathrm{d}\,\mathrm{g}\,[\,\omega(x)\,]$$

where D denotes the bounded domain of the canonical coordinates over the unitary group: for example, in the case of $SU(2)$ we have

$$\mathrm{D}\;=\;\left\{(\omega^1,\omega^2,\omega^3) = \alpha\,\widehat{\mathbf{n}}\,;\,0\le\alpha<2\pi\right\}$$

where $\widehat{\mathbf{n}}$ is the oriented unit vector of the rotation axis. We recall that the $SU(2)$ group manifold is homeomorphic to S_3, *i.e.* the three dimensional sphere of unit radius plunged into the four dimensional Euclidean space with

$$\begin{cases} x_1 = \cos\theta_1 & (\, 0 \le \theta_1 \le \pi\,) \\ x_2 = \sin\theta_1 \cos\theta_2 & (\, 0 \le \theta_2 \le \pi\,) \\ x_3 = \sin\theta_1 \sin\theta_2 \cos\alpha & (\, 0 \le \alpha < 2\pi\,) \\ x_4 = \sin\theta_1 \sin\theta_2 \sin\alpha & \sum_{\imath=1}^{4} x_\imath^2 = 1 \end{cases}$$

so that its invariant measure is provided by

$$\int_{\mathrm{D}} \mathrm{d}\,\mathrm{g}[\,\omega\,] = \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \mathrm{d}\theta_2\, \sin^2\theta_2 \int_0^{\pi} \sin\theta_1\, \mathrm{d}\theta_1 = 2\pi^2 \tag{3.13}$$

because the solid angle in $2\omega-$dimensions is $2\pi^{\,\omega}/\Gamma(\omega)$. More generally, for the unitary groups $SU(N)$ a linear functional $\mathfrak{M}(f)$ exists, called the invariant averaging functional, which fulfill the following properties for any real periodic function $f(\mathrm{g}) \in \mathbb{R}$, $\forall\, \mathrm{g} \in SU(N)$: namely,

1) if $f(\mathrm{g}) > 0\ \forall\, \mathrm{g} \in SU(N)$ then $\mathfrak{M}(f) > 0$;
2) if $f(\mathrm{g}) = 1\ \forall\, \mathrm{g} \in SU(N)$ then $\mathfrak{M}(f) = 1$;
3) if $f_1(\mathrm{g}) = f(\mathrm{g}\cdot\mathrm{g}')$ and $f_2(\mathrm{g}) = f(\mathrm{g}'\cdot\mathrm{g})$ with $\mathrm{g}' \in SU(N)$ then

$$\mathfrak{M}(f_1) = \mathfrak{M}(f_2) = \mathfrak{M}(f) \tag{3.14}$$

In the case of $SU(2)$ we have

$$\mathfrak{M}(f) \equiv \frac{1}{2\pi^2} \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \sin^2\psi\, \mathrm{d}\psi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta\, f(\alpha,\theta,\psi)$$

in such a manner that we can identify for $SU(2)$

$$\mathfrak{M}\,(1)\, 2\pi^2 = \int_{\mathrm{D}} \mathrm{d}\,\mathrm{g}[\,\omega\,]$$

The group invariance property for the measure and the averaging functional is expressed by (3.14) or equivalently

$$\int_{\mathrm{D}} \mathrm{d}\{\mathrm{g}[\,\omega\,]\cdot \mathrm{g}'[\omega']\} = \int_{\mathrm{D}} \mathrm{d}\,\mathrm{g}[\,\omega\,]$$

As a consequence one can readily recognize that

$$\begin{aligned} \triangle_{\Phi}^{-1}[\,A_\mu^\omega\,] &= \int \mathfrak{D}\,\mathrm{g}'_{\omega'}\,\delta\left[\,\Phi\Big(A_\mu^{\omega\,\omega'}\Big)\,\right] \\ &= \int \mathfrak{D}(\mathrm{g}_\omega\,\mathrm{g}'_{\omega'})\,\delta\left[\,\Phi\Big(A_\mu^{\omega\,\omega'}\Big)\,\right] \\ &= \int \mathfrak{D}\,\mathrm{g}_\omega\,\delta\left[\,\Phi\big(A_\mu^{\omega}\big)\,\right] \\ &= \triangle_{\Phi}^{-1}[\,A_\mu\,] \end{aligned} \tag{3.15}$$

which means that $\triangle_{\Phi}^{-1}[\,A_\mu\,]$ is gauge invariant or, more precisely, it is constant along any gauge orbit.

Here it comes the nice trick invented by Ludwig D. Faddeev and Victor N. Popov in 1967: it is expedient to multiply the generating functional (3.6) by 1 (one) though written as

$$\begin{aligned} Z[J] &= \mathcal{N}\int \mathfrak{D}A\,\exp\left\{i\int \mathrm{d}x\,\left[\mathcal{L}_{\mathrm{YM}} + J^{a,\mu}A_\mu^a\right]\right\} \\ &\times\ \triangle_\Phi[\,A_\mu\,]\int \mathfrak{D}\,\mathrm{g}_\omega\,\delta\left[\,\Phi\big(A_\mu^\omega\big)\,\right] \end{aligned}$$

and by exchanging the order of the functional integration we get

$$\begin{aligned} Z[J] &= \mathcal{N}\int \mathfrak{D}\,\mathrm{g}_\omega \int \mathfrak{D}A\,\triangle_\Phi[\,A_\mu\,]\,\delta\left[\,\Phi\big(A_\mu^\omega\big)\,\right] \\ &\times\ \exp\left\{i\int \mathrm{d}x\,\left[\,\mathcal{L}_{\mathrm{YM}} + \tfrac12\,\mathrm{tr}\,(J^\mu A_\mu)\,\right]\right\} \end{aligned}$$

Now, taking into account that the Yang-Mills Lagrangian as well as the quantity $\triangle_\Phi[\,A_\mu\,]$ are gauge invariant expressions and that

$$\int \mathfrak{D}A = \int \mathfrak{D}A^\omega \tag{3.16}$$

since a gauge transformation on the non-Abelian vector potential is nothing but a combination of functional translation and similarity transformations, then by changing the functional integration variable according to $A_\mu \longmapsto A_\mu^\omega$ we definitely obtain

$$\begin{aligned} Z[J] &= \mathcal{N}\left(\textstyle\int \mathfrak{D}\,\mathrm{g}_\omega\right)\int \mathfrak{D}A\,\triangle_\Phi[\,A_\mu\,]\,\delta\,[\,\Phi(A_\mu)\,] \\ &\times\ \exp\left\{i\int \mathrm{d}x\,\left[\,\mathcal{L}_{\mathrm{YM}} + \tfrac12\,\mathrm{tr}\left(J^\mu A_\mu^{-\omega}\right)\,\right]\right\} \end{aligned} \tag{3.17}$$

Now we have

$$\begin{aligned} \mathrm{tr}\big(J^\mu A_\mu^{-\omega}\big) &= \mathrm{tr}\left[\,U_\omega(x)J^\mu(x)U_\omega^\dagger(x)A_\mu(x)\,\right] \\ &- \frac{i}{g}\,\mathrm{tr}\left[\,U_\omega(x)J^\mu(x)\,\partial_\mu U_\omega^\dagger(x)\,\right] \end{aligned}$$

so that we can eventually write

$$Z[\,J'\,] = \int \mathfrak{D}\,\mathrm{g}_\omega\,\exp\left\{\frac{1}{2g}\int \mathrm{d}x\,\mathrm{tr}\left[\,J_\mu'(x)\,\mathrm{g}_\omega(x)\,\partial^\mu\,\mathrm{g}_\omega^{-1}(x)\,\right]\right\}$$

$$
\begin{aligned}
& \times \quad \mathcal{N} \int \mathfrak{D}A \, \triangle_{\Phi}[\, A_{\mu}\,] \, \delta\, [\, \Phi(A_{\mu})\,] \\
& \times \quad \exp\left\{ i \int \mathrm{d}x \, \left[\, \mathcal{L}_{\mathrm{YM}} + \tfrac{1}{2}\, \mathrm{tr}\, \left(J_{\mu}^{\prime}\, A^{\mu} \right) \, \right] \right\} \\
= & \quad \int \mathfrak{D}\, \mathrm{g}_{\omega} \, \exp\left\{ \frac{1}{2g} \int \mathrm{d}x \, \mathrm{tr}\, \left[\, J_{\mu}^{\prime}(x)\, \mathrm{g}_{\omega}(x)\, \partial^{\mu}\, \mathrm{g}_{\omega}^{-1}(x)\, \right] \right\} \\
& \times \quad Z_{\Phi}\, [\, J^{\prime}\,] \qquad (3.18)
\end{aligned}
$$

where I have set

$$
J_{\mu}^{\prime}(x) = U_{\omega}(x) J_{\mu}(x) U_{\omega}^{\dagger}(x) \qquad\qquad U_{\omega}(x) = \mathrm{g}_{\omega}(x)
$$

It follows that by virtue of the Faddeev-Popov trick we can factorize out the integral over the gauge group, in such a manner that we can safely and properly define the generating functional for the Green's functions of the pure Yang-Mills theory as

$$
\begin{aligned}
Z_{\Phi}\, [\, J\,] \quad = & \quad \mathcal{N} \int \mathfrak{D}A \, \triangle_{\Phi}[\, A_{\mu}\,] \, \delta\, [\, \Phi(A_{\mu})\,] \\
& \times \quad \exp\left\{ i \int \mathrm{d}^{4}x \, \left[\, \mathcal{L}_{\mathrm{YM}} + J^{\mu\,,\,a} A_{\mu}^{a} \right] \right\} \qquad (3.19)
\end{aligned}
$$

It is absolutely crucial to observe that from the above derivation based upon the Faddeev-Popov trick it turns out that the generating functional (3.19) formally enjoys the following features:

1. once the orbit volume space has been factorized through the Faddeev-Popov trick the corresponding quantum theory and related Feynman rules will **depend explicitly** upon the specific choice of the subsidiary local functional $\Phi[\, A^{\mu}(x)\,]$. As a matter of fact, all the Green's functions - and specifically the propagators - are actually gauge dependent;

2. it will be eventually understood as a perturbative expansion in powers of the small numerical parameter $g < 1$.

3.1.3 Faddeev-Popov Determinant

Now we turn to the calculation of the invariant quantity (3.12). To this concern I recall the identity which is valid for tempered distributions in $\mathsf{S}^{\prime}(\mathbb{R})$

$$
\int_{-\infty}^{\infty} \mathrm{d}x \; \delta(f(x))\, \varphi(x) = \frac{\varphi(x_{0})}{|\, f^{\prime}(x_{0})\, |} \qquad\qquad [\, f(x_{0}) = 0\,] \qquad (3.20)
$$

where x_0 is assumed to be the single zero of $f(x) \in \mathsf{C}^{\infty}_{x_0}$. The above equality admits a straightforward generalization to the matrix-like and functional framework that yields

$$\begin{aligned} \triangle_{\Phi}^{-1}[\,A_\mu\,] &= \int \mathfrak{D}\,\mathrm{g}_\omega\,\delta\left[\,\Phi\left(A_\mu^\omega\right)\,\right] \\ &= \det{}^{-1}\,\|\,\delta\,\Phi^{\,a}\big(A_\mu^\omega\big)\,/\delta\,\omega_x^{\,b}\,\|_{\,\Phi(A^\omega)=0} \end{aligned} \tag{3.21}$$

However, taking into account that in the generating functional (3.19) the quantity (3.12) is always multiplied by $\delta\,[\,\Phi(A_\mu)\,]$, we can safely recast the very last equality in the form

$$\begin{aligned} \triangle_{\Phi}\,[\,A_\mu\,]\,\delta\,[\,\Phi(A_\mu)\,] &= \delta\,[\,\Phi(A_\mu)\,]\ \det\,\|\,\delta\,\Phi^{\,a}\big(A^\omega_{\mu\,,\,x}\big)\,/\delta\,\omega_y^{\,b}\,\|_{\,\omega\,=\,0} \\ \|\,\mathsf{M}_\Phi(x,y)\,\|^{ab} &\equiv \big\{\delta\,\Phi^{\,a}\big[\,A_\mu^\omega(x)\,\big]\,/\delta\,\omega^{\,b}(y)\big\}_{\,\omega\,=\,0} \end{aligned} \tag{3.22}$$

where $\det\,\|\,\mathsf{M}_\Phi(x,y)\,\|$ is the well-known Faddeev-Popov determinant for the selected subsidiary condition $\Phi[\,A_\mu(x)\,] = 0$.

For the forthcoming applications the following remarks will be useful. First, let me notice that the quantity $\triangle_\Phi\,[\,A_\mu\,]$ inside the functional integral for the generating functional $Z_\Phi\,[\,J_\mu\,]$ can be replaced by $\det\,\|\,\mathsf{M}_\Phi\,\|$, because the auxiliary condition $\Phi(A_\mu) = 0$ is already enforced and endorsed by the $\delta-$functional that fixes the gauge choice. To this concern, let us consider a class of subsidiary conditions of the non-homogeneous kind

$$\Phi[\,A_\mu(x)\,] = \xi\,B(x)$$

where $\xi \in \mathbb{R}$ is called the gauge parameter, while $B(x)$ is any multiplet of arbitrary Lorentz scalar functions on the Minkowski space, which transforms according to the adjoint representation of the gauge gauge: namely,

$$B(x) = B^{\,a}(x)\,\boldsymbol{\tau}_F^{\,a} \quad \longmapsto \quad B^{\,\prime}(x) = U_\omega(x)\,B(x)\,U_\omega^\dagger(x) \tag{3.23}$$

Now we find

$$\begin{aligned} & \{\Phi[\,A_\lambda^\omega(x)\,] - \xi\,B(x)\}\ \delta\,(\Phi[\,A_\lambda\,] - \xi\,B) \\ =\ & \{\Phi[\,A_\lambda(x) + \nabla_\lambda\,\omega(x)\,] - \xi\,B(x)\}\ \delta\,(\Phi[\,A_\lambda\,] - \xi\,B) \\ =\ & \frac{\delta\,\Phi[\,A\,]}{\delta A_\mu^a(x)}\cdot\nabla_\lambda^{ab}\,\omega^{\,b}(x)\ \delta\,(\Phi[\,A_\lambda\,] - \xi\,B) \end{aligned}$$

whence is evident that, for all the auxiliary conditions belonging to the above non-homogeneous class, the quantity $\det\,\|\,\mathsf{M}_\Phi\,\|$ turns out to be the very same, because the auxiliary scalar field $B(x) = B^{\,a}(x)\boldsymbol{\tau}_F^{\,a}$ is unaffected by the non-homogeneous part of the gauge transformations $A_\mu(x) \longmapsto A^\omega_\mu(x)$, so that (3.22) does not depend on B. We take profit of this feature in order to replace the $\delta-$functional in (3.19) by some other functional $\Phi\,[\,A_\mu(x)\,]$ for the gauge-fixing, which might be more convenient for the practical purposes and calculations. The non-homogeneous subsidiary conditions $\Phi\,[\,A_\mu(x)\,] = \xi B(x)$ drive to the generating functional (3.19) which can be rewritten as

$$\begin{aligned} Z_\Phi[\,J^\mu\,;B\,] &= \mathcal{N}\int \mathfrak{D}A\;\det\parallel \mathsf{M}_\Phi \parallel\;\delta\,[\,\Phi(A_\mu)-\xi B\,] \\ &\times\;\exp\left\{i\int \mathrm{d}x\,\left[\,\mathcal{L}_{\mathrm{YM}} + J^{\mu\,,\,a}A^a_\mu\right]\right\} \end{aligned} \tag{3.24}$$

Since the above expression does clearly depend upon Φ and B one can further integrate over $\int\mathfrak{D}B$ with an arbitrary integrable weight functional $\Psi[\,B\,]$; in so doing the result will differ from (3.24) only by an irrelevant overall normalization constant. Taking into account that the $\delta-$functional in (3.24) is the only $B-$dependent term, one obtains

$$\begin{aligned} Z_\Phi[\,J^\mu\,;\xi\,] &= \mathcal{N}'\int \mathfrak{D}A\;\det\parallel \mathsf{M}_\Phi \parallel\;\exp\left\{i\int \mathrm{d}x\,\left[\,\mathcal{L}_{\mathrm{YM}} + J^{\mu\,,\,a}A^a_\mu\right]\right\} \\ &\times\;\int \mathfrak{D}B\;\delta\,[\,\Phi(A_\mu)-\xi B\,]\;\Psi[\,B\,] \end{aligned}$$

A popular and very convenient choice for $\Psi[\,B(x)\,]$ is the Gaussian functional

$$\Psi[\,B\,] = \exp\left\{-\,\frac{i\xi}{2}\int \mathrm{d}x\,B^a(x)\,B^a(x)\right\} \tag{3.25}$$

The scalar fields $B^a(x)$ will be named auxiliary, the functional integration over $\int\mathfrak{D}B$ being equivalent to replace the original Yang-Mills Lagrangian by $\mathcal{L}_{\mathrm{YM}} - (1/2\xi)\Phi^2[\,A_\mu(x)\,]$ which is no longer gauge invariant, in such a manner that perturbation theory can be applied and the Feynman rules can be obtained. Notice also that in the limit $\xi \to 0$ the Gaussian functional approaches the functional $\delta(\partial\cdot A)$, up to some $\xi-$dependent normalization factor which is irrelevant to our purposes: such a (singular) gauge choice is named the Landau gauge. As a consequence, we can eventually set the generating functional for the pure Yang-Mills theory in the two equivalent forms

$$\begin{aligned} Z_\Phi[\,J^\mu\,;\xi\,] &= \mathcal{N}'\int \mathfrak{D}A\;\det\parallel \mathsf{M}_\Phi \parallel \\ &\times\;\exp\left\{i\!\int \mathrm{d}x\left[\mathcal{L}_{\mathrm{YM}} - \frac{1}{2\xi}\,\Phi^2[\,A_\mu\,] + J^{\mu\,,\,a}A^a_\mu\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{N}'' \int \mathfrak{D}A \int \mathfrak{D}B \ \det \| \, \mathsf{M}_{\Phi} \, \| \\
&\times \ \exp\left\{ i \int \mathrm{d}x \left[\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{g.f.}} + J^{\mu\,,\,a} A_{\mu}^{a} \right] \right\}
\end{aligned}
$$

where

$$
\mathcal{L}_{\mathrm{g.f.}} = -\, B^{a}(x)\, \Phi^{a}[\, A_{\mu}(x)\,] + \tfrac{1}{2}\, \xi \, B^{a}(x)\, B^{a}(x) \tag{3.26}
$$

Examples

As a simple example of application of the above procedure, let us first consider the Abelian case of the Maxwell radiation field. Taking the non-homogeneous Lorenz gauge

$$
\Phi[\, A^{\mu}(x)\,] - B(x) \equiv \partial_{\mu} A^{\mu}(x) - B(x) = 0
$$

as the Lorentz invariant subsidiary condition one finds

$$
\{\delta\, \Phi[\, A_{\mu}\,]/\delta\, A_{\nu}(x)\} = g_{\mu\nu}\, \partial_{x}^{\mu} \qquad \nabla_{\mu}\, \omega(y) = \partial_{\mu}\, \omega(y)
$$

that yields

$$
\mathsf{M}(x,y) = -\, \Box_{x}\, \delta(x-y) \tag{3.27}
$$

which shows that the Faddeev-Popov determinant is independent from the vector potential $A_{\mu}(x)$ and can be included into the overall normalization constant. Then the generating functional becomes

$$
\begin{aligned}
Z_{0}[\, J^{\mu}\,; \xi\,] \ &= \ \mathcal{N} \det \| -\Box \, \| \int \mathfrak{D}A \int \mathfrak{D}B \\
&\times \ \exp\left\{ i \int \mathrm{d}x \, [\, \mathcal{L}_{0} + \mathcal{L}_{\mathrm{g.f.}} + J^{\mu} A_{\mu}] \right\} && (3.28) \\
&= \ \mathcal{N}' \int \mathfrak{D}A \ \exp\left\{ i \int \mathrm{d}x \left[\, \mathcal{L}_{0} + \mathcal{L}'_{\mathrm{g.f.}} + J^{\mu} A_{\mu} \right] \right\} && (3.29)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}_{0} \ &= \ -\tfrac{1}{4}\, F^{\mu\nu}(x)\, F_{\mu\nu}(x) && (3.30) \\
\mathcal{L}_{\mathrm{g.f.}} \ &= \ A^{\mu}(x)\, \partial_{\mu} B(x) + \tfrac{1}{2}\, \xi\, B^{2}(x) && (3.31) \\
\mathcal{L}'_{\mathrm{g.f.}} \ &\doteq \ \frac{1}{2\xi}\, A^{\mu}(x)\, \partial_{\mu}\, \partial_{\nu}\, A^{\nu}(x) && (3.32)
\end{aligned}
$$

which is the well known free generating functional of the radiation field in the linear Lorentz invariant gauge that eventually leads to the generally covariant photon propagator

$$
\widetilde{D}_{F}^{\mu\nu}(k\,;\xi) = \frac{i}{k^{2}+i\varepsilon} \left[\left(-\, g^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{k^{2}+i\varepsilon} \right) \ - \ \xi\, \frac{k^{\mu}k^{\nu}}{k^{2}+i\varepsilon} \right] \tag{3.33}
$$

As a further example of kinematic Faddeev-Popov determinant, consider the so called algebraic non-covariant auxiliary conditions in the non-Abelian case of the gauge group $SU(N)$, *viz.*,

$$\Phi^{a}[\,A_{\mu}(x)\,] \equiv n^{\mu}\,A_{\mu}^{a}(x) - B^{a}(x) = 0 \qquad n^{2} \leq 0 \qquad a = 1,\ldots,N^{2}-1 \tag{3.34}$$

where n^{μ} is a fixed space-like or null four-vector which does not change under Lorentz transformations, in such a manner of breaking thereby both the manifest Lorentz and gauge invariances [1]. We have

$$\delta\,\Phi^{a}[\,A^{\omega}\,]/\delta\,\omega^{b}(x)\big\rfloor_{\,\omega=0} = n^{\mu}\,\delta_{b}^{a}$$

so that

$$n^{\mu}\,\delta_{b}^{a}\,\nabla_{\mu}^{bc}\,\omega^{c}(y) = n^{\mu}\,\partial_{\mu}\,\omega^{a}(y) \quad \text{for} \quad n^{\mu}\,A_{\mu}^{a}(x) = B^{a}(x)$$

that yields

$$\|\,\mathsf{M}(x,y)\,\|_{\,ab} = n^{\mu}\,\partial_{\mu\,,\,x}\,\delta(x-y)\,\delta_{\,ab} \tag{3.35}$$

which shows once again that the Faddeev-Popov determinant is independent from the vector potential $A_{\mu}^{a}(x)$ and can be included as well into the overall normalization constant, even in the non-Abelian case, at the price of giving up the manifest Lorentz invariance. It follows that the generating functional for the pure Yang-Mills theory in the algebraic non-covariant gauges takes the form

$$\begin{aligned} Z_{\Phi}[\,J^{\mu}\,;\xi\,] \;&=\; \det\|\,n\cdot\partial\,\|\,\mathcal{N}'\int\mathfrak{D}A \\ &\times\;\exp\left\{i\int\mathrm{d}x\,\left[\,\mathcal{L}_{\mathrm{YM}} - \mathrm{tr}\,(n^{\mu}\,A_{\mu})^{2} + J^{\mu\,,\,a}A_{\mu}^{a}\right]\right\} \end{aligned} \tag{3.36}$$

that drives to the Feynman vector propagator

$$\widetilde{D}_{ab}^{\mu\nu}(k\,;n) = \frac{i\,\delta_{ab}}{k^{2}+i\varepsilon}\left(-\,g^{\mu\nu} + \frac{k^{\mu}\,n^{\nu}+k^{\nu}\,n^{\mu}}{[\,n\cdot k\,]} + \frac{n^{2}}{[\,n\cdot k\,]^{2}}\;k^{\mu}k^{\nu}\right) \tag{3.37}$$

where the prescriptions for the so called spurious singularities $1/[\,n\cdot k\,]$ and $1/[\,n\cdot k\,]^{2}$ must be selected in a suitable way – see A. Bassetto, G. Nardelli, R. Soldati, *Yang-Mills Theories in Algebraic Non-Covariant Gauges* (Canonical Quantization and Renormalization) World Scientific, Singapore, 1991.

[1] The algebraic non-covariant gauges, which include the axial gauge, the light-cone gauge and the planar gauge, have been widely employed and studied in the Literature. For an exhaustive treatment of the quantization and renormalization of the gauge theories in algebraic non-covariant gauges see the textbook [3].

3.1.4 Feynman Rules in Covariant Gauges

Consider now the non-Abelian Lorenz gauge class

$$\Phi^a[\,A_\mu(x)\,] \equiv \partial^\mu A_\mu^a(x) = B^a(x) \qquad [\,a = 1,\ldots,N^2-1\,] \tag{3.38}$$

which leads to the Faddeev-Popov operator

$$\begin{aligned}\|\,\mathsf{M}(x,y)\,\|^{ab} &= \partial_x^\mu\,\nabla_{\mu,y}^{ab}\,\delta(x-y) = \nabla_{\mu,y}^{ab}\,\partial_x^\mu\,\delta(x-y)\\ &= \left[\,-\Box_x\,\delta^{ab} - gf^{abc}A_\mu^c(x)\,\partial_x^\mu\,\right]\delta(x-y)\\ &= -\,\nabla_{\mu,x}^{ba}\,\partial_x^\mu\,\delta(x-y)\end{aligned} \tag{3.39}$$

which is no longer kinematical for it depends upon the non-Abelian vector potentials $A_\mu^c(y)$. The standard tool to deal with the presence of a non-trivial Faddeev-Popov determinant $\det\|-\partial^\mu\,\nabla_\mu\| = \det\|-\nabla_\mu\,\partial^\mu\|$ is to express it as a functional integral over a pair of independent Lorentz scalar Graßmann valued field multiplets $\eta^a(x)$ and $\overline{\eta}^a(x)$, which transform according to the adjoint representation of the gauge group $SU(N)$, called the Faddeev-Popov ghost and anti-ghost fields respectively. Then we can write

$$\begin{aligned}\det\|-\partial^\mu\,\nabla_\mu\| &= \int\mathfrak{D}\,\eta\int\mathfrak{D}\,\overline{\eta}\,\exp\left\{-\textstyle\int\mathrm{d}x\,\overline{\eta}^a(x)\,\partial^\mu\,\nabla_\mu^{ab}\,\eta^b(x)\right\}\\ &\doteq \int\mathfrak{D}\,\eta\int\mathfrak{D}\,\overline{\eta}\,\exp\left\{\,\textstyle\int\mathrm{d}x\,\partial^\mu\,\overline{\eta}^a\,\nabla_\mu^{ab}\,\eta^b\right\}\end{aligned} \tag{3.40}$$

where $\doteq$ means up to an irrelevant boundary term, whereas we understand as usual

$$\int\mathfrak{D}\,\eta\int\mathfrak{D}\,\overline{\eta} \equiv \prod_{a,b=1}^{N^2-1}\,\prod_{x\in\mathcal{M}}\textstyle\int\mathrm{d}\eta_x^a\int\mathrm{d}\overline{\eta}_x^b$$

Putting altogether the previous equality for $Z_{\mathrm{YM}}[\,J\,]$ and $\det\|-\partial^\mu\,\nabla_\mu\|$ and after including the spinor multiplet [2] too into the game, we can eventually write the generating functional of the non-Abelian gauge theories in the linear covariant gauges: namely,

$$\begin{aligned}Z_{\mathrm{YM}}[\,J,\zeta,\overline{\zeta},\beta,\overline{\beta}\,] &= \mathcal{N}\int\mathfrak{D}A\int\mathfrak{D}\,\psi\int\mathfrak{D}\,\overline{\psi}\int\mathfrak{D}B\int\mathfrak{D}\,\eta\int\mathfrak{D}\,\overline{\eta}\\ &\times\ \exp\left\{i\int\mathrm{d}x\,[\,\mathcal{L}_{\mathrm{eff}} + \mathcal{L}_{\mathrm{sources}}\,]\right\}\end{aligned} \tag{3.41}$$

[2] The spinor multiplet $\psi(x)$ of Dirac fields transforms according to the fundamental representation N of the gauge group $SU(N)$, while $\overline{\psi}(x)$ belongs to the other nonequivalent conjugated representation $\overline{\mathsf{N}}$.

where

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{FP}} && (3.42)\\
\mathcal{L}_{\text{YM}} &= -\tfrac{1}{2}\,\mathrm{tr}\,[\,F_{\mu\nu}(x)\,F^{\mu\nu}(x)\,] + \overline{\psi}(x)\,(i\not{\mathcal{D}} - M)\,\psi(x) && (3.43)\\
\mathcal{L}_{\text{g.f.}} &= \partial^{\mu} B^{a}(x)\,A^{a}_{\mu}(x) + \tfrac{1}{2}\,\xi\,B^{a}(x)\,B^{a}(x) && (3.44)\\
\mathcal{L}_{\text{FP}} &= -\,i\,\partial^{\mu}\,\overline{\eta}^{\,a}(x)\,\nabla^{ab}_{\mu}\,\eta^{b}(x) && (3.45)\\
\mathcal{L}_{\text{sources}} &= J^{\mu\,,\,a}A^{a}_{\mu} + \overline{\psi}\,\zeta + \overline{\zeta}\,\psi + \overline{\eta}^{\,a}\,\beta^{\,a} + \overline{\beta}^{\,a}\,\eta^{\,a} && (3.46)
\end{aligned}
$$

where A^{a}_{μ} and $J^{\mu\,,\,a}$ are real functions while all other fields and sources are Graßmann valued functions over the Minkowski space. As already noticed, one can functionally integrate over the auxiliary scalar field multiplet B^{a} and get the gauge fixing Lagrangian

$$
\mathcal{L}'_{\text{g.f.}} = \frac{1}{2\xi}\,A^{\mu}(x)\,\partial_{\mu}\,\partial_{\nu}\,A^{\nu}(x) \qquad (3.47)
$$

Once again the effective Lagrangian can be split into free and interaction parts involving all the gauge, spinor and ghost fields: namely,

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} &= \mathcal{L}_{0} + \mathcal{L}_{\text{int}} && (3.48)\\
\mathcal{L}_{0} &= -\,\tfrac{1}{2}\,g^{\mu\rho}\,g^{\nu\sigma}\Big(\partial_{\mu}\,A^{a}_{\nu} - \partial_{\nu}\,A^{a}_{\mu}\Big)\,\partial_{\rho}\,A^{a}_{\sigma}\\
&+\ \tfrac{1}{2}\,\xi^{-1}\,A^{\mu}\,\partial_{\mu}\,\partial_{\nu}\,A^{\nu} + \bar{\psi}\,(i\not{\partial} - M)\,\psi - i\,\partial^{\mu}\,\overline{\eta}^{\,a}\,\partial_{\mu}\,\eta^{\,a} && (3.49)\\
\mathcal{L}_{\text{int}} &= -\,i\,g\,f^{abc}\,\eta^{\,a}(x)\,A^{c}_{\mu}(x)\,\partial^{\mu}\,\overline{\eta}^{\,b}(x) + g\,\bar{\psi}(x)\,\gamma^{\mu}\,\boldsymbol{\tau}^{\,a}_{F}\,\psi(x)\,A^{a}_{\mu}(x)\\
&-\ g\,f^{abc}\,g^{\rho\mu}\,A^{b}_{\rho}(x)\,A^{c}_{\sigma}(x)\,\partial_{\mu}\,A^{a}_{\nu}(x)\,g^{\nu\sigma}\\
&-\ \tfrac{1}{4}\,g^{2}\,f^{abc}\,f^{ade}\,g^{\mu\rho}\,g^{\nu\sigma}\,A^{b}_{\mu}(x)\,A^{c}_{\nu}(x)\,A^{d}_{\rho}(x)\,A^{e}_{\sigma}(x) && (3.50)
\end{aligned}
$$

Now it is a simple and standard exercise to obtain the Feynman rules for the non-Abelian Yang-Mills theories in a linear covariant gauge. From the free part of the effective Lagrangian one gets the causal Green's functions

$$
\begin{aligned}
\widetilde{D}^{\mu\nu}_{ab}(k\,;\xi) &= \frac{i\,\delta_{ab}}{k^{2}+i\varepsilon}\left[\left(-\,g^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{k^{2}+i\varepsilon}\right) - \ \xi\,\frac{k^{\mu}k^{\nu}}{k^{2}+i\varepsilon}\right] && (3.51)\\
\widetilde{D}_{ab}(k) &= \frac{\delta_{ab}}{k^{2}+i\varepsilon} \qquad (\,a,b = 1,\ldots,N^{2}-1\,) && (3.52)\\
\widetilde{S}^{F}_{\imath\jmath}(\,p\,;M) &= \frac{i\,(\,\delta_{\imath\jmath}\,\not{p} + M_{\imath\jmath}\,)}{p^{2}-M^{2}+i\varepsilon} \qquad (\,\imath,\jmath = 1,\ldots,N\,) && (3.53)
\end{aligned}
$$

while from the interaction part of the effective Lagrangian we readily obtain the vertex involving a vector boson and a spinor fermion-antifermion pair

$$
V^{a,\mu}_{\imath\jmath}(k,p,q) \ = \ ig\,\gamma^{\mu}\ (\boldsymbol{\tau}^{\,a}_{F})_{\imath\jmath} \qquad (\,k+p-q=0\,) \qquad (3.54)
$$

Figure 3.1: the propagator $\widetilde{D}^{\mu\nu}_{ab}(k\,;\xi)$ for non-Abelian vector particles.

Figure 3.2: the propagator $\widetilde{D}_{ab}(k)$ for Faddeev-Popov ghost particles.

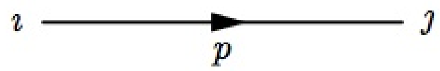

Figure 3.3: the propagator $\widetilde{S}^{F}_{\imath\jmath}(\,p\,;M)$ for non-Abelian spinor particles: the indexes $\imath,\jmath$ refer to the N Dirac spinors in the fermion multiplet belonging to the fundamental representation of $SU(N)$.

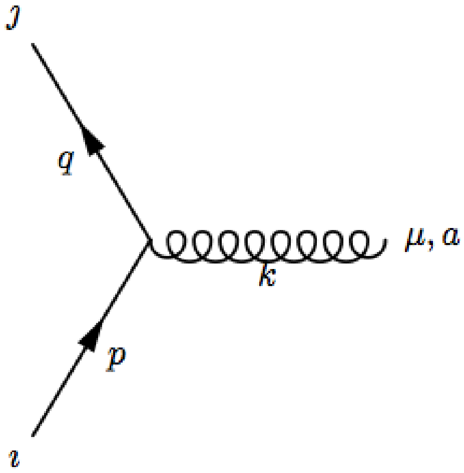

Figure 3.4: the matter fermion-antifermion vector boson vertex $V_{\imath\jmath}^{a,\mu}(k,p,q)$ for non-Abelian gauge theories.

and the Faddeev-Popov ghost-antighost-vector boson vertex

$$V_{\mu}^{\,cab}(p,q) \;=\; gf^{\,abc}\, iq_{\mu} \qquad (\,k+p-q=0\,) \tag{3.55}$$

with the conventional conservation of the non-Abelian charge flux as well as all the energy-momentum vectors being understood as incoming towards the vertexes. Moreover, the Bose-Einstein completely symmetric three and four vector particles elementary vertexes will be given by

$$\begin{aligned}
&V_{\mu\nu\rho}^{abc}(k,p,q) \;=\\
&gf^{\,abc}\,[\,(k-p)_{\rho}\,g_{\mu\nu}+(p-q)_{\mu}\,g_{\nu\rho}+(q-k)_{\nu}\,g_{\mu\rho}\,]\\
&(\,k+p+q=0\,) \qquad [\,\text{three vector boson vertex}\,]
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
&V_{\mu\nu\rho\sigma}^{abcd}(k,p,q,r) \;=\\
&-ig^{2}\,\big[\,f^{\,abp}\,f^{\,pcd}\,(g_{\mu\rho}\,g_{\nu\sigma}-g_{\mu\sigma}\,g_{\nu\rho})+f^{\,bcp}\,f^{\,pad}\,(g_{\mu\nu}\,g_{\rho\sigma}-g_{\mu\rho}\,g_{\nu\sigma})\\
&+f^{\,acp}\,f^{\,db}\,(g_{\mu\sigma}\,g_{\nu\rho}-g_{\mu\nu}\,g_{\rho\sigma})\,\big]\\
&(\,k+p+q+r=0\,) \qquad [\,\text{four vector boson vertex}\,]
\end{aligned} \tag{3.57}$$

all four-momenta being understood as incoming towards the vertexes, as usual. Of course, one has to take carefully into account the (-1) factor for loops involving only spinor propagators and ghost propagators, as well as non-trivial weight and symmetry factors for loops involving only vector particles.

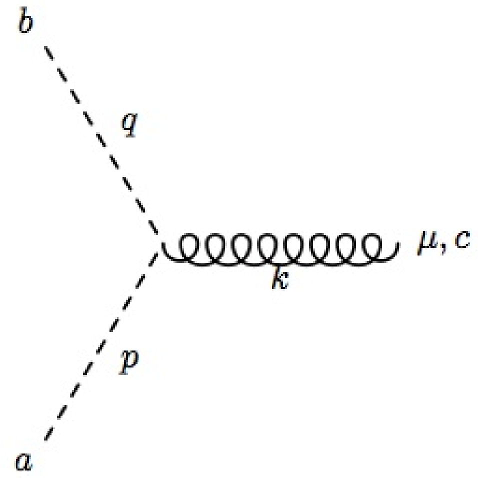

Figure 3.5: the Faddeev-Popov ghost-antighost-vector vertex $V_{\mu}^{cab}(p,q)$.

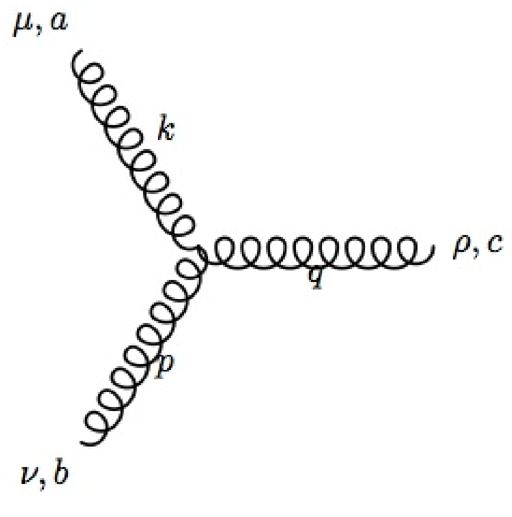

Figure 3.6: the three vector boson vertex $V_{\mu\nu\rho}^{abc}(k,p,q)$.

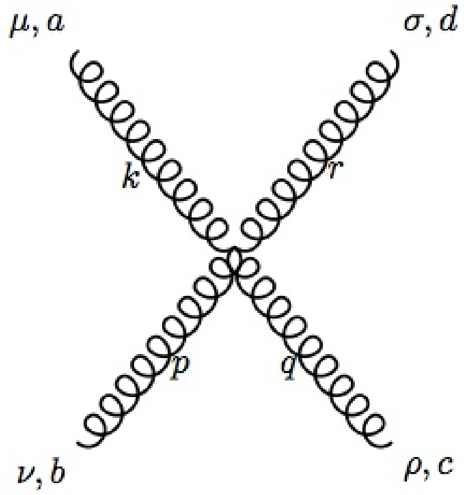

Figure 3.7: the four vector boson vertex $V^{\,abcd}_{\mu\nu\rho\sigma}(k,p,q,r)$.

3.1.5 Becchi-Rouet-Stora-Tyutin Symmetry

It is convenient to write the effective Lagrangian in the form

$$\mathcal{L}_{\,\mathrm{eff}} = -\,\tfrac{1}{4}\,F_{\mu\nu}\cdot F^{\mu\nu} + \overline{\psi}\,(i\not{D} - M)\,\psi + \partial^{\,\mu}B\cdot A_{\mu} + \tfrac{1}{2}\,\xi\,B\cdot B - i\,\partial^{\,\mu}\,\overline{\eta}\cdot\nabla_{\mu}\,\eta$$

where, in order to avoid too many group indices, I have suitably introduced the internal and external products $A\cdot B = A^{a}B^{a}$ and $A\times B = f^{abc}A^{b}B^{c}$ respectively. The above effective Lagrangian is invariant under a remarkable set of continuous symmetry transformations: the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [Carlo Maria Becchi, Alain Rouet & Raymond Stora (1974) *The abelian Higgs Kibble model, unitarity of the S-operator* Physics Letters **52B**, 344] its infinitesimal transformations being given by

$$\delta A_{\mu}(x) = \lambda\,\nabla_{\mu}\,\eta(x) \tag{3.58}$$

$$\delta\eta(x) = -\,\tfrac{1}{2}\,g\,\lambda\,\eta(x)\times\eta(x) \tag{3.59}$$

$$\delta\,\overline{\eta}(x) = i\lambda\,B(x) \qquad\qquad \delta B(x) = 0 \tag{3.60}$$

where λ is an $x-$independent Graßmann number anti-commuting with the FP ghosts fields $\eta(x)$ and $\overline{\eta}(x)$ and commuting with the gauge potentials and auxiliary scalar fields. From Noether's theorem we get the current density four vector

$$\jmath_\mu(x) = B(x) \cdot \nabla_\mu \eta(x) - \eta(x) \cdot \partial_\mu B(x) + \tfrac{1}{2}\, ig\, \partial_\mu \overline{\eta}(x) \cdot \eta(x) \times \eta(x) \tag{3.61}$$

that leads to the conserved charge

$$Q_{\,\mathrm{BRST}} = \int \mathrm{d}\mathbf{x} \left[\, B(\mathbf{x}) \cdot \nabla_0\, \eta(\mathbf{x}) - \eta(\mathbf{x}) \cdot \partial_0 B(\mathbf{x}) + \tfrac{1}{2}\, ig\, \partial_0\, \overline{\eta}(\mathbf{x}) \cdot \eta(\mathbf{x}) \times \eta(\mathbf{x}) \,\right] \tag{3.62}$$

and to the subsidiary condition

$$Q_{\,\mathrm{BRST}}\, |\, \mathrm{phys}\, \rangle = 0 \tag{3.63}$$

which actually select a physical Hilbert subspace of positive semi-definite metric out of the indefinite metric Fock space involving any kind of quanta. The subsidiary condition (3.63) is the suitable non-Abelian generalization of the Nakanishi-Lautrup subsidiary condition

$$B^{(-)}(x)\, |\, \mathrm{phys}\, \rangle = 0 \tag{3.64}$$

to select the physical Hilbert subspace of the indefinite metric Fock space of states of the Abelian case.

From the invariance of the effective Action under the BRST symmetry, one can derive the generalization of the Ward's identities of the Abelian case, called the Slavnov-Taylor identities, which are fundamental in order to prove the renormalizability of the non-Abelian gauge theories. Once this is done, it is possible to calculate the 1-loop $\beta-$function which turns out to be

$$\beta(\,g) = \frac{g^3}{16\pi^2} \left[-\,\frac{11}{3}\, C_A + \frac{4}{3}\, C_F \right] + O(g^5) \tag{3.65}$$

where the so called Dynkin index C_R for the generic irreducible representation R of $SU(N)$ is defined by

$$\delta^{\,ab}\, C_R = \mathrm{tr}\,(\boldsymbol{\tau}^{\,a}_{\,R}\, \boldsymbol{\tau}^{\,b}_{\,R})$$

Now for the realistic case of $SU(3)_c$, which corresponds to color group of quantum chromodynamics, the gauge theory of strong interactions, it turns out that

$$C_A = 3 \qquad \text{for} \quad SU(3) \tag{3.66}$$

$$C_F = \tfrac{1}{2} \qquad \text{for each Dirac spinor} \tag{3.67}$$

$$\beta(\,g) \;=\; \frac{g^3}{16\pi^2} \left[-\,\frac{11}{3} \cdot 3 + \frac{4}{3} \cdot \tfrac{1}{2}\, \mathrm{N_f} \right] + O(g^5) \tag{3.68}$$

which endorses the asymptotic freedom iff the number N_f of quark flavours is equal to six, since $\mathrm{f} = u, d, s, c, b, t$, *i.e.*

$$\beta(g) = -\frac{7g^3}{16\pi^2} + O(g^5) \tag{3.69}$$

Notice that asymptotic freedom holds true up to the presence of eight families of quarks since $\beta(g) < 0 \Leftrightarrow N_f \leq 16$.

3.2 The Standard Model

A quite popular belief tells us that any great progress in Physics occurs when some totally unexpected experimental result does contradict any previously well established theoretical setting up. A paradigmatic and renowned example is the 1887 experiment by Albert Abraham Michelson and Edward Williams Morley that paved the road towards a totally new theoretical framework, the Special Theory of Relativity. Richard Phillips Feynman said: *Progress in Physics is to prove yourself wrong as soon as possible.* The construction of the Standard Model does instead represent a remarkable exception, maybe to confirm the rule.

As a matter of fact, in the late sixties weak interactions were pretty well described by the generalization, called V-A theory, of the current-current Fermi theory[3] of $\beta-$decay and there was no compelling experimental evidence to modify that beautiful and worthy Hamiltonian model. The reasons that pushed to go beyond the V-A theory of the weak interactions were purely theoretical. On the one hand, the V-A theory at high energies - much higher of the rest mass of the nucleons - does violate the unitarity of the Heisenberg scattering operator, which is one of the basic principles of orthodox quantum mechanics. On the other hand, even more important and crucial, the V-A theory is in fact only a phenomenological model and not a true quantum field theory because, in the technical language, it is non-renormalizable. In other words this means that any attempts to compute higher order corrections in powers of the Fermi coupling constant G_F, within the standard quantum mechanical perturbation theory, give rive to meaningless incurable divergent results. Thus, the motivation was æstethic rather than experimental, it was the search of mathematical consistency and theoretical elegance and, to say the truth, at the very beginning of the seventies the data did not seem at all to support those theoretical speculations. But soon after the mid seventies, it started to emerge so strikingly that the gauge principle - *i.e.* the setting up of a

[3]Enrico Fermi (1934) *Tentativo di una teoria dei raggi β*, Il Nuovo Cimento **11** (1) pp.1-19; *Versuch einer Theorie der beta-Strahlen*, Zeitschrift für Physik **88**, p.161.

quantum field theory based upon the invariance under a local and continuous symmetry group - was the true milestone towards the construction of a fully mathematically sound and consistent, as well as highly predictive, quantum field theory: the Standard Model of Electro-Weak and Strong Interactions.

3.2.1 Weak Interactions Before Gauge Theories

On 4 December 1930, Wolfgang Ernst Pauli wrote a famous letter to the Physical Institute of the Federal Institute of Technology, Zürich, in which he proposed the electron neutrino as a potential solution to solve the problem of the continuous beta decay spectrum, as discovered in 1911 by Lise Meitner and Otto Hahn. Soon after the Pauli proposal, during the 1931 international conference in Rome, of a new hypothetical particle called the neutrino, Enrico Fermi proposed his Hamiltonian theory for the $\beta-$decay $n \longrightarrow\ p+e+\bar{\nu}$

$$\mathcal{H}(x) = \frac{G_F}{\hbar^2\sqrt{2}}\left[\,\bar{p}(x)\gamma_\lambda n(x)\,\right]\left[\,\bar{\nu}(x)\gamma^\lambda e(x)\,\right] + \text{c.c.} \tag{3.70}$$

where the fermion field operators are denoted by their particle names, while $G_F/(\hbar c)^3 = 1.1663787(6)\times 10^{-5}\ \text{GeV}^{-2}$ is the Fermi coupling constant, in such a manner that the Hamiltonian density $\mathcal{H}(x)$ has the expected canonical engineering dimensions of $[\mathcal{H}] = \text{eV cm}^{-3}$. The mean lifetime of a free neutron is pretty long $\tau = 880.3 \pm 1.1$ s, about 15 min, and in the ensuing years other decay processes have been discovered, such as the semi-leptonic charged pions decay $\pi^\pm \longrightarrow\ \mu^\pm + \nu_\mu$ and the leptonic muon decay $\mu \longrightarrow\ e + \bar{\nu}_e + \nu_\mu$, with a much shorter mean lifetime of about 2μs.

Hence the concept of a new distinctive class of interactions, called the weak interactions, began to emerge and the truly surprising discovery of the parity non-conservation[4] stimulated a great deal of research that eventually lead at the end of fifties to the formulation of the so called V-A theory[5] by Feynman-Gell-Mann, Marshak-Sudarshan and Sakurai. They suggested that an effective Hamiltonian density operator, quite close to the original one of equation (3.70), does actually describe the weak interactions: namely,

$$\mathcal{H}(x) = \frac{G_F}{\sqrt{2}}\, J_\lambda^\dagger(x)\, J^\lambda(x) + \text{H.c.} \tag{3.71}$$

[4]Tsung-Dao Lee & Chen-Ning Franklin Yang (1956) *Question of Parity Conservation in Weak Interactions*, Phys. Rev. **104**, 254; M.me Chien-Shiung Wu, E. Ambler, R.W. Hayward, D.D. Hoppes, R.P. Hudson (1957) *Experimental Test of Parity Conservation in Beta Decay*, Phys. Rev. **105**, 1413-1415.

[5]Richard P. Feynman & Murray Gell-Mann (1958) *Theory of the Fermi Interaction*, Phys. Rev. **109**, 193; E.C.G. Sudarshan & R.E. Marshak (1958) *Chirality Invariance and the Universal Fermi Interaction*, Phys. Rev. **109**, 1860; Jun John Sakurai (1958) *Mass Reversal and Weak Interactions*, Il Nuovo Cimento **7**, 649.

where the weak current $J_\lambda(x)$ takes the vector-minus-axial form, *i.e.* the V-A theory. If one separates in the weak current the leptonic and hadronic parts

$$J^\lambda(x) = J^\lambda_\ell(x) + J^\lambda_{\mathrm{h}}(x) \tag{3.72}$$

then the leptonic current can be written directly in terms of the solely lepton fields known at that time

$$J^\lambda_\ell(x) = \bar{\nu}_e(x)\gamma^\lambda(1-\gamma_5)\,e(x) + \bar{\nu}_\mu(x)\gamma^\lambda(1-\gamma_5)\,\mu(x) \tag{3.73}$$

while the hadronic current can be written in terms of the quark fields

$$J^\lambda_{\mathrm{h}}(x) = \bar{u}(x)\gamma^\lambda(1-\gamma_5)[\,\cos\theta_c\,d(x) + \sin\theta_c\,s(x)\,] \tag{3.74}$$

where θ_c is the Cabibbo[6] angle of about $13°$, while u, d, s were the quark fields known before the discovery of the charm, the fourth quark flavor, the existence of which was already postulated[7] by Bjorken and Glashow in 1964. Actually, the weak interaction lepton-hadron symmetry and the comparison with equation (3.73) then suggests the generalization

$$J^\lambda_{\mathrm{h}}(x) = \bar{u}(x)\gamma^\lambda(1-\gamma_5)d_\theta(x) + \bar{c}(x)\gamma^\lambda(1-\gamma_5)s_\theta(x) \tag{3.75}$$

where $c(x)$ is the new heavy quark, the charmed quark, whereas

$$\begin{cases} d_\theta(x) = \cos\theta_c\,d(x) + \sin\theta_c\,s(x) \\ s_\theta(x) = -\sin\theta_c\,d(x) + \cos\theta_c\,s(x) \end{cases} \tag{3.76}$$

It turns out that any sensible weak interaction theory must exhibit this extra hadronic current-current term, in order to suppress to an acceptable level the induced flavor-changing neutral current effects (GIM[8] mechanism). As we shall see below, this suppression mechanism, although invented before the wide and general acceptance of gauge theories in weak interactions, can be realized and explained at best within the context of the Standard Model.

Before turning to the Standard Model, it is useful to list some of the basic features of weak interactions, as described by the V-A generalization of the original Fermi theory and encoded in equations (3.73) and (3.75).

---

[6]Nicola Cabibbo (1963) *Unitary Symmetry and Leptonic Decays*, Phys. Rev. Lett. **10**, 531-533.

[7]James Daniel Bjorken & Sheldon Lee Glashow (1964) *Elementary Particles and SU(4)*, Phys. Lett. **11**, 255.

[8]Sheldon Lee Glashow, Ioànnis Iliopoulos and Luciano Maiani (1970) *Weak Interactions with Lepton-Hadron Symmetry*, Phys. Rev. D **2**, 1285.

1. The axial-vector currents $J_\ell^\lambda(x)$ and $J_{\mathrm{h}}^\lambda(x)$ are charged currents that transfer one unit of electric charge; hence, to the lowest order, there are no neutral-current processes such as $n + \nu_\mu \longrightarrow \nu_\mu + n$.

2. The V-A weak currents are bi-linear in the fermion fields involving the left-handed helicity projector $\mathbb{P}_L = \frac{1}{2}(\,\mathbb{I} - \gamma_5\,)$: thus, only left-handed fermions are present in the weak currents. It follows the rule that parity violation is maximal in weak interactions, which has a very simple interpretation in terms of the lepton and quark fields, for in current-current weak interactions the fundamental dynamical matter degrees of freedom are only two-component left-handed fermion fields.

3. The V-A classical effective Lagrangian
$$\mathcal{L}_{\text{eff}} = -\,\frac{G_F}{\sqrt{2}}\,J^\lambda(x)J_\lambda(x) + \text{c.c} \tag{3.77}$$
and the corresponding Feynman rules, which involve the four-fermion interaction vertexes, are extremely successful in describing a pretty wide class of low-energy experimental data and phenomena, provided we restrict ourselves to the leading order in the Fermi constant. The exceptions are few: it is not clear whether the non-leptonic weak decays are correctly described by the coupling $-G_F J_{\mathrm{h}}^\lambda(x)J_{\mathrm{h}\lambda}(x)/\sqrt{2}$, owing to our inability to perform reliable non-perturbative calculations to take into account the strong interaction effects which recombine quarks to yield hadrons in the final states.

4. The $\mathcal{CP}$ violation that was discovered in the K-meson system[9] can not be included in the V-A theory in any simple and convincing fashion. A successful phenomenological model endorsing the $\mathcal{CP}$ violation has been put forward by Wolfenstein[10], who postulated the existence of a new *ad hoc* super-weak interaction.

In short, the V-A generalization of the original Fermi theory does correctly describe the domain of weak interaction phenomena, within the range in which it is expected to be applicable. In other words, it can be trusted as an effective Lagrangian or Hamiltonian phenomenological model, up to the lowest order in the Fermi constant. Nonetheless the Lagrange density (3.77) can not at all be trusted as a physically reliable and mathematically consistent quantum field theory of the weak interactions because:

[9] J.H. Christenson, J.W. Cronin, V.L. Fitch and R. Turlay (1964) *Evidence for 2 pi Decay of the K(2)0 Meson*, Phys. Rev. Lett. **13**, 138.

[10] Lincoln Wolfenstein (1964) *Violation of the CP Invariance and the Possibility of Very Weak Interactions*, Phys. Rev. Lett. **13**, 352.

- it is non-renormalizable by power counting;
- even at the lowest order in G_F it violates unitarity of the $S-$matrix for certain processes at sufficiently high energies of about 300 GeV. This can be readily gathered by considering, for example, the reaction $\nu_\mu e^- \longrightarrow \mu^- \nu_e$ as described by the effective Lagrangian (3.77). On the one hand, at sufficiently high energies, when the electron and muon rest masses can be disregarded, the cross section will be proportional to $\sigma \propto s\, G_F^2$, where $s \cong 2m_e E$, E being the ν_μ energy in the electron rest frame. On the other hand, unitarity requires, just by dimensional reasons, that when energy increases and all masses can be disregarded, then $\sigma \propto (\hbar c)^2/s$. Thus, for energies above $\sqrt{s} \simeq G_F^{-1/2} \approx 300$ GeV the theoretical cross section from (3.77) does violate unitarity of the scattering matrix.

It turns out that those two troubles are closely and deeply related one each other. If the lowest order scattering amplitudes have a bad high energy behavior that violates unitarity, *i.e.* probability conservation in quantum mechanics, then the divergences in the higher order radiative corrections can not be systematically and safely removed to all order in perturbation theory. The ultimate reason for this breakdown is that the Fermi constant is not a (small) number but it carries on nontrivial dimensions, since it is proportional[11], in natural units, to the inverse square of the charged vector boson mass $M_W = 80.385 \pm 0.015$ GeV/c^2. The same obstacle does actually occur, *mutatis mutandis*, in the hopeless attempt to setting up a reliable perturbative approach to quantum gravity, the coupling being the inverse square of the Planck mass

$$M_P = \sqrt{\hbar c/G_N} = 1.22093(7) \times 10^{19}\ \mathrm{GeV}/c^2$$

It follows that the ratio between the electro-weak and quantum gravity scales G_F/G_N is of about 34 orders of magnitude, which means that any quantum gravity phenomenology is very far away from the present and future realm of experimental investigations.

3.2.2 Preliminary Building Program

In the sequel I will essentially follow the Lecture Notes of Ioannis Jean John Iliopoulos, *Introduction to the STANDARD MODEL of the Electro-Weak Interactions*, Lectures given at the 2012 CERN Summer School, June 2012, Angers (France) arXiv:1305.6779v1 [hep-th].

[11] As we shall see later on, it turns out that the following relationships hold true: namely, $G_F/\sqrt{2} = g^2/8M_W^2 = 1/2v^2$, with $v \simeq 246$ GeV/c^2.

- Step 1: the so called Standard Model for the weak and electromagnetic interactions is empirically based upon the non-semi-simple and unitary gauge group

$$G = SU(2) \times U(1) \doteqdot U(2)$$

where $\doteqdot$ means isomorphic as continuous Lie groups. The non-Abelian gauge vector bosons $V_\nu^a \quad (a = 1, 2, 3)$ and the Abelian vector gauge field B_μ, respectively for the SU(2) and U(1) factors, do eventually describe and carry on the exchange of the weak and electromagnetic interactions, with the corresponding dimensionless coupling g and g'.

As we know, the number as well as the interaction properties of the gauge vector bosons are established by the gauge group, see [22] §**2.5**. This is no longer true for the matter fields: in principle, they might appear in any number and transform according to any representation of the gauge group. Hence, the special choice that will be described in the next Step 2 is merely dictated by phenomenology.

- Step 2: the left-handed components of the lepton fields are elementary doublets in one of the fundamental representations of SU(2) - the same for all the three lepton species - denoted by

$$\Psi_\iota = \begin{pmatrix} \nu_\iota \\ \ell_\iota^- \end{pmatrix} \qquad \ell_\iota^- = \tfrac{1}{2}(\mathbb{I} - \gamma_5)\,\ell_\iota \qquad (\iota = 1, 2, 3) \tag{3.78}$$

Sometimes the lepton family index is equivalently understood to run over the particle labels $(\iota = e, \mu, \tau)$ instead of $(\iota = 1, 2, 3)$. The right-handed lepton fields

$$\ell_\iota^+ = \tfrac{1}{2}(\mathbb{I} + \gamma_5)\,\ell_\iota \qquad (\iota = 1, 2, 3) \tag{3.79}$$

are SU(2) singlets, *i.e.* they transform according to the 1-dimensional trivial representation of SU(2). Actually, to be complete, we can also add the right-handed components $\nu_{\iota\mathrm{R}}(x)$ for the neutrino fields as SU(2) singlets, the presence of which, however, is still to be confirmed by the experimental data. I shall drop them for the moment but I may come back later to this subject.

In the classical Action of the Standard Model there are three lepton families and one single complex, *i.e.* charged, Higgs doublet

$$\phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

- Step 3: the SU(2) Hermitean generators are provided by

$$T_{\pm} = \sum_{\iota=e,\mu,\tau} \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{x}\, \bar{\psi}_\iota(t,\mathbf{x})\, \gamma^0\, \boldsymbol{\tau}_\pm\, \psi_\iota(t,\mathbf{x})$$

$$T_{+} = \sum_{\iota=e,\mu,\tau} \int \mathrm{d}\mathbf{x}\, \bar{\nu}_\iota(t,\mathbf{x})\, \gamma^0\, \ell_\iota^-(t,\mathbf{x}) \tag{3.80}$$

$$T_{-} = \sum_{\iota=e,\mu,\tau} \int \mathrm{d}\mathbf{x}\, \overline{\ell_\iota^-}(t,\mathbf{x})\, \gamma^0\, \nu_\iota(t,\mathbf{x}) = T_+^\dagger \tag{3.81}$$

$$T_{3} = \sum_{\iota=e,\mu,\tau} \int \mathrm{d}\mathbf{x}\, \bar{\psi}_\iota(t,\mathbf{x})\, \gamma^0\, \boldsymbol{\tau}_3\, \psi_\iota(t,\mathbf{x}) \tag{3.82}$$

$$= \sum_{\iota=e,\mu,\tau} \int \mathrm{d}\mathbf{x}\, \tfrac{1}{2}\, \left[\, \bar{\nu}_\iota(t,\mathbf{x})\, \gamma^0\, \nu_\iota(t,\mathbf{x}) - \overline{\ell_\iota^-}(t,\mathbf{x})\, \gamma^0\, \ell_\iota^-(t,\mathbf{x})\, \right]$$

where

$$\boldsymbol{\tau}_a = \tfrac{1}{2}\,\sigma_a \quad (\, a = 1,2,3\,) \qquad\qquad \boldsymbol{\tau}_\pm \equiv \boldsymbol{\tau}_1 \pm i\boldsymbol{\tau}_2$$

while the U(1) Hermitean generator is customarily denoted by Y , which is called hyper-charge. Hence, the electric charge operator turns out to be a linear combination of T_3 and Y

$$Q = T_3 + \tfrac{1}{2}\, Y \tag{3.83}$$

Notice that the normalization of the generators of a non-Abelian group is fixed by the commutation relations, *i.e.* the Lie algebra, while that one for the Abelian generator is arbitrary. The above relationship for the charge operator was actually the historical choice, although it might fail to be the most natural and convenient one from the group theory perspective. The ensuing transformation rules under local SU(2) and U(1) symmetry groups are

$$\begin{aligned} \Psi_\iota^\prime(x) &= \exp\{ig\omega_a(x)\boldsymbol{\tau}_a - \tfrac{1}{2}\, ig^\prime\omega(x)\}\, \Psi_\iota(x) \\ &\equiv U_\omega(x)\, \Psi_\iota(x)\, \exp\left\{-\tfrac{1}{2}\, ig^\prime\omega(x)\right\} \end{aligned} \tag{3.84}$$

$$\ell_\iota^{+\prime}(x) = \exp\{-\, ig^\prime\, \omega(x)\}\, \ell_\iota^+(x) \qquad\qquad (\, \iota, a = 1,2,3\,) \tag{3.85}$$

These transformation rules, together with the normalization (3.83) for the electric charge Q in units of the proton charge $e > 0$, lead to the following hyper-charge assignments (in units of $g^\prime > 0$)

$$Y(\Psi_\iota) = -\,1 \qquad\qquad Y(\ell_\iota^+) = -\,2 \qquad\qquad (\, \iota = 1,2,3\,) \tag{3.86}$$

Notice that if any electrically neutral right-handed neutrino existed, it would carry the null hyper-charge $Y(\nu_{\iota R}) = 0$, in such a manner that it wouldn't be coupled to any gauge vector boson.

We are left with the choice of the Higgs scalar field and we shall choose, according to Nature, the simplest possibility involving the minimal number of field degrees of freedom. Actually, we must give masses to three vector bosons and keep mass-less the fourth one, that will be identified with the photon field. By the way, we have to remind - see §1.2.3 - that for every vector boson acquiring a mass a corresponding scalar field must decouple from Dynamics. Now, since we started with four gauge vector fields, two charged and two charge-less, it follows that the minimum number of scalar field to deal with is four, again with a pair of charged ones and a pair of electrically neutral ones. Then we can naturally arrange them into a complex doublet in one of the two non-equivalent fundamental representations: namely,

$$\phi = \begin{pmatrix} \phi^{+} \\ \phi^{0} \end{pmatrix} \qquad \phi^{\dagger} = \begin{pmatrix} \phi^{-} & \phi_0^{*} \end{pmatrix} \qquad \phi^{-} = (\phi^{+})^{*} \tag{3.87}$$

with the transformation property

$$\phi(x) \quad \longrightarrow \quad \phi'(x) = \exp\{\tfrac{1}{2}\, ig'\omega(x)\}\, U_{\omega}(x)\, \phi(x) \tag{3.88}$$

the corresponding U(1) hyper-charge being $Y(\phi) = 1$ in units of g'. Notice that another nonequivalent irreducible representation of SU(2) does actually exist, called the conjugated one, wich acts on the other complex doublet

$$\widetilde{\phi}(x) = i\sigma_2\phi^{*}(x) = \begin{pmatrix} \phi_0^{*}(x) \\ -\,\phi^{-}(x) \end{pmatrix} \tag{3.89}$$

that will be necessary to the aim of extending the present construction to the hadronic sector, as we shall see later on, the hyper-charge of which is opposite, *viz.,* $Y(\widetilde{\phi}) = -\,1$ in units of g'.

Rather surprisingly, after those three basic Steps the Model is essentially complete. All further developments are merely technical and in fact uniquely determined. From the beginning, we already know - see [22] §**2.5** - that the gauge potentials, the field strengths and the covariant derivatives acting on the lepton and Higgs field are defined by

$$V^{\nu}(x) \;\; \equiv \;\; V_a^{\nu}(x)\, \boldsymbol{\tau}_a \tag{3.90}$$

$$
\begin{aligned}
D_\mu \Psi_\iota(x) &\equiv \left[\, \partial_\mu - igV_\mu(x) + \tfrac{1}{2}\, ig' B_\mu(x) \,\right] \Psi_\iota(x) && (3.91)\\
D_\mu \ell_\iota^+(x) &= [\, \partial_\mu + ig' B_\mu(x) \,]\, \ell_\iota^+(x) && (3.92)\\
D_\mu \phi(x) &\equiv \left[\, \partial_\mu - igV_\mu(x) - \tfrac{1}{2}\, ig' B_\mu(x) \,\right] \phi(x) && (3.93)\\
B_{\mu\nu}(x) &= \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) && (3.94)\\
G_{\mu\nu}(x) &= \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) - ig\, [\, V_\mu(x)\,,\, V_\nu(x)\,] && (3.95)
\end{aligned}
$$

The gauge transformations of the vector potentials are

$$
V_\mu(x) \mapsto V_\mu^\omega(x) = U_\omega(x)\, V_\mu(x)\, U_\omega^\dagger(x) - \frac{i}{g}\, [\, \partial_\mu U_\omega(x)\,]\, U_\omega^\dagger(x) \qquad (3.96)
$$

$$
B_\mu(x) \mapsto B_\mu^\omega(x) = B_\mu(x) + \partial_\mu \omega(x) \qquad (3.97)
$$

Moreover we know that the fundamental requirement of power counting renormalizability implies that all the terms appearing in the Lagrangian must be local monomials in the fields and their derivatives of canonical engineering dimensions equal to four in natural units. The unique necessary result is

$$
\begin{aligned}
\mathcal{L} &= -\tfrac{1}{2}\, \mathrm{tr}\, [\, G_{\mu\nu}\, G^{\mu\nu}\,] - \tfrac{1}{4}\, B_{\mu\nu}\, B^{\mu\nu} + (\, D^\mu \phi\,)^\dagger D_\mu \phi - \mathcal{V}(\phi)\\
&+ i\, \overline{\Psi}_\iota\, \not{D}\, \Psi_\iota + i\, \overline{\ell_\iota^+}\, \not{D}\, \ell_\iota^+ - y_\iota \left(\overline{\Psi}_\iota \cdot \phi\, \ell_\iota^+ + \text{c.c.} \right) \qquad (3.98)
\end{aligned}
$$

where the Higgs potential is necessarily given by

$$
\mathcal{V}(\phi) = -\, \mu^2\, \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi \right)^2 \qquad (3.99)
$$

while the very last term in (3.98) is a Yukawa-like coupling between the SU(2) doublets of the scalar Higgs and the left-handed lepton fields, with $y_\iota \in \mathbb{R}$ $(\, \iota = 1, 2, 3\,)$. To comment the above lepton Lagrangian some remarks are in order.

1. Notice that, in writing the above Lagrangian, the separate conservation of the three lepton numbers - one for each family - is tacitly assumed, since the whole matter Action is diagonal with respect to the lepton family index $\iota = 1, 2, 3$.

2. If right-handed neutrinos existed, then a further Yukawa-like term would be there, in which ℓ_ι^+ will be replaced by $\nu_{\iota R}$ while the Higgs scalar doublet ϕ by the related doublet $\widetilde{\phi} = i\sigma_2 \phi^*$ in the other complex conjugated 2-dimensional representation of SU(2).

3. As expected, it turns out that all the vector gauge bosons as well as all the lepton fields appear to be mass-less. This is not at all surprising, because the assumed different transformation laws for the right-handed

and left-handed lepton components do actually forbid a standard SU(2) invariant Dirac mass term or, *a fortiori*, any SU(2) invariant Majorana-like mass term for leptons. Actually, the only constant mass parameter in the Standard Model (3.98) appears to be μ^2, *i.e.* the dimensional parameter in the Higgs potential (3.99). Thus, every particle mass in the Standard Model is expected to be proportional to the absolute value of μ. The next development of the present construction will consist in taking the Higgs scalar mass term negative definite, in such a manner to give rise to the phenomenon of the spontaneous symmetry breaking and of the Higgs mechanism to generate masses.

3.2.3 Higgs Mechanism: Non-Abelian Case

In close analogy with the Abelian case (1.2.3), it turns out that the classical Hamiltonian which follows from the Lagrangian (3.98) reaches its minimum for vanishing lepton and gauge vector fields, but for non-vanishing constant values of the SU(2) Higgs scalar doublet

$$\phi_o^\dagger \phi_o = \frac{\mu^2}{2\lambda} \tag{3.100}$$

Then the U(2) gauge symmetry is spontaneously broken by selecting

$$\langle\, 0 \,|\, \phi(x) \,|\, 0 \,\rangle = \langle\, 0 \,|\, \phi(0) \,|\, 0 \,\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon \end{pmatrix} \qquad \upsilon^2 = \frac{\mu^2}{\lambda} \tag{3.101}$$

where we know nowadays[12] that $\upsilon \approx 246$ GeV. Let us now turn to the so called polar decomposition and shift the scalar doublet with respect to its minimum constant value: namely,

$$\begin{aligned} \phi(x) \;&=\; \frac{1}{\sqrt{2}}\, U_\chi(x)[\,\upsilon + H(x)\,] \begin{pmatrix} 0 \\ 1 \end{pmatrix} &(3.102)\\ U_\chi(x) \;&\equiv\; \exp\{i\chi_a(x)\boldsymbol{\tau}_a/\upsilon\} &(3.103)\end{aligned}$$

where $H(x)$ is the neutral and spin-less Higgs field, while $\chi_a(x)$ $(a = 1, 2, 3)$ are Goldstone-like neutral scalar fields that will completely disappear from the Dynamics, thanks to the Higgs mechanism outlined here below. Notice that the total number of the classical real scalar wave fields that constitute the Higgs scalar doublet is always equal to four, as it does. Moreover, we can suitably define

$$V_\mu(x) \;\equiv\; U_\chi(x)\, W_\mu(x)\, U_\chi^{-1}(x) - \frac{i}{g}\, [\,\partial_\mu U_\chi(x)\,]\, U_\chi^{-1}(x) \tag{3.104}$$

[12] See the Particle Data Group, http://pdg.lbl.gov

After the polar and shifted representation (3.103) has been introduced, the covariant derivatives of the Higgs scalar doublet becomes

$$\begin{aligned}
\sqrt{2}\,D_\mu\,\phi(x) &= \left[\,\partial_\mu - igV_\mu(x) - \tfrac{1}{2}\,ig'B_\mu(x)\,\right]\,U_\chi(x)\begin{pmatrix}0\\ \upsilon+H(x)\end{pmatrix}\\
&= \partial_\mu\left\{U_\chi(x)\begin{pmatrix}0\\ \upsilon+H(x)\end{pmatrix}\right\} - [\,\partial_\mu U_\chi(x)\,]\,U_\chi^\dagger(x)\,\phi(x)\sqrt{2}\\
&- \left[ig\,U_\chi(x)W_\mu(x)U_\chi^\dagger(x) + \tfrac{1}{2}\,ig'B_\mu(x)\right]U_\chi(x)\begin{pmatrix}0\\ \upsilon+H(x)\end{pmatrix}\\
&= U_\chi(x)\left[\,\partial_\mu - igW_\mu(x) - \tfrac{1}{2}\,ig'B_\mu(x)\,\right]\begin{pmatrix}0\\ \upsilon+H(x)\end{pmatrix}
\end{aligned}$$

which shows that the threesome Goldstone-like scalar fields $\chi_a(x)$ can be factorized into a local unitary matrix and eventually decoupled from the Action thanks to gauge invariance. As a matter of fact we have

$$\begin{aligned}
\mathcal{V}[\phi(x)] &= -\tfrac{1}{2}\,\mu^2\,[\,\upsilon+H(x)\,]^2 + \tfrac{1}{4}\,\lambda\,[\,\upsilon+H(x)\,]^4\\
&= \mu^2\,H^2(x) + \mu\sqrt{\lambda}\,H^3(x) + \tfrac{1}{4}\,\lambda\,H^4(x) - (\,\mu^4/4\lambda\,)\\
&= \mathcal{V}[H(x)] \;+\; \mathcal{V}_o
\end{aligned}\tag{3.105}$$

Moreover we notice that

$$W_\mu(x) = \frac{1}{2}\begin{pmatrix} W_\mu^3(x) & W_\mu^+(x)\sqrt{2}\\ W_\mu^-(x)\sqrt{2} & -W_\mu^3(x)\end{pmatrix}$$

where we have set

$$W_\mu^\pm = \frac{W_\mu^1 \mp iW_\mu^2}{\sqrt{2}}$$

in such a manner that we can eventually write

$$W_\mu(x)W^\mu(x) = \tfrac{1}{4}\,g^{\mu\nu}\sum_{a=1}^{3} W_\mu^a(x)W_\nu^a(x)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Then we obtain

$$\begin{aligned}
&[\,D^\mu\phi(x)\,]^\dagger D_\mu\,\phi(x) \;=\\
&\tfrac{1}{2}\begin{pmatrix}0 & 1\end{pmatrix}\left[\,\partial^\mu + igW^\mu(x) + \tfrac{1}{2}\,ig'B^\mu(x)\,\right][\,\upsilon+H(x)\,]\\
&\cdot\left[\,\partial_\mu - igW_\mu(x) - \tfrac{1}{2}\,ig'B_\mu(x)\,\right][\,\upsilon+H(x)\,]\begin{pmatrix}0\\ 1\end{pmatrix}\\
=\;& \tfrac{1}{2}\,\partial_\mu H(x)\,\partial^\mu H(x) + \tfrac{1}{2}\,[\,\upsilon+H(x)\,]^2
\end{aligned}$$

$$
\begin{aligned}
\times & \quad \begin{pmatrix} 0 & 1 \end{pmatrix} [\, gW^{\mu}(x) + \tfrac{1}{2}\, g' B^{\mu}(x)\,][\, gW_{\mu}(x) + \tfrac{1}{2}\, g' B_{\mu}(x)\,] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
= & \quad \tfrac{1}{2}\, \partial^{\mu} H(x) \partial_{\mu} H(x) + \tfrac{1}{2}\, [\, \upsilon + H(x)\,]^2 \\
\times & \quad \tfrac{1}{2} \left[\, g^2\, W^{\mu}(x) W_{\mu}(x) + \tfrac{1}{4}\, g'^2 B^{\mu}(x) B_{\mu}(x) - \tfrac{1}{2}\, g g'\, W^3_{\mu}(x) B^{\mu}(x)\, \right] \\
= & \quad \tfrac{1}{2}\, \partial^{\mu} H(x) \partial_{\mu} H(x) + \tfrac{1}{4}\, g^2 [\, \upsilon + H(x)\,]^2\, W^{+}_{\mu}(x) W^{\mu}_{-}(x) \\
+ & \quad \tfrac{1}{8}\, [\, g W^{\mu}_3(x) - g' B^{\mu}(x)\,] \left[\, g W^3_{\mu}(x) - g' B_{\mu}(x)\, \right] [\, \upsilon + H(x)\,]^2
\end{aligned}
$$

From the above expression it follows that both the charged vector bosons $W^{\pm}_{\mu}$ have mass

$$
m_W = \tfrac{1}{2}\, \upsilon g \tag{3.106}
$$

while the neutral vector bosons B_{μ} and W^3_{μ} share a non-diagonal 2×2 mass matrix. In order to find the mass eigenstates let us set

$$
\begin{pmatrix} W^3_{\mu}(x) \\ B_{\mu}(x) \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z^0_{\mu}(x) \\ A_{\mu}(x) \end{pmatrix} \tag{3.107}
$$

Then we get the mass term in the Lagrangian

$$
\begin{aligned}
& \tfrac{1}{8}\, \upsilon^2 \left[\, (\, g\cos\theta_W + g' \sin\theta_W) Z^0_{\mu}(x) + (\, g\sin\theta_W - g' \cos\theta_W) A_{\mu}(x)\, \right] \\
\times & \quad \left[\, (\, g\cos\theta_W + g' \sin\theta_W) Z^0_{\nu}(x) + (\, g\sin\theta_W - g' \cos\theta_W) A_{\nu}(x)\, \right] g^{\,\mu\nu}
\end{aligned}
$$

which takes the diagonal form, for g, g' and θ_W all positive, if and only if

$$
\tan\theta_W = \frac{g'}{g} \tag{3.108}
$$

This corresponds to the mass eigenvalues

$$
m_A = 0 \qquad\qquad m_Z = \tfrac{1}{2}\, \upsilon \sqrt{g^2 + g'^2} = \frac{m_W}{\cos\theta_W} \tag{3.109}
$$

As expected, one of the neutral vector bosons appears to be mass-less and will be identified with the photon field. It turns out that the Higgs mechanism breaks the original U(2) gauge symmetry according to

$$
\mathrm{SU}(2) \times \mathrm{U}(1)_Y \quad \longmapsto \quad \mathrm{U}(1)_Q
$$

the local phase invariance relic symmetry being understood as the usual gauge invariance of QED. The parameter $\theta_W = \arctan(g'/g)$ is the angle between the original broken hyper-charge Abelian gauge symmetry $\mathrm{U}(1)_Y$ and the eventual electromagnetic charge gauge symmetry $\mathrm{U}(1)_Q$ which is left unbroken. This angle was first introduced by Sheldon Lee Glashow (1961) *Partial Symmetries of Weak Interactions*, Nuclear Physics **22**, 579, although it is often and curiously referred to as the "Weinberg angle".

As a consequence, we can definitely recast the Higgs field Lagrangian in the Standard Model form: namely,

$$
\begin{aligned}
\mathcal{L}_{\rm Higgs} &= (\,D^{\mu}\,\phi\,)^{\dagger} D_{\mu}\,\phi - \mathcal{V}(\phi) \\
&= \tfrac{1}{2}\left[\,\partial^{\mu}H(x)\partial_{\mu}H(x) - m_H^2\,H^2(x)\,\right] + \tfrac{1}{4}\,\lambda v^2 - \lambda v\,H^3(x) - \tfrac{1}{4}\,\lambda\,H^4(x) \\
&+ \left[\,m_W^2\,W_{+}^{\mu}(x)W_{\mu}^{-}(x) + \tfrac{1}{2}\,m_Z^2\,Z_0^{\mu}(x)Z_{\mu}^{0}(x)\,\right]\left[\,1 + H(x)/v\,\right]^2
\end{aligned}
\tag{3.110}
$$

$$
\left.\begin{array}{c} m_H = \mu\sqrt{2} \simeq 126 \text{ GeV}/c^2 \\ v = \mu c^2/\sqrt{\lambda} \approx 246 \text{ GeV} \end{array}\right\} \quad \Rightarrow \quad \lambda \simeq 0.13 \tag{3.111}
$$

Vacuum Density of the Higgs Field. If we turn back to the vacuum energy density of the Higgs field

$$
\mathcal{V}_o = -\,\frac{\mu^4}{4\lambda} = -\,\frac{\lambda v^4}{4(\hbar c)^3}
$$

we can readily obtain its approximate value. Actually, from the conversion factors

$$
\hbar c \simeq 0.2 \text{ GeV fm} \qquad 1 \text{ GeV}/c^2 \simeq 1.8 \times 10^{-27} \text{ kg}
$$

we immediately get

$$
-\,\mathcal{V}_o \simeq 1.1 \times 10^{54} \text{ GeV m}^{-3} \quad \Longleftrightarrow \quad \rho_{\rm Higgs} = (\,-\,\mathcal{V}_o/c^2\,) \simeq 1.8 \times 10^{44} \text{ kg m}^{-3}
$$

a really huge value. Actually, it might be amusing to compare the order of magnitudes of several benchmark densities in units of kg/m^3: the critical density of the Universe, the perfect gas density in standard conditions, the nuclear matter density, the above zero-point Higgs mass density and the zero-point mass density of a quantum field with the Planck mass as an ultra-violet cutoff, because Quantum Gravity, if any, is certainly outside the framework of the perturbative renormalizable and unitary quantum field theories. We find

10^{-26}	10^{-2}	10^{18}	10^{44}	10^{94}
$\rho_{\rm crit}$	gas	nuclei	$\rho_{\rm Higgs}$	$\rho_{\rm Planck}$

To provide a *rationale* to this intriguing hierarchy of values, spread over 120 orders of magnitude, is a truly formidable challenge for present days Theoretical Physics.

As a final comment I'd like to stress once more that three of the four real scalar field involved in the $\phi(x)$ complex doublet have been absorbed by the Higgs mechanism in order to allow for the three gauge vector bosons $W_{\mu}^{\mp}(x)$ and $Z_{\mu}(x)$ to acquire their masses. The fourth real scalar field, *i.e.* the Higgs field which corresponds to $\sqrt{\phi^0\phi_0^{\dagger}}$, remains physical. Its mass is provided by the coefficient of the quadratic part of the Higgs potential (3.105) and reads

$$
m_H = \sqrt{2\mu^2} = \sqrt{2\lambda v^2} \tag{3.112}
$$

From now on it will be utmost convenient to write all expressions in terms of the gauge fields mass eigenstates $W_\mu^\mp(x), Z_\mu(x)$ and $A_\mu(x)$. For example, the covariant derivatives can be suitably recast in the suggestive form

$$\begin{aligned}
iD_\mu\Psi'_\iota(x) &= \left[\, i\partial_\mu + gV_\mu^a(x)\,\boldsymbol{\tau}_a - \tfrac{1}{2}\,g' B_\mu(x)\,\right] U_\chi(x)\Psi_\iota(x) \\
&= U_\chi(x)\Big\{ i\partial_\mu\Psi_\iota(x) + \frac{g}{\sqrt{2}}\left[\, W_\mu^+(x)\,\boldsymbol{\tau}_+ + W_\mu^-(x)\,\boldsymbol{\tau}_-\,\right]\Psi_\iota(x) \\
&+ \left[\, gW_\mu^3(x)\,\boldsymbol{\tau}_3 - \tfrac{1}{2}\,g' B_\mu(x)\,\right]\Psi_\iota(x)\Big\} \\
&= U_\chi(x)\Big\{ i\partial_\mu\Psi_\iota(x) + \frac{g}{\sqrt{2}}\left[\, W_\mu^+(x)\,\boldsymbol{\tau}_+ + W_\mu^-(x)\,\boldsymbol{\tau}_-\,\right]\Psi_\iota(x) \\
&+ g\left[\,\cos\theta_W\, Z_\mu^0(x) + \sin\theta_W\, A_\mu(x)\,\right]\boldsymbol{\tau}_3\,\Psi_\iota(x) \\
&- \tfrac{1}{2}\,g'\left[\,-\sin\theta_W\, Z_\mu^0(x) + \cos\theta_W\, A_\mu(x)\,\right]\Psi_\iota(x)\Big\} \\
&= U_\chi(x)\Big\{ i\partial_\mu\Psi_\iota(x) + \frac{g}{\sqrt{2}}\left[\, W_\mu^+(x)\,\boldsymbol{\tau}_+ + W_\mu^-(x)\,\boldsymbol{\tau}_-\,\right]\Psi_\iota(x) \\
&+ Z_\mu^0(x)\left[\, g\cos\theta_W\,\boldsymbol{\tau}_3 - \tfrac{1}{2}\,g'\sin\theta_W\, Y(\Psi_\iota)\,\right]\Psi_\iota(x) \\
&+ A_\mu(x)\left[\, g\sin\theta_W\,\boldsymbol{\tau}_3 + \tfrac{1}{2}\,g'\cos\theta_W\, Y(\Psi_\iota)\,\right]\Psi_\iota(x)\Big\}
\end{aligned}$$

where use has been made of the transformation rule (3.107). Now, since $g' = g\tan\theta_W$, we can write

$$g\sin\theta_W\,\boldsymbol{\tau}_3 + \tfrac{1}{2}\,g'\cos\theta_W\, Y(\Psi_\iota) = g\sin\theta_W\left[\,\boldsymbol{\tau}_3 + \tfrac{1}{2}\,Y(\Psi_\iota)\,\right] = g\sin\theta_W\; Q(\Psi_\iota)$$

and thereby $e = g\sin_W$ must coincide with the proton charge. Moreover we obviously have by direct inspection

$$\begin{aligned}
iD_\mu\ell_\iota^+(x) &= i\partial_\mu\ell_\iota^+(x) + g'[\,\sin\theta_W\, Z_\mu^0(x) - \cos\theta_W\, A_\mu(x)\,]\ell_\iota^+(x) \\
&= i\partial_\mu\ell_\iota^+(x) + \sin\theta_W[\, g' Z_\mu^0(x) - g\, A_\mu(x)\,]\ell_\iota^+(x) \qquad (3.113)
\end{aligned}$$

It follows that, after the field replacements (3.103), (3.104) and (3.107) we can eventually recast the lepton Lagrangian in the Standard Model form

$$\begin{aligned}
\mathcal{L}_{\text{lepton}} &= \sum_{\iota=1}^{3}\Big\{\, i\,\overline{\Psi}{}'_\iota\,\not{D}\,\Psi'_\iota + i\,\overline{\ell_\iota^+}\,\not{D}\,\ell_\iota^+ - y_\iota\Big[\,\overline{\Psi}{}'_\iota\,\phi\,\ell_\iota^+ + \text{c.c.}\,\Big]\Big\} \\
&= \sum_{\iota=1}^{3}\Big\{\, i\,\overline{\Psi}_\iota\,\not{\partial}\;\Psi_\iota + i\,\overline{\ell_\iota^+}\,\not{\partial}\;\ell_\iota^+ - m_\iota\,(1 + H/v)\,\overline{\ell}_\iota\,\ell_\iota\Big\} \\
&+ g\left(W_\mu^+\, J_+^\mu + W_\mu^-\, J_-^\mu + Z_\mu^0\, J_0^\mu\right) - eA_\mu\, J^\mu \qquad (3.114)
\end{aligned}$$

where $vy_\iota/\sqrt{2} = m_\iota = (\, m_e, m_\mu, m_\tau\,)$ whereas we have conventionally set

$$J_+^\mu(x) = \frac{1}{\sqrt{2}}\sum_{\iota=1}^{3}\overline{\nu}_\iota(x)\,\gamma^\mu\,\ell_\iota^-(x) = [\, J_-^\mu(x)\,]^* \qquad (3.115)$$

$$
\begin{aligned}
J_0^\mu(x) &= \sec\theta_W \sum_{\iota=1}^{3} \Big\{ \tfrac{1}{2}\,\overline{\nu}_\iota(x)\,\gamma^\mu\,\nu_\iota(x) - \tfrac{1}{2}\,\overline{\ell_\iota^-}(x)\,\gamma^\mu\,\ell_\iota^-(x) \\
&+ \sin^2\theta_W\ \overline{\ell_\iota}(x)\,\gamma^\mu\,\ell_\iota(x)\Big\} \qquad (3.116) \\
J^\mu(x) &= \sum_{\iota=1}^{3} \overline{\ell}_\iota(x)\,\gamma^\mu\,\ell_\iota(x) \qquad (3.117)
\end{aligned}
$$

the latter current being the familiar QED electromagnetic current coupled to the photon field. The tetra-current densities, in physical units,

$$
\jmath_\pm^\mu(x) \equiv \frac{gc}{\hbar}\,J_\pm^\mu(x) \qquad \jmath_0^\mu(x) \equiv \frac{gc}{\hbar}\,J_0^\mu(x) \tag{3.118}
$$

are called the charged and neutral currents[13] of the weak interactions. The various and different coupling produced by the extension to the hadronic sector of the Lagrangian for the Standard Model will be discussed in the next Section. To conclude the present Section, let me recall the experimental values of masses for the vector bosons and Higgs particles: namely,

$$
\begin{gathered}
m_W = 80.385 \pm 0.015\ \mathrm{GeV}/c^2 \\
m_Z = 91.1876 \pm 0.0021\ \mathrm{GeV}/c^2 \\
m_H = 125.7 \pm 0.4\ \mathrm{GeV}/c^2
\end{gathered} \tag{3.119}
$$

so that

$$
\cos\theta_W = \frac{m_W}{m_Z} = 0.881 \quad\Longrightarrow\quad \theta_W \simeq 28^\circ \quad\Rightarrow\quad \sin^2\theta_W = 0.22 \tag{3.120}
$$

$$
\tan\theta_W = \frac{g'}{g} = 0.53 \qquad g^2 = \frac{e^2}{\sin^2\theta_W} = \frac{4\pi\alpha}{\sin^2\theta_W} \simeq 0.42 \tag{3.121}
$$

3.2.4 Extension to Hadrons

The introduction of hadrons in the Standard Model shares with leptons some basic features but presents as well some novel important aspects. They are mainly due to the fact that the individual quark quantum numbers are not separately conserved by the interactions, at variance with the lepton's world. There is nowadays a wide consensus concerning the choice of the *elementary* constituents of Matter: besides the six leptons there are six quarks, which turn out to be fractionally charged and appear each in three colors.

[13] The discovery of weak neutral currents was done by Gargamelle experiment at CERN: F.J. Hasert *et al.*, Phys. Lett. **46B** (1973) 38.

The empirically observed lepton-hadron universality property suggests us to arrange the quarks degrees of freedom into SU(2) doublets and singlets. The first important novelty I mentioned above is that all quarks appear to have non-vanishing Dirac masses, so that we must introduce right-handed singlets for both members of each family. A naïve assignment would be to write the analogue of equations (3.78) and (3.79) in the form

$$q_\iota^-(x) = \tfrac{1}{2}(\mathbb{I} - \gamma_5) \begin{pmatrix} u_\iota(x) \\ d_\iota(x) \end{pmatrix} \tag{3.122}$$

$$u_\iota^+(x) = \tfrac{1}{2}(\mathbb{I} + \gamma_5)\, u_\iota(x) \qquad d_\iota^+(x) = \tfrac{1}{2}(\mathbb{I} + \gamma_5)\, d_\iota(x) \tag{3.123}$$

where the index $\iota = 1, 2, 3$ is again running over the three families

$$u_\iota(x) = \{u(x), c(x), t(x)\} \qquad d_\iota(x) = \{d(x), s(x), b(x)\}$$

respectively. An additional threesome color index (blue, yellow and green) is also necessary for quarks, which are sensitive to the strong interactions, but it will be understood in the sequel for simplicity. The above assignments determine the SU(2) transformation properties of the quark fields, just like in equation (3.84). Moreover, they also fix as well their hyper-charges and the consequent U(1) transformation law in accordance with equation (3.85). As a matter of fact, from equation (3.83) and the well known fractional electric charges of the quark fields $Q(u_\iota) = \frac{2}{3}$, $Q(d_\iota) = -\frac{1}{3}$, in units of the proton charge, it follows that

$$Y(q_\iota^-) = \tfrac{1}{3} \qquad Y(u_\iota^+) = \tfrac{4}{3} \qquad Y(d_\iota^+) = -\tfrac{2}{3} \tag{3.124}$$

The actual presence in each family of two right-handed quark singlets has a crucial consequence. It turns out that, even in the case of one single family, we have in fact two distinct Yukawa-like terms involving quark and Higgs fields: namely,

$$y_d\, \bar{q}_\iota^-(x) \cdot \phi(x)\, d_\iota^+(x) + y_u\, \bar{q}_\iota^-(x) \cdot \widetilde{\phi}(x)\, u_\iota^+(x) + \text{c.c.} \qquad (\, y_d\,,\, y_u \in \mathbb{R}\,) \tag{3.125}$$

where the family index $\iota = 1, 2, 3$ is not summed over, while the scalar doublet

$$\widetilde{\phi}(x) = i\sigma_2\, \phi^*(x) = \begin{pmatrix} \phi_0^*(x) \\ -\phi^-(x) \end{pmatrix}$$

belongs to the other nonequivalent, irreducible, fundamental representation of SU(2) with opposite hyper-charge $Y(\bar{\phi}) = -1$. If there were only one family, this would have been the end of the story: for any single value of the family index, the Lagrangian for the hadronic matter fields is the same as in (3.98) with quark fields replacing lepton fields and additional Yukawa-like term for the further right-handed element, *viz.*,

$$
\begin{aligned}
\mathcal{L}^{\iota}_{\text{quark}} &= \sum_{\text{colors}} \Big\{ i\,\bar{u}_\iota \not{D}\, u_\iota + i\,\bar{d}_\iota \not{D}\, d_\iota \\
&- \Big[y_d \left(\bar{q}_\iota^{\,-} \cdot \phi \right) d_\iota^{+} + y_u \left(\bar{q}_\iota^{\,-} \cdot \widetilde{\phi} \right) u_\iota^{+} + \text{c.c.} \Big] \Big\} \\
\mathcal{L}_{\text{lepton}} &= \sum_{\iota=1}^{3} \left\{ i\,\overline{\Psi}_\iota \not{D}\, \Psi_\iota + i\,\overline{\ell_\iota^{+}} \not{D}\, \ell_\iota^{+} - y_\iota \left[\left(\overline{\Psi}_\iota \cdot \phi \right) \ell_\iota^{+} + \text{c.c.} \right] \right\}
\end{aligned}
$$

The complication I was warning about before arises from the presence of three families of quarks. In this case, the Lagrangian for the quark fields is not the sum over the family index of $\mathcal{L}^{\iota}$, because in Nature there is no conservation of the quark quantum numbers, in contrast to lepton number conservation. Then, for example, when I started from the doublets (3.122) I had implicitly assumed a particular pairing of the quark fields in each family: $u(x)$ with $d(x)$, then $c(x)$ with $s(x)$ and finally $t(x)$ with $b(x)$. However, there is no compelling reason for that choice, neither theoretical nor phenomenological [14]. Hence, in general, we can choose any basis in the family space but, as we have two Yukawa-like terms in the quark Lagrangian, we cannot put both of them simultaneously into diagonal form. It follows that the most general quark Action must contain two non-diagonal matrices that correspond to the unavoidable family mixing in arbitrary family bases.

Thus, the correct most general form of the Yukawa-like Lagrangian for quark matter fields is provided by

$$
-\sum_{\iota=1}^{3}\sum_{\jmath=1}^{3} \Big[\bar{q}_\iota^{\,-}(x) \cdot \phi(x)\, Y^{d}_{\iota\jmath}\, d_\jmath^{+}(x) + \bar{q}_\iota^{\,-}(x) \cdot \widetilde{\phi}(x)\, Y^{u}_{\iota\jmath}\, u_\iota^{+}(x) + \text{c.c.} \Big] \tag{3.126}
$$

where the Yukawa-like coupling for the up and down quarks have become general complex matrices with matrix elements $Y^{u}_{\iota\jmath}$, $Y^{d}_{\iota\jmath}$ ($\iota, \jmath = 1, 2, 3$) in the family space. Now, after the redefinition (3.103) of the Higgs complex scalar doublet, the Dirac-like mass term are produced for the up and down quarks - from now on the sum over color indexes is understood

$$
-\sum_{\iota=1}^{3}\sum_{\jmath=1}^{3} \frac{v}{\sqrt{2}} \left[\bar{d}_\iota^{\,-}(x)\, Y^{d}_{\iota\jmath}\, d_\jmath^{+}(x) + \bar{u}_\iota^{\,-}(x)\, Y^{u}_{\iota\jmath}\, u_\iota^{+}(x) + \text{c.c.} \right] \tag{3.127}
$$

[14] Actually, owing to the non-perturbative mechanism called quark confinement, quarks can not even been detected as particles on their mass-shell unless, perhaps, some very high temperatures and densities are reached for hadronic matter, so that the quark-gluon plasma might be eventually formed.

Before turning to put the above expression into its diagonal form, an important remark is in order concerning the $\mathcal{CP}$ discrete symmetry. Consider for example the term

$$\begin{aligned} & \overline{d_\imath^-}(x)\, Y_{\imath\jmath}^d\, d_\jmath^+(x) + \text{c.c.} = d_\imath^\dagger(x) P^-\gamma^0\, Y_{\imath\jmath}^d\, P^+ d_\jmath(x) + \text{c.c.} \\ = \; & d_{\imath L}^\dagger(x)\, d_{\jmath R}(x)\, Y_{\imath\jmath} + d_{\imath R}^\dagger(x)\, d_{\jmath L}(x)\, Y_{\imath\jmath}^* \end{aligned} \tag{3.128}$$

where I have set $P^\mp = \frac{1}{2}(\,\mathbb{I}\mp\gamma_5\,) = (P^\mp)^\dagger$ to denote the left- and right-handed projectors, whereas the chiral Weyl 2-component spinors have been explicitly written in the very last line. Now the $\mathcal{C}$ and $\mathcal{P}$ symmetry transformations act on the classical Graßmann-valued Dirac bispinors as follows: namely,

$$\psi(x) \quad \longmapsto \quad \psi'(x) = \mathcal{C}\psi(x) = \gamma^2\psi^*(x) \tag{3.129}$$

$$\psi(x) \quad \longmapsto \quad \psi'(x') = (\mathcal{P}\psi)(x') = \gamma^0\psi(x) \tag{3.130}$$

$$\{\psi(x), \psi(y)\} = \{\psi(x), \overline{\psi}(y)\} = \left\{\overline{\psi}(x), \overline{\psi}(y)\right\} = 0$$

where the chiral representation for the $\gamma-$matrices is understood, without any loss of generality. Thus, for example, the $\mathcal{CP}$ transformation acts on (3.128) as

$$\begin{aligned} \mathcal{CP}\left\{\overline{d_\imath^-}(x)\, Y_{\imath\jmath}^d\, d_\jmath^+(x)\right\} &= Y_{\imath\jmath}^d\,[\,\gamma^0\gamma^2 d_\imath^*(x)\,]^\dagger P^-\gamma^0\, P^+\gamma^0\gamma^2 d_\jmath^*(x) \\ = Y_{\imath\jmath}^d\, d_\imath^\top(x)\,\gamma^{2\dagger}\gamma^0 P^-\gamma^0\, P^+\gamma^0\gamma^2 d_\jmath^*(x) &= -\,Y_{\imath\jmath}^d\, d_\imath^\top(x)\, P^-\gamma^0\, P^+ d_\jmath^*(x) \\ = Y_{\imath\jmath}^d\, d_\jmath^\dagger(x)\, P^+\gamma^0\, P^- d_\imath(x) = Y_{\imath\jmath}^d\left[\,\bar{d}_\imath^-(x)\, d_\jmath^+(x)\,\right]^* &= \left[\,\overline{d_\imath^-}(x)\, Y_{\imath\jmath}^{d*}\, d_\jmath^+(x)\,\right]^* \end{aligned}$$

whence it is apparent that $\mathcal{CP}-$symmetry holds true iff $Y_{\imath\jmath}^d = (Y_{\imath\jmath}^d)^*$, which is not the case in the present context where $Y_{\imath\jmath}^d$ are a collection of arbitrary complex numbers. As a consequence, any Dirac-like mass term of the kind (3.128) does unavoidably leads to $\mathcal{CP}-$violation. It is utmost convenient to work in a quark field space where all the Dirac mass terms appear to be diagonal, in such a manner to clearly identify their masses. This can be readily done by means of four unitary 3×3 matrices V_-^u, V_+^u, V_-^d, V_+^d such that

$$\frac{\upsilon}{\sqrt{2}}\, V_-^{d\dagger}\, Y^d\, V_+^d = \mathrm{diag}(\,m_d, m_s, m_b\,)$$

$$\frac{\upsilon}{\sqrt{2}}\, V_-^{u\dagger}\, Y^u\, V_+^u = \mathrm{diag}(\,m_u, m_c, m_t\,)$$

`Proof.` It is not difficult to prove that any general non-degenerate complex matrix Y of rank N, *i.e.* $Y\in GL(N,\mathbb{C})$ with $\det Y \neq 0$, can always be set into diagonal form by means of a bi-unitary transformation: namely, there exist unitary matrices U, V of rank N such that

$$V^\dagger Y U = \mathrm{diag}(\,y_1,\ldots,y_N\,) \qquad y_k > 0 \quad \forall\, k = 1,2,\ldots,N$$

The main ingredients for the proof are that any complex matrix Y belonging to the general linear group $GL(N,\mathbb{C})$ can be always written as the product of a Hermitean and a unitary matrices

$$Y = \rho\,\mathrm{e}^{\,i\Theta} \qquad\quad \rho = \rho^\dagger\,,\ \Theta = \Theta^\dagger \qquad \det\rho > 0$$

and in turn the Hermitean matrix ρ can always be set in its diagonal form by means of a further unitary matrix. The proof goes as follows: the matrix $Y^\dagger Y$ is Hermitean and positive, so that it can be put in diagonal form by means of the unitary matrix V and thereby

$$V^\dagger(Y^\dagger Y)V = \Upsilon^2$$

where

$$\Upsilon^2 = \mathrm{diag}\,\left(\,y_1^2, y_2^2, \ldots, y_N^2\,\right)$$

It is worthwhile to remark that the unitary matrix V is unique up to an arbitrary diagonal phase matrix

$$\Omega = \mathrm{diag}\,\left(\,\mathrm{e}^{i\phi_1}, \mathrm{e}^{i\phi_2} \ldots, , \mathrm{e}^{i\phi_N}\,\right)$$

since it can be immediately verified by direct inspection that

$$(V\Omega)^\dagger(Y^\dagger Y)V\Omega = \Upsilon^2$$

Let me define now a further Hermitean non-singular matrix

$$\rho \equiv V\Upsilon V^\dagger$$

where $\Upsilon \equiv \sqrt{\Upsilon^2} = \mathrm{diag}(\,y_1, \ldots, y_N\,)$ with $y_k > 0\ \forall\, k = 1,\ldots,N$, in such a manner that

$$\mathrm{e}^{\,i\Theta} \equiv \rho^{-1}Y \qquad\quad \mathrm{e}^{-\,i\Theta} = Y^\dagger\rho^{-1}$$

is indeed a unitary matrix because

$$\rho^{-1}YY^\dagger\rho^{-1} = \rho^{-1}V\Upsilon^2V^{-1}\rho^{-1} = \rho^{-1}(V\Upsilon V^{-1})(V\Upsilon V^{-1})\rho^{-1} = \rho^{-1}\rho\,\rho\rho^{-1} = \mathbb{I}$$

Hence, from the definitions $\rho \equiv V\Upsilon V^{-1}$ and $\mathrm{e}^{\,i\Theta} \equiv \rho^{-1}Y$ we eventually obtain

$$\Upsilon = V^\dagger\rho V = V^{-1}Y\mathrm{e}^{-\,i\Theta}V$$

and thereby

$$V^\dagger YU = \mathrm{diag}(\,y_1, y_2, \ldots, y_N\,) \qquad\quad U = \mathrm{e}^{-\,i\Theta}V$$

where U is unitary, as previously claimed. $\square$

Then the relations between gauge and mass eigenstates do actually follow

$$u^\mp(x) = V^u_\mp\,U^\mp(x) \qquad\quad d^\mp(x) = V^d_\mp\,D^\mp(x) \tag{3.131}$$

that yield

$$\begin{aligned} \frac{\upsilon}{\sqrt{2}}\,\bar{d}^{\,-}(x)Y^{\,d}d^{\,+}(x) + \text{c.c.} &= \frac{\upsilon}{\sqrt{2}}\,\bar{D}^{\,-}(x)V_-^{\,d\dagger}Y^{\,d}V_+^{\,d}D^{\,+}(x) + \text{c.c.} \\ &= \sum_{\iota=d,s,b} m_\iota\,\overline{D}_\iota(x)\,D_\iota(x) \end{aligned} \tag{3.132}$$

and quite analogous relation for up quark flavors. Now one is always free to choose the mass eigenstates or physical basis in the family space, in such a manner that the Dirac-like mass term and the Yukawa couplings for the quark fields take the usual form, which turns out to be diagonal in flavor.

$$\left[1+\frac{H(x)}{v}\right]\left\{-\sum_{\iota=u,c,t} m_\iota\,\overline{U}_\iota(x)\,U_\iota(x)-\sum_{\iota=d,s,b} m_\iota\,\overline{D}_\iota(x)\,D_\iota(x)\right\} \tag{3.133}$$

The transformations (3.131), which correspond to the choice of the physical bases in the family space, thus convert quarks to their mass-flavor common eigenstates. Moreover, since the left-handed and right-handed up and down quarks have identical couplings to the QCD color gauge fields, then the four unitary matrices V_-^u, V_+^u, V_-^d, V_+^d do all commute with the color covariant derivatives. As a consequence it turns out that in the physical bases the kinetic terms, the mass terms, the color covariant derivatives and the Yukawa coupling to the Higgs scalar field are all diagonal in flavor space and conserve $\mathcal{C},\mathcal{P}$ and $\mathcal{T}$ symmetries.

Conversely, it tuns out that in the very same physical bases the flavor covariant derivatives are neither diagonal nor $\mathcal{CP}$ invariant. In fact, it is an instructive exercise to repeat the derivation of the charged and neutral weak currents (3.115 -3.116), as well as of the electromagnetic current (3.117), for the hadronic sector: the result reads

$$J_+^\mu(x) = \frac{1}{\sqrt{2}}\sum_{\iota=1}^{3}\overline{U}_\iota^{\,-}(x)\,\gamma^\mu\left(V_-^{u\dagger}V_-^d\right)_{\iota\jmath}D_\jmath^-(x)=[\,J_-^\mu(x)\,]^* \tag{3.134}$$

$$\begin{aligned} J_0^\mu(x) &= \sec\theta_W\sum_{\iota=1}^{3}\Big\{\tfrac12\,\overline{u}_\iota^{\,-}(x)\,\gamma^\mu\,u_\iota^-(x)-\tfrac12\,\overline{d}_\iota^{\,-}(x)\,\gamma^\mu\,d_\iota^-(x) \\ &+ \sin^2\theta_W\left[\,\tfrac13\,\overline{d}_\iota(x)\,\gamma^\mu\,d_\iota(x)-\tfrac23\,\overline{u}_\iota(x)\,\gamma^\mu\,u_\iota(x)\,\right]\Big\} \end{aligned} \tag{3.135}$$

$$J^\mu(x) = \sum_{\iota=1}^{3}\left[\,\tfrac13\,\overline{d}_\iota(x)\,\gamma^\mu\,d_\iota(x)-\tfrac23\,\overline{u}_\iota(x)\,\gamma^\mu\,u_\iota(x)\,\right] \tag{3.136}$$

From the above expressions it appears to be manifest that the only items in which the choice of the bases in the family space turns out to be relevant and meaningful are the hadronic weak charged currents $J_\mp^\mu(x)$. Hence, the flavor changing weak interactions mix the three up quarks with the threesome down quarks by means of the unitary matrix of rank three, *viz.,*

$$V\equiv V_-^{u\dagger}V_-^d \tag{3.137}$$

which is known as the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix [15]. A unitary matrix belonging to the U(N) Lie group depends upon N^2 real parameters, *i.e.* $\frac{1}{2}$N(N-1) rotation angles and $\frac{1}{2}$N(N+1) phase factors. In particular, for the present case we have three rotation angles and six phases. However, phases can be reabsorbed by making phase transformations on the left-handed quarks fields

$$U_\imath^-(x) \quad \longmapsto \quad \tilde{U}_\imath^-(x) = \exp\{i\alpha_\imath^u\}\, U_\imath^-(x) \tag{3.138}$$

$$D_\imath^-(x) \quad \longmapsto \quad \tilde{D}_\imath^-(x) = \exp\{i\alpha_\imath^d\}\, D_\imath^-(x) \tag{3.139}$$

However, owing to the specific structure (3.137) of the CKM matrix, it turns out that the phases in the above quark phase change are constrained to satisfy

$$\sum_{\imath=1}^{3} \left(\alpha_\imath^d - \alpha_\imath^u\right) = 0$$

since an overall phase change on the quark fields is always allowed and does not produce any phase reabsorbing. Thus, five of the six phases of the unitary matrix of rank three can always be reabsorbed by the quark phase change, but one is left and usually denoted by δ . Hence, the unitary CKM matrix is customarily labeled by three Euler-like angles and only one arbitrary phase, the $\mathcal{CP}-$violating Kobayashi-Maskawa phase. Of the infinitely many possible conventions, a standard choice [16] is traditionally written in the form

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{3.140}$$

with the notation $c_{\imath\jmath} = \cos\theta_{\imath\jmath}$, $s_{\imath\jmath} = \sin\theta_{\imath\jmath}$ $(\imath, \jmath = 1, 2, 3)$, while δ is the phase which is responsible for all the $\mathcal{CP}-$violating phenomena in the flavor changing processes of the Standard Model. The angle $\theta_{\imath\jmath}$ can be chosen to lie in the first quadrant, so that $s_{\imath\jmath} > 0$, $c_{\imath\jmath} > 0$. Hence it appears quite evident that in so doing a natural mechanism is available to open the possibility of breaking the $\mathcal{CP}-$symmetry in terms of the phase parameter δ , a highly nontrivial and far reaching circumstance. It is known experimentally that $s_{13} \ll s_{23} \ll s_{12} \ll 1$ and it turns out very convenient to take this hierarchy

[15] Makoto Kobayashi & Toshihide Maskawa (1973) *CP-Violation in the Renormalizable Theory of Weak Interactions*, Progress of Theoretical Physics **49**, 652-657.

[16] Ling-Lie Chau & Wai-Yee Keung (1984) *Comments on the Parametrization of the Kobayashi-Maskawa Matrix*, The Physical Review Letters **53**, 1802.

into account in terms of the four Wolfenstein parameters [17]

$$s_{12} = \lambda_W \qquad s_{23} = A\lambda_W^2 \tag{3.141}$$

$$s_{13}\,\mathrm{e}^{i\delta} = \frac{A\lambda_W^3(\,\bar\rho + i\bar\eta\,)\sqrt{(1 - A\lambda_W^2)(1 + A\lambda_W^2)}}{\sqrt{1-\lambda_W^2}\,[\,1 - A^2\lambda_W^4(\,\bar\rho + i\bar\eta\,)\,]} \tag{3.142}$$

The up-to-date best fit for the Wolfenstein parameters yields

$$\lambda_W = 0.2253\,\pm\,0.0007 \qquad A = 0.808\,^{+0.022}_{-0.015} \tag{3.143}$$

$$\bar\rho = 0.132\,^{+0.022}_{-0.014} \qquad \bar\eta = 0.341\,\pm\,0.013 \tag{3.144}$$

3.2.5 The Standard Model Free Parameters

It is somewhat customary to choose by default the physical basis for the up quarks, in such a manner that the full Lagrange density of the Standard Model, before the translation and gauge transformation on the Higgs field, does actually read

$$\begin{aligned}
\mathcal{L} \;&=\; -\tfrac{1}{2}\,\mathrm{tr}\,[\,G_{\mu\nu}\,G^{\mu\nu}\,] - \tfrac{1}{4}\,B_{\mu\nu}\,B^{\mu\nu} + (\,D^{\mu}\,\phi\,)^{\dagger}D_{\mu}\,\phi - \mathcal{V}(\phi)\\
&+\; \sum_{\iota=1}^{3}\Big\{ i\,\overline{\Psi}_{\iota}\,\not{D}\,\Psi_{\iota} + i\,\overline{\ell_{\iota}^{+}}\,\not{D}\,\ell_{\iota}^{+} - y_{\iota}\left(\,\overline{\Psi}_{\iota}\cdot\phi\,\ell_{\iota}^{+} + \mathrm{c.c.}\,\right)\Big\}\\
&+\; \sum_{\mathrm{colors}}\sum_{\iota=1}^{3}\Big\{ i\,\overline{U}_{\iota}\,\not{D}\,U_{\iota} + i\,\overline{d}_{\iota}\,\not{D}\,d_{\iota}\\
&-\; \Big[\,\sum_{\jmath=1}^{3} Y_{\iota\jmath}^{d}\,(\bar q_{\iota}^{\,-}\cdot\phi)\,d_{\jmath}^{+} + y_{\iota}^{U}\,\left(\bar q_{\iota}^{\,-}\cdot\widetilde{\phi}\right)\,U_{\iota}^{+} + \mathrm{c.c.}\Big]\Big\}
\end{aligned} \tag{3.145}$$

where the flavor covariant derivatives on the quark fields are given by

$$D_{\mu}\,q_{\iota}^{\,-}(x) \;\equiv\; \left[\,\partial_{\mu} - igV_{\mu}(x) - \tfrac{1}{6}\,ig'B_{\mu}(x)\,\right]q_{\iota}^{\,-}(x) \tag{3.146}$$

$$D_{\mu}\,U_{\iota}^{+}(x) \;=\; \left[\,\partial_{\mu} - \tfrac{2}{3}\,ig'B_{\mu}(x)\,\right]U_{\iota}^{+}(x) \tag{3.147}$$

$$D_{\mu}\,d_{\iota}^{+}(x) \;=\; \left[\,\partial_{\mu} + \tfrac{1}{3}\,ig'B_{\mu}(x)\,\right]d_{\iota}^{+}(x) \tag{3.148}$$

On the other side, it turns out that the classical Lagrangian after the shift and gauge transformation of the Higgs isodoublet (3.103) takes the form we have previously investigated: namely,

$$\mathcal{L}' \;=\; \tfrac{1}{4}\,\lambda v^2 - \tfrac{1}{2}\,\mathrm{tr}\,[\,W_{\mu\nu}\,W^{\mu\nu}\,] - \tfrac{1}{4}\,B_{\mu\nu}\,B^{\mu\nu}$$

[17] Lincoln Wolfenstein (1983) *Parametrization of the Kobayashi-Maskawa Matrix*, The Physical Review Letters **51**, 1945.

$$
\begin{aligned}
&+ \ \tfrac{1}{2}\,\partial^{\mu}H\partial_{\mu}H - \tfrac{1}{2}\,m_H^2\,H^2 - \lambda\upsilon\,H^3 - \tfrac{1}{4}\,\lambda\,H^4 \\
&+ \ \left[\,m_W^2\,W_+^{\mu}W_{\mu}^- + \tfrac{1}{2}\,m_Z^2\,Z_0^{\mu}Z_{\mu}^0\,\right](1 + H/\upsilon)^2 \\
&+ \ i\sum_{\iota=1}^{3}\Big[\,\overline{\Psi}_{\iota}\,\partial\!\!\!/\,\Psi_{\iota} + \overline{\ell_{\iota}^{+}}\,\partial\!\!\!/\,\ell_{\iota}^{+} + \sum_{\text{colors}}\Big(\overline{U}_{\iota}\,\partial\!\!\!/\,U_{\iota} + \overline{d}_{\iota}'\,\partial\!\!\!/\,d_{\iota}'\Big)\,\Big] \\
&- \ \sum_{\iota=1}^{3}\Big[\,m_{\iota}^{\ell}\,\bar{\ell}_{\iota}\ell_{\iota} + \sum_{\text{colors}}\Big(m_{\iota}^{u}\,\overline{U}_{\iota}U_{\iota} + m_{\iota}^{d}\,\overline{d}_{\iota}'d_{\iota}'\Big)\,\Big](1 + H/\upsilon) \\
&+ \ A_{\mu}J^{\mu} + Z_{\mu}^{0}J_0^{\mu} + W_{\mu}^{-}J_{+}^{\mu} + W_{\mu}^{+}J_{-}^{\mu} \qquad (3.149)
\end{aligned}
$$

where I have made use of the alternative $'$ notation for the down quarks in the physical basis $D_{\iota}(x) = d_{\iota}'(x)$ to avoid confusion with the covariant derivatives, while the fermion masses and Yukawa-like couplings are connected by the relations $m_{\iota}^{\ell} = \upsilon\,y_{\iota}^{\ell}/\sqrt{2}$ and so on. The vector tetra-current densities are the sum of lepton (3.115)-(3.117) and quark currents (3.134)-(3.136)

$$
\begin{aligned}
J^{\mu} &= -\,e\sum_{\iota=1}^{3}\Big[\,\bar{\ell}_{\iota}\,\gamma^{\mu}\,\ell_{\iota} + \sum_{\text{colors}}\Big(\tfrac{1}{3}\,\overline{d}_{\iota}'\,\gamma^{\mu}\,d_{\iota}' - \tfrac{2}{3}\,\overline{U}_{\iota}\,\gamma^{\mu}\,U_{\iota}\Big)\,\Big] \\
J_{+}^{\mu} &= \frac{g}{\sqrt{2}}\sum_{\iota=1}^{3}\Big[\,\overline{\nu}_{\iota}\,\gamma^{\mu}\,\ell_{\iota}^{-} + \sum_{\text{colors}}\overline{U}_{\iota}^{\,-}\,\gamma^{\mu}\,\tfrac{1}{2}\,(\mathbb{I} - \gamma_5)V_{\iota\jmath}\,d_{\jmath}'\Big] \qquad J_{-}^{\mu} = [\,J_{+}^{\mu}\,]^{*} \\
J_0^{\mu} &= \tfrac{1}{2}\sec\theta_W\sum_{\iota=1}^{3}\Big\{\,\overline{\nu}_{\iota}\,\gamma^{\mu}\,\nu_{\iota} - \overline{\ell_{\iota}^{-}}\,\gamma^{\mu}\,\ell_{\iota}^{-} + 2\sin^2\theta_W\,\overline{\ell_{\iota}}\,\gamma^{\mu}\,\ell_{\iota} \\
&+ \ \overline{U}_{\iota}^{\,-}\,\gamma^{\mu}\,U_{\iota}^{-} - \tfrac{1}{2}\,\overline{d}_{\iota}'\,\gamma^{\mu}(\mathbb{I} - \gamma_5)d_{\iota}' + \sin^2\theta_W\Big[\,\tfrac{2}{3}\,\overline{d}_{\iota}'\,\gamma^{\mu}\,d_{\iota}' - \tfrac{4}{3}\,\overline{U}_{\iota}\,\gamma^{\mu}\,U_{\iota}\,\Big]\,\Big\}
\end{aligned}
$$

Remark: it should be gathered here that the two Lagrange densities $\mathcal{L}$ and $\mathcal{L}'$ are absolutely equivalent. They both describe the dynamics of the very same physical system because a change of variables, such as (3.103) and (3.131), can not evidently change physics. However, the perfect equivalence is actually true only if we can solve the dynamics exactly. In such a case we could pass from one kind of exact solution of the field equations to the other one by merely change the quantum field variables. Unfortunately we do not know any solutions of the complicated quantum field equations, so that we can only proceed by means of perturbation theory, which is an approximation tool, but in so doing that perfect equivalence is no longer guaranteed at all. As a matter of fact, perturbation theory has a much better chance to produce a sensible and reliable result starting from one Lagrangian rather than from the other one. Specifically, had we taken the form $\mathcal{L}$ to develop quantum field theory and to evaluate physical quantities starting from the quadratic terms of $\mathcal{L}$ as free non-interacting part, we immediately see that we shall fall into a non-sense. Actually, the spectrum of the free non-interacting Hamiltonian would consist of tachyon-like quanta with negative square mass and no higher order corrections arising from the perturbative treatment of the interactions could ever be able to modify that inconsistency. Conversely, it is clear that the quadratic part of $\mathcal{L}'$ gives rise to a zeroth order free Hamiltonian with a quite acceptable and reasonable spectrum, in such a manner that perturbation theory is expected to produce in turn reasonable and reliable results.

The classical Lagrangian (3.145) contains 17 FREE REAL PARAMETERS:

- two gauge couplings g and g'
- two parameters λ and μ^2 in the Higgs potential $\mathcal{V}(\phi)$
- three Yukawa couplings y_ι for the lepton families
- six Yukawa couplings and four KM parameters for the quark families

15 out of those 17 parameters are directly connected with the Higgs sector. It turns out that the translation of the Higgs field as in equation (3.103) and the setting into diagonal form of the resulting down quark matrix do actually provide the mass terms for fermions and bosons as well as the interaction terms. One can readily write those ones which involve the physical fields corresponding to the whole set of observable and detectable particles [18].

1. **Gauge Bosons to Fermion Couplings:** these are all the ones which generate the known electromagnetic and weak interactions. The gauge vector potential $A_\mu(x)$ is coupled to the charged fermions through the usual electromagnetic spinor current (3.117) plus (3.136)

$$-\,eA_\mu\sum_{\iota=1}^{3}\left[\,\overline{\ell}_\iota\,\gamma^\mu\,\ell_\iota+\sum_{\text{colors}}\left(\tfrac{1}{3}\,\overline{D}_\iota\,\gamma^\mu\,D_\iota-\tfrac{2}{3}\,\overline{U}_\iota\,\gamma^\mu\,U_\iota\right)\right]\qquad(3.150)$$

$$e=g\sin\theta_W=g'\cos\theta_W=\frac{gg'}{\sqrt{g^2+g'^2}}=\sqrt{4\pi\alpha}\qquad(3.151)$$

[18] In order to complete the quantum process leading to a well defined and fully consistent Lorentz covariant perturbation theory for the Standard Model, one has to further introduce nonphysical field such as the auxiliary scalar multiplet together with the so called Faddeev-Popov ghost and anti-ghost fields, beyond the above discussed quantum fields for physical detectable particles, as we shall see in the next Section.

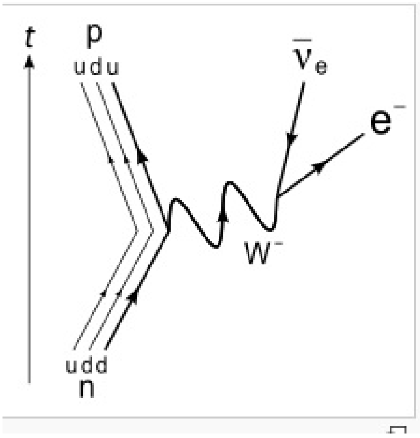

Figure 3.8: Feynman diagram for the $\beta-$decay of a neutron into proton, electron and electron anti-neutrino through a charged vector boson W^-.

In a quite similar way, the couplings between the charged vector boson and the weak currents are given by

$$\frac{g}{\sqrt{2}}\, W_\mu^+ \sum_{\iota=1}^{3} \left[\overline{\nu}_\iota\, \gamma^\mu\, \ell_\iota^- + \sum_{\text{colors}} \left(\overline{U}_\iota^-\, \gamma^\mu\, V_{\iota\jmath}\, D_\jmath^- \right) \right] + \text{c.c.} \tag{3.152}$$

As expected, only left-handed fermions are involved into the charged weak current. Moreover, after setting into diagonal form the quark mass matrix one necessarily introduces into the charged weak hadronic current the off-diagonal mixing terms, which are provided by the quark linear combinations encoded in the CKM matrix (3.137).

Nuclear beta decay. Consider now a process such as the nuclear $\beta-$decay $n \longrightarrow p\, e^-\, \bar{\nu}_e$ or the muon decay $\mu^- \longrightarrow e^-\, \bar{\nu}_e\, \nu_\mu$, where the momentum transfer is extremely small with respect to the $W_\mu^\pm$ mass m_W. As a matter of fact, in the neutron rest frame, after the decay the proton is also essentially at rest - up to some tiny irrelevant recoil - so that the Mandelstam variable t, *i.e.* the square of the momentum transfer, becomes $t \simeq (m_n - m_p)^2 \simeq 1.7\ \text{MeV}^2$ that yields $t/M_W^2 \simeq 2.5 \times 10^{-10}$. The charge-less neutron nucleon has the quark structure $n = ddu$ while the positively charged proton nucleon has $p = uud$. Hence, in the $\beta-$decay one of the quarks d of the neutron turns into one of the quarks u of the proton *via* the emission of a virtual W^-, owing to charge conservation, the remaining quarks being relegated to mere spectators – see Fig.(3.8). It follows that the relevant interaction term in the Lagrangian reduces to

$$\frac{g}{2\sqrt{2}}\, W_\mu^- \,\left[\,\overline{e}\,\gamma^\mu(\mathbb{I}-\gamma_5)\,\nu_e + 3\,\overline{d}\,\gamma^\mu(\mathbb{I}-\gamma_5)\,u\,\cos\theta_c\,\right] \tag{3.153}$$

where 3 stands for the number of colors and θ_c is the Cabibbo's angle. It turns out that the lowest order amplitude for the above process can be obtained from the LSZ reduction formulas and reads

$$-\,\tfrac{3}{8}\,ig^2\cos\theta_c\,\overline{u}_r(p)\,\gamma^\mu(\mathbb{I}-\gamma_5)\,d_s(q)\,\overline{e}_{r'}(p')\,\gamma^\nu(\mathbb{I}-\gamma_5)\,\upsilon_{s'}(q')\,\widetilde{W}^-_{\mu\nu}(q-p)$$

Here I denoted by (q,s) the incoming neutron $d-$quark momentum and polarization, (p,r) the corresponding quantities for the outgoing proton $u-$quark, (p',r') for the outgoing electron momentum and (q',s') for the electron emerging anti-neutrino; moreover $d_s(q)$ is the incoming quark spin state, $\overline{u}_r(p)$ the spin state for the outgoing quark, while $\overline{e}_{r'}(p')$ and $\upsilon_{s'}(q')$ those ones for the outgoing electron and electron anti-neutrino respectively. Finally, $\widetilde{W}_{\mu\nu}(q-p)$ stands for the Fourier transform of the Feynman propagator of the charged vector boson: namely,

$$\begin{aligned} W_{\mu\nu}(x-y) &= \langle 0|\,\mathrm{T}\,W_\mu^-(x)W_\nu^+(y)\,|0\rangle \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\widetilde{W}_{\mu\nu}(k)\,\exp\{-\,ik\cdot(x-y)\} \\ \widetilde{W}_{\mu\nu}(k) &= \frac{-\,i}{k^2-m_W^2+i\varepsilon}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{m_W^2}\right) \end{aligned}$$

Now, for $k_\mu = (q-p)_\mu \simeq (m_n - m_p, 0, 0, 0)$ it turns out that we can safely approximate

$$\widetilde{W}^\pm_{\mu\nu}(q-p) \simeq g_{\mu\nu}\,\frac{i}{m_W^2} \tag{3.154}$$

in such a manner that the lowest order amplitude for the $\beta-$decay reduces to the expression

$$\tfrac{3}{8}\,g^2\cos\theta_c\,\overline{u}_r(p)\,\gamma^\mu(\mathbb{I}-\gamma_5)\,d_s(q)\,\overline{e}_{r'}(p')\,\gamma_\mu(\mathbb{I}-\gamma_5)\,\upsilon_{s'}(q')/m_W^2$$

which coincides with the amplitude of the old-standing $V-A$ theory (3.77) of the weak interactions, provided we identify

$$\frac{G_F}{\sqrt{2}} \simeq \frac{g^2}{8m_W^2} = \frac{1}{2\upsilon^2} \quad\Rightarrow\quad \upsilon = 1/\sqrt{G_F\sqrt{2}} \simeq 246\ \text{GeV} \tag{3.155}$$

where use has been made of the relation (3.106).

Contrary to the charged weak current in equation (3.152), the coupling of the Z^0 boson to the neutral weak current involves both right- and left-handed fermions: namely,

$$
\begin{aligned}
& \frac{g}{2}\, Z^0_\mu \sec\theta_W \sum_{\iota=1}^{3} \Big\{ \overline{\nu}_\iota\, \gamma^\mu\, \nu_\iota - \overline{\ell_\iota^-}\, \gamma^\mu\, \ell_\iota^- + 2\sin^2\theta_W\, \overline{\ell_\iota}\, \gamma^\mu\, \ell_\iota \\
+ \quad & \sum_{\text{colors}} \Big[\overline{U}_\iota^-\, \gamma^\mu\, U_\iota^- - \overline{D}_\iota^-\, \gamma^\mu\, D_\iota^- \\
+ \quad & 2\sin^2\theta_W \left(\tfrac{1}{3}\, \overline{D}_\iota\, \gamma^\mu\, D_\iota - \tfrac{2}{3}\, \overline{U}_\iota\, \gamma^\mu\, U_\iota \right) \Big] \Big\} \qquad (3.156)
\end{aligned}
$$

This beautiful formula manifestly exhibits the key feature of the neutral weak current to be diagonal in the quark flavour space, *i.e.* NO FLAVOR CHANGING NEUTRAL WEAK CURRENTS.

GIM mechanism. We have already seen in the previous section that, owing to the presence of the CKM matrix, the flavor space quark mass matrices are not diagonal – see (3.127). This means that quark mass eigenstates are different from quark gauge eigenstates, *i.e.* those ones having definite transformation properties under the action of the gauge group $\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ and do appear in the covariant derivatives. As we have already seen, this fact just produces the CKM mixing of the quarks in the charged weak currents (3.134). To the aim of shorten and simplify the calculations, but without loss of generality, let us restrict to the two flavors case $\iota = 1, 2$ that means

$$
q_1^-(x) = \tfrac{1}{2}(\mathbb{I} - \gamma_5) \begin{pmatrix} u(x) \\ d(x) \end{pmatrix} \qquad q_2^-(x) = \tfrac{1}{2}(\mathbb{I} - \gamma_5) \begin{pmatrix} c(x) \\ s(x) \end{pmatrix}
$$

$$
\begin{aligned}
u_1^+(x) &= \tfrac{1}{2}(\mathbb{I} + \gamma_5)\, u(x) \qquad & d_1^+(x) &= \tfrac{1}{2}(\mathbb{I} + \gamma_5)\, d(x) \\
u_2^+(x) &= \tfrac{1}{2}(\mathbb{I} + \gamma_5)\, c(x) \qquad & d_2^+(x) &= \tfrac{1}{2}(\mathbb{I} + \gamma_5)\, s(x)
\end{aligned}
$$

in such a manner that the (12)-minor of the CKM matrix (3.137) just reduces to the Cabibbo rotation matrix

$$
R(\theta_c) = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \qquad (3.157)
$$

for $\theta_1 = \theta_c\,,\, \theta_2 = \theta_3 = \delta = 0$. It follows that the restricted charged weak currents take the form

$$
J_+^\mu(x) \;=\; \sum_{\text{colors}} \Big[\bar{u}^-(x)\gamma^\mu d^-(x) + \bar{c}^-(x)\gamma^\mu s^-(x) \Big]
$$

$$= \sum_{\rm colors} \sum_{\iota=1,2} \bar{u}_\iota^{\,-}(x)\,\gamma^\mu\, d_\iota^{\,-}(x) \tag{3.158}$$

$$J_-^\mu(x) = \sum_{\rm colors} \sum_{\iota=1,2} \bar{d}_\iota^{\,-}(x)\,\gamma^\mu\, u_\iota^{\,-}(x) = [\, J_+^\mu(x)\,]^* \tag{3.159}$$

where the column-down and row-up quark-doublet have been suitably introduced

$$d_\iota^{\,-}(x) = \left(\begin{array}{c} d^{\,-}(x) \\ s^{\,-}(x) \end{array} \right) \qquad \bar{u}_\iota^{\,-}(x) = \left(\begin{array}{cc} \bar{u}^{\,-}(x) & \bar{c}^{\,-}(x) \end{array} \right) \tag{3.160}$$

and since we have

$$\bar{u}_\iota^{\,-}(x) = \sum_{\jmath=1,2} \overline{U}_\jmath^{\,-}(x)\, \left(V_-^{u\dagger} \right)_{\iota\jmath}$$
$$d_\iota^{\,-}(x) = \sum_{\varkappa=1,2} \left(V_-^{\,d} \right)_{\iota\varkappa} D_\varkappa^{\,-}(x)$$

we eventually obtain

$$\begin{aligned} J_+^\mu(x) &= \sum_{\rm colors} \sum_{\iota=1,2} \sum_{\jmath=1,2} \overline{U}_\jmath^{\,-}(x)\, \left(V_-^{u*} \right)_{\jmath\iota} \gamma^\mu \sum_{\varkappa=1,2} \left(V_-^{\,d} \right)_{\iota\varkappa} D_\varkappa^{\,-}(x) \\ &= \sum_{\rm colors} \sum_{\jmath=1,2} \sum_{\varkappa=1,2} \overline{U}_\jmath^{\,-}(x)\,\gamma^\mu \left(V_-^{u\dagger}\, V_-^{\,d} \right)_{\jmath\varkappa} D_\varkappa^{\,-}(x) \end{aligned} \tag{3.161}$$

However, the 2×2 matrices product $V_-^{u\dagger}\, V_-^{\,d}$ is nothing but the reduced CKM matrix, *i.e.* the Cabibbo matrix $R(\theta_c)$ that yields

$$\begin{aligned} J_+^\mu(x) &= \sum_{\rm colors} \Big\{ \overline{U}^{\,-}(x)\,\gamma^\mu \left[\, D^{\,-}(x)\, \cos\theta_c + S^{\,-}(x)\, \sin\theta_c \,\right] \\ &+ \overline{C}^{\,-}(x)\,\gamma^\mu \left[\, -D^{\,-}(x)\, \sin\theta_c + S^{\,-}(x)\, \cos\theta_c \,\right] \Big\} \\ &= \tfrac{1}{2} \sum_{\rm colors} \Big[\, \overline{U}(x)\,\gamma^\mu\,(\mathbb{I}-\gamma_5)\, D_\theta(x) + \overline{C}(x)\,\gamma^\mu\,(\mathbb{I}-\gamma_5)\, S_\theta(x) \Big] \end{aligned} \tag{3.162}$$

the very last expression just coinciding - up to the factor $\frac{1}{2}$, the sum over colors and the addition of its complex conjugated term - with the old-standing V-A hadronic charged weak current $J_{\rm h}^\mu(x)$ in (3.75). This simple though nontrivial derivation gives us the key to grasp the main tool supporting the GIM mechanism. In order to allow the presence of the CKM flavor mixing matrix $V = V_-^{u\dagger} V_-^{\,d}$ in the hadronic electroweak currents, it is NECESSARY by construction to deal with quark bi-linears which involve up and down components of the families. It turns out that only the off-diagonal flavor matrices $\boldsymbol{\tau}_\pm$ do the job, the other two matrices $\boldsymbol{\tau}_3$ and $\boldsymbol{\tau}_3 + \frac{1}{2}\mathbb{I}$ being diagonal and unable to mix up and down flavor components. Actually, this originally motivated Sheldon Lee Glashow, Ioànnis Iliopoulos and Luciano Maiani to set the charmed quark coupled to $S_\theta(x)$ in such a manner that $S_\theta(x)$ has the same SU(2)×U(1) quantum numbers as $D_\theta(x)$ in order to cancel strangeness changing neutral currents. Otherwise, it would be allowed to see neutral current processes of order G_F with $\Delta S \neq 0$, such as $K_L \longrightarrow \mu^+\mu^-$ or $K^\pm \longrightarrow \pi^\pm\nu\bar{\nu}$, what is ruled out by phenomenology. It is worth to stress that GIM achieve this suppression without any *ad hoc* adjustment of the parameters involved in the neutral current processes. In fact, the GIM mechanism means much more than the above lowest order cancellation: it also provides the suppression of the $\Delta S \neq 0$ neutral current contributions that are induced by higher order loop diagrams, *i.e.* by radiative corrections. This is mandatory, for those radiatively induced amplitudes would be of order $\alpha\, G_F$, without GIM, while the experimental data about those processes are much smaller, typically of order $(mG_F)^2$, where m is a characteristic mass scale of the order of few eV. A paradigmatic example of this occurrence is in the computation of the $K_L - K_S$ mass difference $\Delta m = 0.35 \times 10^{-5}$ eV.

Turning back the neutral current expression in (3.156), a further remark is that the axial part of its hadronic piece is provided by

$$\tfrac{1}{4}\sin\theta_W \sum_{\iota=1}^{3} \sum_{\text{colors}} \left[\overline{U}_\iota(x)\,\gamma^\mu\,\gamma_5\,U_\iota(x) - \overline{D}_\iota(x)\,\gamma^\mu\,\gamma_5\,D_\iota(x) \right]$$

This very special form of the pseudo-vector axial neutral current is quite relevant for phenomenological applications in nuclear and atomic Physics, such as the induced parity violating effects in atoms and nuclei.

Gauge bosons self-couplings.

One of the main characteristic features of the Yang-Mills theory [19] is the peculiar form of the self-couplings among the various gauge bosons. It follows from the pure Yang-Mills part of the Lagrangian, *i.e.* that one which involves the non-Abelian field strengths: namely,

$$\mathcal{L}_{\text{YM}} \;=\; -\tfrac{1}{2}\,\text{tr}\,\left[\,G_{\mu\nu}\,G^{\mu\nu}\,\right] \;=\; -\tfrac{1}{2}\,\text{tr}\,\left[\,W_{\mu\nu}\,W^{\mu\nu}\,\right] \tag{3.163}$$

[19] Chen-Ning Franklin Yang & Robert Laurence Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, The Physical Review **96**, 191 (1954).

where

$$
\begin{aligned}
G_{\mu\nu}(x) &= \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) - ig\,[\,V_\mu(x)\,,\,V_\nu(x)\,] \\
W_{\mu\nu}(x) &= \partial_\mu W_\nu(x) - \partial_\nu W_\mu(x) - ig\,[\,W_\mu(x)\,,\,W_\nu(x)\,] \\
V^{\,\mu}(x) &= V_a^{\,\mu}(x)\,\boldsymbol{\tau}_a \qquad\quad W^{\,\mu}(x) = W_a^{\,\mu}(x)\,\boldsymbol{\tau}_a \\
U_\chi(x) &\equiv \exp\{i\chi_a(x)\boldsymbol{\tau}_a/\upsilon\} \\
V_\mu(x) &\equiv U_\chi(x)\,W_\mu(x)\,U_\chi^{-1}(x) - \frac{i}{g}\,[\,\partial_\mu U_\chi(x)\,]\,U_\chi^{-1}(x) \\
G_{\mu\nu}(x) &= U_\chi(x)\,W_{\mu\nu}(x)\,U_\chi^{-1}(x) \\
W_\mu(x) &= \frac{1}{2}\begin{pmatrix} W_\mu^3(x) & W_\mu^+(x)\sqrt{2} \\ W_\mu^-(x)\sqrt{2} & -W_\mu^3(x) \end{pmatrix} \\
W_\mu^3(x) &= \cos\theta_W\,Z_\mu^0(x) + \sin\theta_W\,A_\mu(x)
\end{aligned}
$$

Thus, expressed in terms of the physical vector fields $W_\mu^\pm, Z_\mu^0, A_\mu$, the gauge bosons self-coupling reads

$$
\begin{aligned}
&- \; i\left[\,W_-^{\,\nu}(x)\,w_{\mu\nu}^+(x) - W_+^{\,\nu}(x)\,w_{\mu\nu}^-(x)\,\right][\,eA^\mu(x) + g\cos\theta_W Z_0^\mu(x)\,] \\
&- \; i\,[\,eF^{\,\mu\nu}(x) + g\cos\theta_W z_0^{\mu\nu}(x)\,]\,W_\mu^{\,-}(x)W_\nu^{\,+}(x) \\
&- \; W_\mu^{\,+}(x)W_-^{\,\mu}(x)\,[\,eA(x) + g\cos\theta_W Z_0(x)\,]^2 \\
&+ \; W_\mu^{\,+}(x)W_\nu^{\,-}(x)\,[\,eA^\mu(x) + g\cos\theta_W Z_0^\mu(x)\,] \\
&\times \; [\,eA^\nu(x) + g\cos\theta_W Z_0^\nu(x)\,] - \tfrac{1}{2}\,g^{\,2}\left[\,W_\mu^{\,+}(x)W_-^{\,\mu}(x)\,\right]^2 \\
&+ \; \tfrac{1}{2}\,g^{\,2}\left[\,W_\mu^{\,+}(x)W_\nu^{\,-}(x)\,\right]\left[\,W_+^{\mu}(x)W_-^{\,\nu}(x)\,\right] \qquad (3.164)
\end{aligned}
$$

where I have used the small letters for the Abelian-like parts of the field strengths: namely, $w_{\mu\nu}^\pm = \partial_\mu\,W_\nu^\pm - \partial_\nu\,W_\mu^\pm$ and $z_{\mu\nu}^0 = \partial_\mu\,Z_\nu^0 - \partial_\nu\,Z_\mu^0$. Let us concentrate on the two γW^+W^- trilinear couplings of the first two lines. The first one is of the form $eA^\mu J_\mu$, which is nothing but the electromagnetic minimal coupling, while the second one is of the kind $-\,\mathrm{g}\,\mu_W F^{\,\mu\nu}S_{\mu\nu}$ which produces, in the low-energy and non-relativistic limit, a magnetic coupling with magnetic moment μ_W and gyro-magnetic factor g. The specific prediction which follows from the SU(2) gauge invariance is that the gyro-magnetic factor for the charged gauge bosons is 2 while the corresponding magnetic moment is $\mu_W = e\hbar/2m_W c$, as it does.

Higgs boson to fermion couplings.

These interaction terms are just provided by the Yukawa-like coupling

$$\begin{aligned} &- \frac{H(x)}{\sqrt{2}} \sum_{\iota=1}^{3} y_\iota^\ell \, \bar{\ell}_\iota(x)\ell_\iota(x) \\ &- \frac{H(x)}{\sqrt{2}} \sum_{\iota=1}^{3} \sum_{\text{colors}} \left[y_\iota^u \, \overline{U}_\iota(x)U_\iota(x) + y_\iota^d \, \overline{D}_\iota(x)D_\iota(x) \right] \end{aligned} \tag{3.165}$$

where the Yukawa-like couplings – the very same that generate the fermion masses after spontaneous symmetry breaking and shift of the scalar doublet – are expressed by the ratios

$$y_a/\sqrt{2} = m_a/v \qquad (\, a = e, \mu, \tau, u, c, t, d, s, b\,) \tag{3.166}$$

It turns out that the physical Higgs scalar field $H(x)$ couples to quarks and leptons with a strength which is proportional to the fermion mass. Hence, the prediction is that the Higgs particle will decay predominantly to the heaviest possible fermion compatible with the allowed phase space. This property provides a TYPICAL SIGNATURE for the Higgs particle identification. In the very same light, it appears that the top quark is the *most natural fermion* in the sense that $y_3^t = y_t = 0.99 \simeq 1$. Moreover, there is no direct coupling among the Higgs particle and neutrinos, as long as the neutrino masses can be disregarded.

Higgs scalar to gauge vector bosons couplings.

These couplings arise from the covariant derivative terms $(\, D^\mu \phi\,)^\dagger D_\mu \phi$ in the Standard Model Lagrangian: after shift to the physical Higgs field they read

$$\begin{aligned} & \left[\, 2m_W^2 \, W_+^\mu(x)W_\mu^-(x) + m_Z^2 \, Z_0^\mu(x)Z_\mu^0(x) \,\right] \left[\frac{H(x)}{v} + \frac{H^2(x)}{2v^2} \right] \\ = \;& gH(x) \left[m_W \, W_+^\mu(x)W_\mu^-(x) + \frac{m_Z}{2\cos\theta_W} \; Z_0^\mu(x)Z_\mu^0(x) \right] \\ + \;& \tfrac{1}{4}\, g^2 H^2(x) \Big[\, W_+^\mu(x)W_\mu^-(x) + \tfrac{1}{2} \sec^2\theta_W \, Z_0^\mu(x)Z_\mu^0(x) \Big] \end{aligned} \tag{3.167}$$

This leads to a 3-point direct vertexes HW^+W^- and HZ^0Z^0 which has been revealed very useful in the experimental Higgs particle searches.

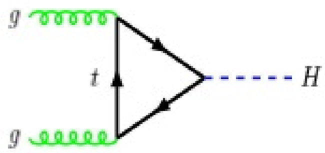

Figure 3.9: Feynman diagram $gg \longrightarrow H$ for the Higgs production at LHC.

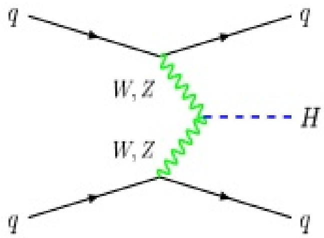

Figure 3.10: Feynman diagram $qq \longrightarrow qqH$ for the Higgs production at LHC.

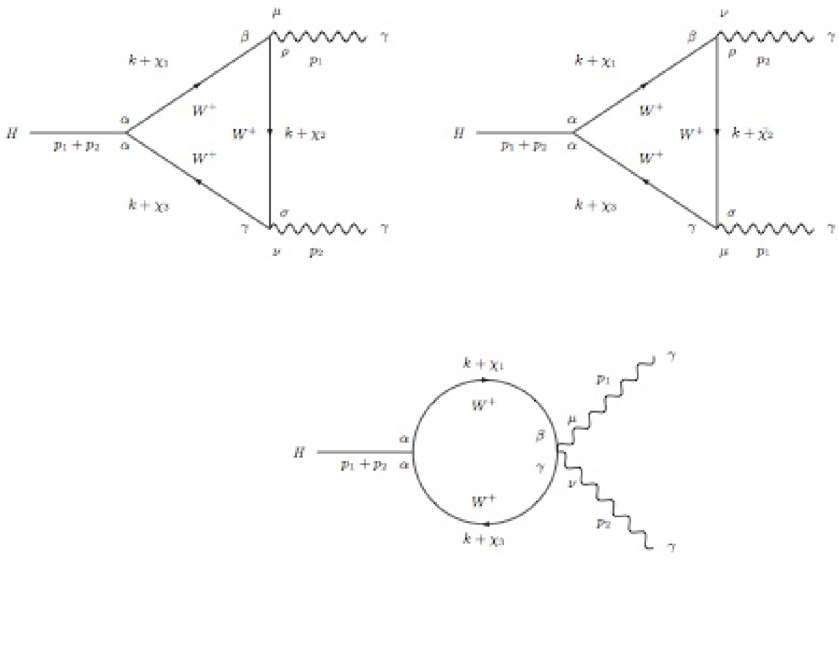

Figure 3.11: Feynman diagrams for the Higgs decay into a photon pair.

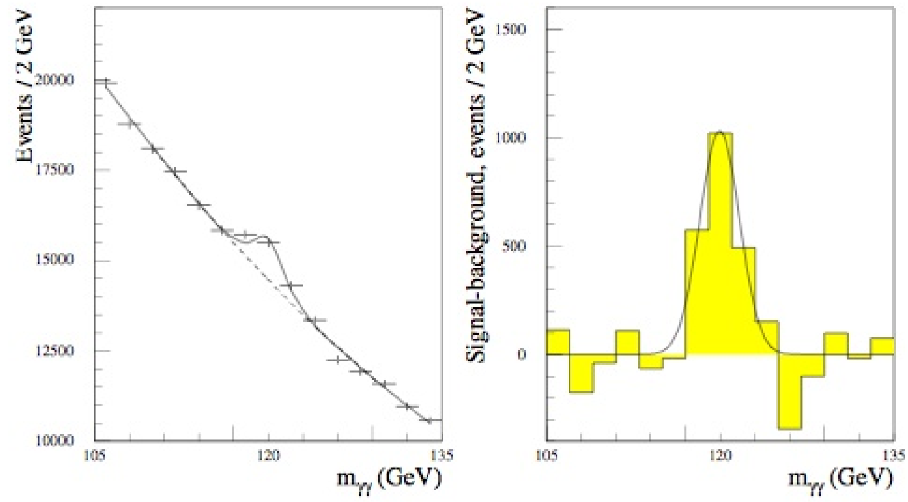

Figure 3.12: Higgs particle discovery through resonant peak $H \longrightarrow \gamma\gamma$.

Higgs boson self-couplings

These are provided by the scalar potential after the shift of the scalar doublet: there are a 3-point vertex and a 4-point vertex for the physical Higgs scalar self-couplings

$$\lambda v\, H^3(x) + \tfrac{1}{4}\,\lambda\, H^4(x) \tag{3.168}$$

and since we have

$$v^2 = 1/G_F\sqrt{2} = \mu^2/\lambda = m_H^2/2\lambda \quad \Leftrightarrow \quad \lambda \simeq G_F\, m_H^2/\sqrt{2} \approx 0.13$$

it follows that in the tree approximation the scalar self-coupling λ is proportional to square of the Higgs boson mass. This could provide a very neat test of the Standard Model Higgs, but it will not be easy to measure.

The Standard Model building up program is now complete for both lepton and quarks. The seventeen parameters of the model have been all determined by the experiments. Although the number of the free parameters to be fixed by the renormalization prescriptions looks to be pretty large, it should be kept in mind that it concerns all the mass and coupling parameters, just like the electron mass or the fine structure constant in Quantum Electro-Dynamics. The reason why we have more than these is because the Standard Model describes in a unified framework a much much larger number of particles and interactions.

Our confidence in this beautiful model is simply justified on the basis of its astonishing capability to quite accurately describe the bulk of the present day experimental data and even more on its enormous success in predicting new phenomena. Let me mention few of them following the historical order.

- The discovery [20] of the weak neutral currents by Gargamelle in 1972

  $$\nu_\mu + e^- \quad \longrightarrow \quad \nu_\mu + e^- \qquad\qquad \nu_\mu + N \quad \longrightarrow \quad \nu_\mu + X$$

  their strengths and properties being predicted by the Standard Model.

- The discovery [21] of the J/ψ charmed resonance at BNL-SLAC in 1974. The very existence of the charm quark was postulated by Sheldon Lee Glashow ten years before and it is essential to ensure the absence of the strangeness changing neutral currents processes like $K_0 \to \mu^+\mu^-$. The main property of the charmed mesons is to decay predominantly into strange particles owing to

  $$\begin{aligned} J_+^\mu(x) &= \sum_{\text{colors}} \Big\{ \overline{U}^{\,-}(x)\,\gamma^\mu \left[\,D^-(x)\,\cos\theta_c + S^-(x)\,\sin\theta_c\,\right] \\ &+ \overline{C}^{\,-}(x)\,\gamma^\mu \left[\,-D^-(x)\,\sin\theta_c + S^-(x)\,\cos\theta_c\,\right] \Big\} \end{aligned}$$

  below the bottom creation threshold.

- A necessary condition for the theoretical consistency of the Standard Model is that inside each family the following relation holds true

  $$\tfrac{1}{2}\,\mathrm{tr}\,Y_\iota \;=\; \mathrm{tr}\,Q_\iota \;=\; 0 \qquad\quad \forall\,\iota = 1,2,3 \tag{3.169}$$

  where the symbol tr stands for the sum over all fermionic hypercharge or electric charge values for leptons and quarks of a given family. This is known [22] as the ANOMALY CANCELLATION CONSTRAINT. If we recall that

  $$Y(\Psi_\iota) = -\,1 \qquad\quad Y(\ell_\iota^+) = -\,2$$

---

[20] A. Lagarrigue, P. Mousset, A. Rousset *et al.* (1973) *Observation of neutrino-like interactions without muon or electron in the GARGAMELLE neutrino experiment*, Physics Letters **46B**, 38.

[21] Samuel Chao Chung Ting *et al.* (1974) *Experimental Observation of a Heavy Particle J*, The Physical Review Letters **33**, 1404; Burton Richter *et al.* (1974) *Discovery of a Narrow Resonance in e⁺e⁻ Annihilation*, The Physical Review Letters **33**, 1406.

[22] C. Bouchiat, J. Iliopoulos and Ph. Meyer (1972) *An anomaly-free version of the Weinberg's model*, Physics Letters **38B**, 519-523.

$$Y(q_\iota^-) = \frac{1}{3} \qquad Y(u_\iota^+) = \frac{4}{3} \qquad Y(d_\iota^+) = -\frac{2}{3}$$

we get in fact

$$\mathrm{tr}\, Y_\iota = (-1) \times 2 - 2 + 3 \left(\tfrac{1}{3} \times 2 + \tfrac{4}{3} - \tfrac{2}{3} \right) = 0 \qquad \forall\, \iota = 1, 2, 3$$

When the $\tau-$lepton was discovered by Perl [23] and collaborators at SLAC-LBNL in a series of experiments between 1974 and 1977, then the prediction was entailed for the existence of a new family of t and b quarks with the right electric charges, the b quark being actually soon observed in the Υ bottomed resonance [24] by Leon Max Lederman and collaborators at Fermilab in 1977.

- The discovery by Carlo Rubbia and the UA1 group [25]of the massive vector bosons $W^\pm$ and Z^0 at CERN in 1883 with the right masses as theoretically predicted. Moreover, the characteristic relation $m_Z = m_W / \cos\theta_W$, from the Standard Model Higgs mechanism triggered by an SU(2) doublet, has been actually checked up to very high accuracy, including radiative corrections.

- The top quark was *foreseen* at LEP through its effects in radiative corrections to e^+e^- annihilation, before its actual joint discovery [26]at Fermilab in 1995 by two groups.

- The vector bosons self-couplings γW^+W^- and $Z^0W^+W^-$ have been measured at LEP and confirm the Yang-Mills prediction concerning the gyro-magnetic factor and the magnetic moment.

- The recent discovery of a new boson resonance which could be very likely the Higgs boson of the Standard Model is the last one of this astonishing series or successes.

All the above discoveries should not make us to forget that the Standard Model has been equally successful in fitting a huge number of experiments. The conclusion is obvious:

[23]Martin Lewis Perl *et al.* (1975) *Evidence for Anomalous Lepton Production in e^+e^- Annihilation*, The Physical Review Letters **35** 1489.

[24]Leon Max Lederman *et al.* (1977) *Observation of a Dimuon Resonance at 9.5 GeV in 400 GeV Proton-Nucleus Collisions*, The Physical Review Letters **39**, 252.

[25]Carlo Rubbia *et al.* (1983) *Experimental observation of isolated large transverse energy electrons with associated missing energy at $\sqrt{s} = 540$ GeV*, Physics Letters **122B**, 103-116.

[26]F. Abe *et al.* (1995) *Observation of Top Quark Production in $\bar{p}p$ Collisions with the Collider Detector at Fermilab*, The Physical Review Letters **74**, 2626.

THE STANDARD MODEL IS A CONSISTENT THEORETICAL CONSTRUCTION WHICH HAS BEEN ENORMOUSLY SUCCESSFUL IN EXPLAINING ESSENTIALLY ALL THE PRESENT DAY PHENOMENOLOGY OF THE HIGH ENERGY AND PARTICLE PHYSICS.

3.2.6 Unitarity versus Renormalizability: R_ξ gauges

The classical Lagrange density (3.149) does actually involve the physical fields and their interactions. On the one hand, the unbroken electromagnetic sector is described by the Maxwell-Lorentz theory, where the radiation field exhibits gauge invariance, *i.e.* masslessness, as well as two transverse polarization states. On the other hand, the free quadratic part of the Lagrangian for the massive vector bosons is of the Proca kind, which entails three polarization states for the $W^\pm$ and Z^0 particles and leads to the momentum space causal Green's function

$$D_{\mu\nu}(k) = \frac{i}{k^2 - m^2 + i\varepsilon}\left(-\,g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\right) \tag{3.170}$$

where m stands for m_W and/or m_Z. It turns out that the above propagator does not scale homogeneously for large wave numbers, as it is well known, thereby preventing *a priori* the possibility to determine the superficial degree of divergence of a generic Feynman diagram – see § 2.1.1. As a consequence, it appears that the so called manifestly unitary Lagrangian (3.149), which involves only physical fields with physical polarization states and all their interactions dictated by gauge invariance and naïve dimensional analysis [27] is not at all suitable to develop perturbation theory because

- it does not allow to derive the photon propagator
- the Proca propagators do not supply the necessary condition for power counting renormalizability

Thus, in order to overcome the above two main obstacles, one has to introduce a sensible number of nonphysical fields and polarization states to the aim of providing a Lorentz covariant photon propagator and a Stückelberg-like propagator for the massive vector fields – see § 1.2.3. This can be done by a suitable nontrivial modification of the Lagrangian known as the R_ξ gauges

[27]This means that, according to the general criterion of renormalizability by power counting, couplings with the dimensions of negative powers of any mass scale are forbidden.

quantization procedure. The highly nontrivial task became then to proof [28] that:

i) the theory is actually renormalizable, *i.e.* all the divergences of the Green's functions can be removed order-by-order in the perturbation theory by a suitable redefinition of the free parameters, *i.e.* couplings, masses and fields amplitudes, and that the renormalized electroweak theory still enjoys the U(2) gauge invariance

ii) all the nonphysical degrees of freedom that must be introduced in the R_ξ gauges do eventually decouple from the gauge invariant observables, *e.g.* the cross-sections of collision processes and the decay mean life-time involving physical particles

iii) in other words, a physical sub-space $\mathfrak{H}_{\mathrm{phys}}$ of the entire Fock space $\mathfrak{F}$ of the quantum states can always be defined, such that the restriction to $\mathfrak{H}_{\mathrm{phys}}$ of the scattering matrix is a unitary operator

iv) the physical sub-space is a Hilbert space with a positive semi-definite norm, while the entire Fock space of the states in an indefinite metric linear space

v) the main technical tool to prove order-by-order the gauge invariance of the renormalized observables is provided by the Slavnov-Taylor [29] identities, the generalization of the QED Ward identities – see § 2.3.1 – to the non-Abelian case.

The perturbative quantum theory of the Standard Model, that will drive to define a rich and exhaustive set of Feynman rules for propagators and vertexes of all the involved quantum fields, both physical and nonphysical, will be developed by a suitable generalization, *mutatis mutandis*, to the non-Abelian case of the perturbative quantum theory for the original Abelian Higgs model in a general linear covariant gauge $\partial \cdot V(x) - \xi B(x) = 0$ – see paragraph 1.2.3. Needless to say, as we have thoroughly discussed in Section 3.1, a suitable set of auxiliary scalar fields and Faddeev-Popov ghost fields has to be necessarily introduced, in spite of being nonphysical, in order to

[28]Gerard 't Hooft & Martin J.G. Veltman (1972) *Regularization and Renormalization of Gauge Fields*, Nucl. Phys. **B44**, 189-213; Benjamin W. Lee & Jean Zinn-Justin (1972) *Spontaneously Broken Gauge Symmetries*, Phys. Rev. D**5**, 3121-3160.

[29]Andrei Alexeevic Slavnov (1972) *Ward identities in gauge theories*, Theor. Math. Phys. **10**, 99; John Cecil Taylor (1971) *Ward identities and charge renormalization of the Yang-Mills fields*, Nucl. Phys. **B33**, 436; J.C. Taylor (1976) *Gauge Theories of Weak Interactions*, Cambridge University Press, Cambridge (UK).

realize a manifestly covariant and power-counting renormalizable quantum theory for the electro-weak non-Abelian gauge theory.

On the one hand, it is worthwhile to remind that, in order to set up the manifestly Lorentz invariant quantization of the gauge invariant Abelian Higgs model and to derive the related Feynman rules in the so called R_ξ gauge, one has:

$i)$ to redefine the field functions as follows: namely,

$$|\langle 0|\phi(x)|0\rangle| = \frac{\upsilon}{\sqrt{2}} \equiv \frac{\mu}{\sqrt{2\lambda}}$$

$$\phi(x) = \frac{1}{\sqrt{2}}[\upsilon + \eta(x)]\exp\{i\chi(x)/\upsilon\}$$
$$A_\mu(x) \equiv V_\mu(x) + \partial_\mu\chi(x)/\upsilon g$$

$ii)$ to add a gauge fixing: for example, the simplest choice is the linear and Lorentz invariant gauge fixing Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{g.f.}} &= V^\nu\partial_\nu B + \tfrac{1}{2}\xi B^2 \qquad (3.171)\\ &= \left(A^\nu - \frac{\sqrt{\lambda}}{g\mu}\partial^\nu\chi\right)\partial_\nu B + \tfrac{1}{2}\xi B^2 \end{aligned}$$

leading to the non-homogeneous Lorenz condition $\partial\cdot V = \xi B$, where $B(x)$ is the auxiliary scalar field and $\xi \in \mathbb{R}$ the gauge-fixing parameter.

On the other hand, we should as well recall that the generating functional of the non-Abelian gauge theories – with unbroken gauge symmetry like QCD – in the linear covariant gauges actually reads

$$\begin{aligned} Z_{\text{YM}}[J,\zeta,\overline{\zeta},\beta,\overline{\beta}] &= \mathcal{N}\int\mathfrak{D}A\int\mathfrak{D}\psi\int\mathfrak{D}\overline{\psi}\int\mathfrak{D}B\int\mathfrak{D}\eta\int\mathfrak{D}\overline{\eta}\\ &\times\ \exp\left\{\frac{i}{\hbar c}\int \mathrm{d}^4x\,[\mathcal{L}_{\text{eff}} + \mathcal{L}_{\text{sources}}]\right\} \qquad (3.172) \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{FP}} &(3.173)\\ \mathcal{L}_{\text{YM}} &= -\tfrac{1}{2}\,\mathrm{tr}\,[F_{\mu\nu}(x)\,F^{\mu\nu}(x)] + \overline{\psi}(x)\,(i\not{D} - M)\,\psi(x) &(3.174)\\ \mathcal{L}_{\text{g.f.}} &= \partial^\mu B^a(x)\,A^a_\mu(x) + \tfrac{1}{2}\,\xi\,B^a(x)\,B^a(x) &(3.175)\\ \mathcal{L}_{\text{FP}} &= -\,i\,\partial^\mu\,\overline{\eta}^a(x)\,\nabla^{ab}_\mu\,\eta^b(x) &(3.176)\\ \mathcal{L}_{\text{sources}} &= J^{\mu,a}A^a_\mu + \overline{\psi}\,\zeta + \overline{\zeta}\,\psi + \overline{\eta}^a\,\beta^a + \overline{\beta}^a\,\eta^a &(3.177) \end{aligned}$$

where A^a_μ and $J^{\mu,a}$ are real functions while all other fields and sources are Graßmann valued functions over the Minkowski space. As already noticed, one can functionally integrate over the auxiliary scalar field multiplet B^a – or equivalently substitute the constraint $\partial \cdot A = \xi B$ in the gauge-fixing Lagrangian – and get

$$\mathcal{L}_{\text{g.f.}} = \frac{1}{2\xi}\, A^\mu(x)\, \partial_\mu\, \partial_\nu\, A^\nu(x) \tag{3.178}$$

Hence, one has to safely and wisely melt the two above recipes to obtain the manifestly Lorentz invariant perturbation theory for the spontaneously broken non-Abelian gauge theory underlying the Standard Model, in such a manner to fulfill the crucial requirement of renormalizability by power counting. This can be done within the framework of the so called R_ξ gauges, as I will briefly discuss [30] in what follows. To this purpose, let me come back to the SU(2) symmetry breaking vacuum choice (3.103) for the Higgs iso-doublet in the polar representation, *viz.,*

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}}\, U_\chi(x)[\, v + H(x)\,] \begin{pmatrix} 0 \\ 1 \end{pmatrix} && (3.179) \\ U_\chi(x) &\equiv \exp\{i\chi_a(x)\boldsymbol{\tau}_a/v\} && (3.180) \end{aligned}$$

where $H(x)$ is the neutral and spin-less Higgs field, while $\chi_a(x)$ $(a = 1, 2, 3)$ are Goldstone-like neutral scalar fields that will completely disappear from the Dynamics, thanks to the above discussed Higgs mechanism. Remember that the total number of the classical real scalar wave fields that constitute the Higgs scalar doublet is still equal to four, as it does. Moreover, we did suitably define

$$\begin{aligned} V_\mu(x) &\equiv U_\chi(x)\, W_\mu(x)\, U_\chi^{-1}(x) - \frac{i}{g}\, [\, \partial_\mu U_\chi(x)\,]\, U_\chi^{-1}(x) && (3.181) \\ W_\mu(x) &= \frac{i}{g}\, U_\chi^{-1}(x)\, \partial_\mu U_\chi(x) + U_\chi^{-1}(x)\, V_\mu(x)\, U_\chi(x) && (3.182) \end{aligned}$$

and if we assume that the Fock vacuum is gauge and translation invariant

$$U_\chi(x)\,|\,0\,\rangle \;=\; |\,0\,\rangle \qquad\qquad P_\mu\,|\,0\,\rangle \;=\; 0$$

we get the SU(2) symmetry breaking condition

$$\langle\, 0\,|\,\phi(x)\,|\,0\,\rangle \;=\; v \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;=\; \langle\, 0\,|\,\phi(0)\,|\,0\,\rangle$$

[30]The definition and treatment of the so called renormalizable R_ξ gauges, that will be outlined here below, is a little bit different and simpler with respect to the more conventional one reported in the textbooks [7] and [16].

It follows therefrom that the straightforward generalization of the Abelian gauge fixing Lagrangian (3.171) to the non-Abelian case takes the form

$$\begin{aligned}\mathcal{L}'_{\text{g.f.}} &= W^a_\nu(x)\,\partial^\nu B^a(x) + \tfrac{1}{2}\,\xi B^a(x)B^a(x) \\ &= 2\,\mathrm{tr}\,\{[\,V_\nu(x) - \partial_\nu\chi(x)/g\upsilon\,]\,\partial^\nu \mathrm{B}(x)\} + \xi\,\mathrm{tr}\,[\,\mathrm{B}(x)\mathrm{B}(x)\,]\end{aligned} \tag{3.183}$$

where the auxiliary scalar field $\mathrm{B}(x) = B^a(x)\,\boldsymbol{\tau}_a$ transforms according to the adjoint representation of the gauge group

$$\mathrm{B}'(x) = U_\chi(x)\,\mathrm{B}(x)\,U_\chi^{-1}(x) \tag{3.184}$$

whereas $\xi \in \mathbb{R}$ is still the gauge-fixing parameter, in such a manner that we are lead again to the non-homogeneous non-Abelian Lorenz condition

$$\partial\cdot W(x) = \xi\,\mathrm{B}(x) \tag{3.185}$$

Furthermore we have to fix the gauge also for the Abelian part of the gauge potential, which is left untouched by the spontaneous symmetry breaking mechanism: namely,

$$\mathcal{L}''_{\text{g.f.}} = B^\mu(x)\,\partial_\mu B(x) + \tfrac{1}{2}\,\xi B^2(x) \tag{3.186}$$

$$\partial_\mu B^\mu(x) = \xi\,B(x) \tag{3.187}$$

Putting altogether

$$\begin{aligned}\mathcal{L}_{\text{g.f.}} &= \mathcal{L}'_{\text{g.f.}} + \mathcal{L}''_{\text{g.f.}} \\ &= W^a_\nu(x)\partial^\nu B^a(x) + B^\mu(x)\,\partial_\mu B(x) + \tfrac{1}{2}\,\xi[\,B^a(x)B^a(x) + B^2(x)\,]\end{aligned}$$

Now we recall the definitions

$$W^\pm_\mu(x) = \frac{W^1_\mu(x) \mp iW^2_\mu(x)}{\sqrt{2}}$$

$$\begin{pmatrix} Z^0_\mu(x) \\ A_\mu(x) \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu(x) \\ B_\mu(x) \end{pmatrix}$$

so that if we set

$$B^\pm(x) = \frac{B^1(x) \mp iB^2(x)}{\sqrt{2}} \tag{3.188}$$

$$\begin{pmatrix} B^3(x) \\ B(x) \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B^0(x) \\ \overline{B}(x) \end{pmatrix} \tag{3.189}$$

then we can rewrite the gauge-fixing Lagrangian in the suitable diagonal form

$$\begin{aligned}\mathcal{L}_{\text{g.f.}} &= W_\nu^+(x)\,\partial^\nu B^-(x) + W_\nu^-(x)\,\partial^\nu B^+(x) + \xi B^+(x)B^-(x) \\ &+ Z_\nu^0(x)\partial^\nu B^0(x) + A^\nu(x)\partial_\nu \overline{B}(x) + \tfrac{1}{2}\,\xi\left[\,B_0^2(x) + \overline{B}^2(x)\right]\end{aligned} \tag{3.190}$$

which drives to the non-Abelian subsidiary conditions

$$\partial\cdot W^{\pm} = \xi B^{\pm} \qquad \partial\cdot Z^0 = \xi B^0 \qquad \partial\cdot A = \xi\overline{B} \tag{3.191}$$

The finite and infinitesimal gauge transformations for the gauge potentials are as follows:

$$W_\mu(x) \longrightarrow W_\mu^{\,\omega}(x) = U_\omega(x)\,W_\mu(x)\,U_\omega^{-1}(x) - \frac{i}{g}\,[\,\partial_\mu U_\omega(x)\,]\,U_\omega^{-1}(x)$$

$$U_\omega(x) = \exp\{ig\,\omega^{\,a}(x)\boldsymbol{\tau}_a\}$$

$$B_\mu(x) \longrightarrow B_\mu^{\,\omega}(x) = B_\mu(x) - \partial_\mu\,\omega(x)$$

$$\begin{aligned}\delta W_\mu^a(x) &= \partial_\mu\,\delta\omega^{\,a}(x) + g\varepsilon^{\,abc}\,W_\mu^c(x)\,\delta\omega^{\,b}(x) \\ &= \left[\,\delta^{\,ab}\partial_\mu - g\varepsilon^{\,acb}\,W_\mu^c(x)\,\right]\delta\omega^{\,b}(x) \equiv \nabla_\mu^{\,ab}\delta\omega^{\,b}(x) \\ \delta B_\mu(x) &= \partial_\mu\,\delta\omega(x)\end{aligned}$$

where $f^{\,abc} = -\,\varepsilon^{\,abc}$ ($\varepsilon^{\,123} = \varepsilon_{123} = 1$) are the structure constant of the SU(2) gauge group, so that

$$\begin{aligned}\delta W_\mu^{\pm}(x) &= \partial_\mu\,\delta\omega^{\pm}(x) \pm igW_\mu^{\mp}(x)\,\delta\omega^3(x) \pm igW_\mu^3(x)\,\delta\omega^{\pm}(x) \\ \delta W_\mu^3(x) &= \partial_\mu\,\delta\omega^3(x) + igW_\mu^+(x)\,\delta\omega^-(x) - igW_\mu^-(x)\,\delta\omega^+(x) \\ \delta B_\mu(x) &= \partial_\mu\,\delta\omega(x)\end{aligned}$$

and consequently

$$\begin{aligned}\delta Z_\mu^0(x) &= ig\cos\theta_W\left[\,W_\mu^+(x)\,\delta\omega^-(x) - W_\mu^-(x)\,\delta\omega^+(x)\right] \\ &+ \cos\theta_W\,\partial_\mu\,\delta\omega^3(x) - \sin\theta_W\,\partial_\mu\,\delta\omega(x) \\ \delta A_\mu(x) &= \cos\theta_W\,\partial_\mu\,\delta\omega + \sin\theta_W\,\partial_\mu\,\delta\omega^3(x) \\ &+ ig\sin\theta_W\left[\,W_\mu^+(x)\,\delta\omega^-(x) - W_\mu^-(x)\,\delta\omega^+(x)\right]\end{aligned}$$

in such a manner that if we set

$$\begin{pmatrix} \omega^3(x) \\ \omega(x) \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} \omega^0(x) \\ \varpi(x) \end{pmatrix}$$

we eventually obtain the infinitesimal gauge transformations for the physical vector boson fields, *viz.*,

$$
\begin{aligned}
\delta W_\mu^\pm(x) &= \partial_\mu\,\delta\omega^\pm(x) \pm igW_\mu^\mp(x)\left[\cos\theta_W\,\delta\omega^0(x)+\sin\theta_W\,\delta\overline{\omega}(x)\right] \\
&\pm\ ig\left[\cos\theta_W\,Z_\mu^0(x)+\sin\theta_W\,A_\mu(x)\right]\delta\omega^\pm(x) \\
\delta Z_\mu^0(x) &= \partial_\mu\,\delta\omega^0(x)+ig\cos\theta_W\Big[W_\mu^+(x)\,\delta\omega^-(x)-W_\mu^-(x)\,\delta\omega^+(x)\Big] \\
\delta A_\mu(x) &= \partial_\mu\,\delta\overline{\omega}(x)+ig\sin\theta_W\,\Big[W_\mu^+(x)\,\delta\omega^-(x)-W_\mu^-(x)\,\delta\omega^+(x)\Big]
\end{aligned}
$$

Consider now the infinitesimal gauge variation of the subsidiary conditions

$$
\begin{aligned}
&\delta\left(\partial\cdot W^\pm-\xi B^\pm\right)=\Box\,\delta\omega^\pm\pm ig\,\xi B^\mp\left(\cos\theta_W\,\delta\omega^0+\sin\theta_W\,\delta\overline{\omega}\right) \\
&\pm\ ig\,\xi\left(\cos\theta_W\,B^0+\sin\theta_W\,B\right)\delta\omega^\pm-\xi\,\delta B^\pm \\
&\pm\ ig\,W_\mp^\mu\left(\cos\theta_W\,\partial_\mu\,\delta\omega^0+\sin\theta_W\,\partial_\mu\,\delta\overline{\omega}\right) \\
&\pm\ ig\left(\cos\theta_W\,Z_0^\mu+\sin\theta_W\,A^\mu\right)\partial_\mu\,\delta\omega^\pm
\end{aligned}
$$

Since we have

$$
\delta B^\pm=\pm ig\left[B^\mp\left(\cos\theta_W\,\delta\omega^0+\sin\theta_W\,\delta\overline{\omega}\right)+\left(\cos\theta_W\,B^0+\sin\theta_W\,B\right)\delta\omega^\pm\right]
$$

we actually find

$$
\begin{aligned}
\delta\left(\partial\cdot W^\pm-\xi B^\pm\right) &= \Box\,\delta\omega^\pm\pm\ ig\,W_\mp^\mu\left(\cos\theta_W\,\partial_\mu\,\delta\omega^0+\sin\theta_W\,\partial_\mu\,\delta\overline{\omega}\right) \\
&\pm\ ig\left(\cos\theta_W\,Z_0^\mu+\sin\theta_W\,A^\mu\right)\partial_\mu\,\delta\omega^\pm \qquad (3.192)
\end{aligned}
$$

and in a quite analogous way

$$
\delta\left(\partial\cdot Z^0-\xi B^0\right) = \Box\,\delta\omega^0+ig\cos\theta_W\Big[W^+\cdot\partial\,\delta\omega^--W^-\cdot\partial\,\delta\omega^+\Big] \qquad (3.193)
$$

$$
\delta(\partial\cdot A-\xi B) = \Box\,\delta\overline{\omega}+ig\sin\theta_W\,\Big[W^+\cdot\partial\,\delta\omega^--W^-\cdot\partial\,\delta\omega^+\Big] \qquad (3.194)
$$

To write down the Faddeev-Popov determinant in the basis of the physical vector boson fields $Z_\mu^0, W_\mu^\pm, A_\mu$, it is expedient to introduce a 4×4 matrix with entries labeled by capital Latin letters indexes $A, B, C, \ldots = (\,0, +, -, \overline{\mathrm{o}}\,)$. We find

$$
\mathsf{M}_{AB}=\delta_{AB}\,\Box-ig\,\Delta_{AB} \qquad (3.195)
$$

where $\Delta =$

$$
\begin{pmatrix}
0 & \cos\theta_W\,W^-\cdot\partial & -\cos\theta_W\,W^+\cdot\partial & 0 \\
-\cos\theta_W\,W^-\cdot\partial & -\cos\theta_W\,Z^0\cdot\partial-\sin\theta_W\,A\cdot\partial & \cos\theta_W\,Z^0\cdot\partial+\sin\theta_W\,A\cdot\partial & -\sin\theta_W\,W^-\cdot\partial \\
\cos\theta_W\,W^+\cdot\partial & \cos\theta_W\,Z^0\cdot\partial+\sin\theta_W\,A\cdot\partial & -\cos\theta_W\,Z^0\cdot\partial-\sin\theta_W\,A\cdot\partial & \sin\theta_W\,W^+\cdot\partial \\
0 & \sin\theta_W\,W^-\cdot\partial & -\sin\theta_W\,W^+\cdot\partial & 0
\end{pmatrix}
$$

It follows that if we introduce the four components column vectors for the physical vector bosons and the auxiliary scalar fields, *viz.,*

$$\Upsilon_\mu(x) \equiv \left(\begin{array}{c} Z^0_\mu(x) \\ W^+_\mu(x) \\ W^-_\mu(x) \\ A_\mu(x) \end{array} \right) \qquad \Omega(x) \equiv \left(\begin{array}{c} B^0(x) \\ B^+(x) \\ B^-(x) \\ B(x) \end{array} \right) \tag{3.196}$$

then we can rewrite the non-Abelian non-homogeneous Lorenz-like subsidiary conditions in the form

$$\Phi_A[\,\Upsilon_\mu(x)\,] \equiv \partial\cdot\Upsilon_A(x) = \xi\,\Omega_A(x) \qquad (\,A = 0,+,-,\bar{\mathrm{o}}\,) \tag{3.197}$$

which leads to the Faddeev-Popov operator

$$\|\,\mathsf{M}(x,y)\,\|^{AB} \;=\; \left[\,-\,\Box\,\delta^{AB} + ig\Delta^{AB}\,\right]\delta(x-y) \tag{3.198}$$

As usual, the standard tool to express the Faddeev-Popov determinant in terms of a local Action functional is by means of a functional integral over a pair of independent Lorentz scalar Graßmann valued field multiplets $\eta_A(x)$ and $\overline{\eta}_B(x)$, the Faddeev-Popov ghost and anti-ghost fields respectively. Then we can write

$$\det\mathsf{M} = \int\mathfrak{D}\,\eta\int\mathfrak{D}\,\overline{\eta}\ \exp\textstyle\int\mathrm{d}x\ \left[\,\partial^\mu\overline{\eta}_A(x)\,\partial_\mu\eta_A(x) + ig\overline{\eta}_A(x)\Delta_{AB}\,\eta_B(x)\,\right]$$

where we understand as usual

$$\int\mathfrak{D}\,\eta\int\mathfrak{D}\,\overline{\eta} \equiv \prod_{A,B}\ \prod_{x\,\in\,\mathcal{M}}\ \textstyle\int\mathrm{d}\eta^A_x\int\mathrm{d}\overline{\eta}^B_x$$

Putting altogether we can eventually write the generating functional of the electroweak Standard Model in the R_ξ gauge: namely,

$$\begin{aligned} Z_{\mathrm{SM}}[\,J,\zeta,\overline{\zeta},\beta,\overline{\beta}\,] \;&=\; \mathcal{N}\int\mathfrak{D}\Upsilon\int\mathfrak{D}H\int\mathfrak{D}\,\psi\int\mathfrak{D}\,\overline{\psi}\int\mathfrak{D}\,\eta\int\mathfrak{D}\,\overline{\eta} \\ &\times\;\exp\left\{i\int\mathrm{d}x\,[\,\mathcal{L}_{\mathrm{eff}} + \mathcal{L}_{\,\mathrm{sources}}\,]\right\} \end{aligned} \tag{3.199}$$

where $\mathcal{N}^{-1} = Z_{\mathrm{SM}}[\,0\,]$ whereas

$$\mathcal{L}_{\mathrm{eff}} \;=\; \mathcal{L}^{\,\prime} + \mathcal{L}_{\,\mathrm{g.f.}} + \mathcal{L}_{\,\mathrm{FP}} \tag{3.200}$$

$$\mathcal{L}_{\,\mathrm{g.f.}} \;=\; -\,\partial\cdot\Upsilon_A(x)\,\partial\cdot\Upsilon_A(x)/2\xi \tag{3.201}$$

$$\mathcal{L}_{\,\mathrm{FP}} \;=\; -\,i\,\partial^\mu\overline{\eta}_A(x)\,\partial_\mu\eta_A(x) + g\,\overline{\eta}_A(x)\Delta_{AB}\,\eta_B(x) \tag{3.202}$$

$$\mathcal{L}_{\,\mathrm{sources}} \;=\; J^A_\mu\,\Upsilon^\mu_A + JH + \overline{\psi}_f\,\zeta^{\,f} + \overline{\zeta}^{\,f}\,\psi_f + \overline{\eta}_A\,\beta^A + \overline{\beta}^{\,A}\,\eta_A \tag{3.203}$$

Here the sum over repeated indexes is understood while a new collective index f has been introduced and running over all fermion species, that means

$$f = e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d', c, s', t, b'$$

the dummy quarks color indexes being also tacitly included, while $\mathcal{L}'$ is the Standard Model Lagrangian (3.149) which is suitable for perturbation theory, whereas the integration boson field variables Υ_μ^A, H and related sources J_A^μ, J are real or complex functions, all other fields and sources being Graßmann valued over the Minkowski space [31] so that, as usual,

$$\int \mathfrak{D}H \equiv \prod_{x\,\in\,\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}H_x \qquad\qquad \int \mathfrak{D}\Upsilon \equiv \prod_{A,\,\mu}\ \prod_{x\,\in\,\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}\,\Upsilon_{\mu,\,x}^A$$

$$\int \mathfrak{D}\,\psi \int \mathfrak{D}\,\overline{\psi} \equiv \prod_f\ \prod_{x\,\in\,\mathcal{M}} \int \mathrm{d}\,\psi_{f,\,x} \int \mathrm{d}\,\overline{\psi}_{f,\,x}$$

It is worthwhile to remark that the gauge fixing Lagrangian can be written in different albeit equivalent ways, under the substitutions of the subsidiary conditions (3.191). Thus, in so doing, the gauge fixing Lagrange density (3.190) for the so called R_ξ gauge can be written in the different equivalent forms

$$\begin{aligned} \mathcal{L}_{\text{g.f.}} &= \xi\,\mathrm{tr}\,[\,\Omega(x)\,\Omega(x)\,] + 2\,\mathrm{tr}\,[\,\Upsilon(x)\cdot\partial\,\Omega(x)\,] && (3.204)\\ &\doteq \Upsilon_A^\mu(x)\,\partial_\mu\partial_\nu\Upsilon_A^\nu(x)/2\xi = \xi^{-1}W_-^\mu(x)\,\partial_\mu\partial_\nu W_+^\nu(x) \\ &+ \ [\,Z_0^\mu(x)\,\partial_\mu\partial_\nu Z_0^\nu(x) + A^\mu(x)\,\partial_\mu\partial_\nu A^\nu(x)\,]\,/2\xi && (3.205) \end{aligned}$$

up to boundary and subsidiary conditions.

Once again, as customary, the effective Lagrangian $\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ can be split into free quadratic part and the remaining higher degree interaction part, involving all the gauge, spinor and ghost fields. The interaction parts for the physical boson and fermion fields have been written and somewhat discussed and commented in the previous section, while the interaction part involving the Faddeev-Popov ghosts and the vector bosons is encoded in the $g-$dependent matrix of equation (3.195). Now it is a straightforward exercise to obtain the Feynman rules for the electroweak Standard Model in the general R_ξ linear covariant gauge. The interaction parts give raise to a number of 3-point and 4-point vertexes among scalar, spinor and vector

[31] It looks like a curious coincidence that if we disregard the quarks color indexes, which are admittedly dummy within the realm of the electroweak interactions, the number of the fundamental fields is again seventeen, just like the number of free parameters in the Standard Model.

fields, which can be easily and directly read off from the Lagrangian (3.149) and the Faddeev-Popov matrix (3.195), up to the weight factors encoded in the completely symmetric Bose-Einstein identical particle nature of the scalar and vector legs. Moreover, from the free and quadratic part of the effective Lagrangian one gets the causal Green's functions. Concerning the free non-interacting part of the effective Lagrangian, including gauge fixing, we definitely get

$$\begin{aligned}
\mathcal{L}_0 &= \tfrac{1}{2}\,\partial_\mu H\,\partial^\mu H - \tfrac{1}{2}\,m_H^2\,H^2 \\
&+ W_-^\mu(x)\left[\,g^{\mu\nu}\left(\Box + m_W^2\right) - (1-\xi^{-1})\,\partial_\mu\partial_\nu\,\right]W_-^\nu(x) \\
&+ \tfrac{1}{2}\,Z_0^\mu(x)\left[\,g^{\mu\nu}\left(\Box + m_Z^2\right) - (1-\xi^{-1})\,\partial_\mu\partial_\nu\,\right]Z_0^\nu(x) \\
&+ \tfrac{1}{2}\,A^\mu(x)\left[\,g^{\mu\nu}\,\Box - (1-\xi^{-1})\,\partial_\mu\partial_\nu\,\right]A^\nu(x) \\
&+ \sum_f \overline{\psi}_f\,(i\not\partial - m_f)\,\psi_f - i\sum_A \partial^\mu\,\overline{\eta}^{\,A}\,\partial_\mu\,\eta^{\,A}
\end{aligned} \tag{3.206}$$

that drives to the momentum space propagators

$$\frac{i}{k^2 - m_H^2 + i\varepsilon} \qquad (\,\text{Higgs particle}\,)$$

$$\frac{i\,(\not p + m_f)}{p^2 - m_f^2 + i\varepsilon} \qquad (\,f - \text{species fermion particle}\,)$$

$$\frac{\delta^{\,AB}}{\ell^{\,2} + i\varepsilon} \qquad (\,\text{Faddeev} - \text{Popov ghost particle}\,)$$

$$\frac{i}{k^2 + i\varepsilon}\left[\,-\,g_{\mu\nu} + \frac{(\,1-\xi\,)\,k_\mu\,k_\nu}{k^2 + i\varepsilon}\,\right] \qquad (\,\text{photon}\,)$$

$$\frac{i}{k^2 - m_A^2 + i\varepsilon}\left\{-\,g_{\mu\nu} + \frac{(\,1-\xi\,)\,k_\mu\,k_\nu}{k^2 - \xi\,m_A^2 + i\varepsilon}\right\} \qquad (\,\text{massive vector particles}\,)$$

where $m_A = m_W$ for the charged bosons $W^{\pm}$, $m_A = m_Z$ for the neutral boson Z^0 and $m_A = 0$ for photons. The key observation is that in the R_ξ gauges the propagators for the massive vector particles take the Stückelberg form, which guarantees the correct power counting behavior $O(1/k^2)$ for large wave numbers. Of course, the simplest choice to perform perturbative calculations is still the Feynman gauge $\xi = 1$, although for any finite choice of the gauge parameter ξ the theory is renormalizable, in the sense that all the divergences in the Green's functions – and *a fortiori* in the $S-$matrix elements – are removed with a finite set of counter-terms. Moreover, the Slavnov-Taylor identities just endorse that the only required counter-terms are those ones which keep untouched the original gauge symmetry. The finite parts of the counter-terms are fixed by the measured values of the seventeen free parameters of the Standard Model. Finally, the restriction of the renormalized $S-$matrix to the physical Hilbert sub-space, as well as the matrix elements of gauge invariant operators, turn out to be independent from the gauge parameter ξ, as expected. Hence, in the perturbative calculation of a renormalized physical quantity, we can eventually take the limit $\xi \longrightarrow \infty$ and turn back to the so called unitary gauge, in which only the physical spectrum and polarization do actually occur and contribute. This kind of miraculous expedient, which allows to reconcile renormalizability and unitarity, is the benchmark of the powerful and unique mechanism which lies at the ground of the Standard Model: non-Abelian gauge invariance and spontaneous symmetry breaking *via* the Higgs mechanism are in fact THE ONLY AVAILABLE AND FULLY CONSISTENT THEORETICAL TOOL to understand the electromagnetic, weak and strong interactions within the unified framework of a Lorentz invariant, causal, renormalizable and unitary quantum field theory.

Standard Model Vademecum

Basic definitions and conventions: the summation over repeated indexes is normally understood unless explicitly stated.

Lie Algebra

$$\boldsymbol{\tau}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{\tau}_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \boldsymbol{\tau}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\tau}_a\, \boldsymbol{\tau}_b = \tfrac{1}{4}\,\delta_{ab}\,\mathbb{I} + \tfrac{1}{2}\, i\varepsilon_{abc}\, \boldsymbol{\tau}_c \qquad (\, a, b, c, \ldots = 1, 2, 3\,; \quad \varepsilon_{123} = \varepsilon^{123} = 1\,)$$

$$[\, \boldsymbol{\tau}_a \,,\, \boldsymbol{\tau}_b \,] = i\varepsilon_{abc}\, \boldsymbol{\tau}_c$$

Gauge Potentials

$$W_\mu(x) = W_\mu^{\ a}(x)\, \boldsymbol{\tau}_a \qquad [\, W_\mu(x)\,,\, W_\nu(x)\,] = i\varepsilon_{abc}\, W_\mu^{\ a}(x) W_\nu^{\ b}(x)\, \boldsymbol{\tau}_c$$

$$W_\mu^{\ \pm}(x) = \frac{W_\mu^1(x) \mp i W_\mu^2(x)}{\sqrt{2}}$$

$$\begin{pmatrix} Z_\mu^0(x) \\ A_\mu(x) \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^{\ 3}(x) \\ B_\mu(x) \end{pmatrix}$$

$$W_\mu^1(x) W_\nu^{\ 2}(x) - W_\nu^{\ 1}(x) W_\mu^{\ 2}(x) = i W_\mu^{\ -}(x) W_\nu^{\ +}(x) - i W_\nu^{\ -}(x) W_\mu^{\ +}(x)$$

Field Stregths

$$\begin{aligned} B_{\mu\nu}(x) &= \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) \qquad F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \\ W_{\mu\nu}(x) &= \partial_\mu W_\nu(x) - \partial_\nu W_\mu(x) - ig[\, W_\mu(x), W_\nu(x)\,] \end{aligned}$$

$$\begin{aligned} W_{\mu\nu}^{\ a}(x) &= \partial_\mu W_\nu^{\ a}(x) - \partial_\nu W_\mu^{\ a}(x) + g\varepsilon_{\,abc} W_\mu^{\ b}(x) W_\nu^{\ c}(x) \\ &\equiv w_{\mu\nu}^{\ a}(x) + g\varepsilon_{\,abc} W_\mu^{\ b}(x) W_\nu^{\ c}(x) \end{aligned}$$

$$\begin{aligned} W_{\mu\nu}^{\ \pm}(x) &= w_{\mu\nu}^{\ \pm}(x) \mp i \left\{ W_\nu^{\ \pm}(x) \left[\, g\cos\theta_W\, Z_\mu^0(x) + e\, A_\mu(x) \,\right] - \mu \rightleftarrows \nu \,\right\} \\ w_{\mu\nu}^{\ \pm}(x) &= \partial_\mu W_\nu^{\ \pm}(x) - \partial_\nu W_\mu^{\ \pm}(x) \qquad z_{\mu\nu}^0(x) = \partial_\mu Z_\nu^0(x) - \partial_\nu Z_\mu^0(x) \end{aligned}$$

$$W_{\mu\nu}^{\ 3}(x) = \cos\theta_W z_{\mu\nu}^0(x) + \sin\theta_W F_{\mu\nu}(x) + ig\left[W_\mu^{\ -}(x) W_\nu^{\ +}(x) - W_\nu^{\ -}(x) W_\mu^{\ +}(x) \right]$$

Yang-Mills Lagrangian

$$w_{\mu\nu}^3(x)\, w_3^{\mu\nu}(x) + B_{\mu\nu}(x) B^{\,\mu\nu}(x) = z_{\mu\nu}^0(x)\, z_0^{\mu\nu}(x) + F_{\mu\nu}(x) F^{\,\mu\nu}(x)$$

$$-\tfrac{1}{2}\,\mathrm{tr}\,[\,W^{\,\mu\nu}(x)W_{\mu\nu}(x)\,] = -\,\mathrm{tr}\,[\,W^{-}_{\mu\nu}(x)W^{\mu\nu}_{+}(x)\,] - \,\tfrac{1}{2}\,\mathrm{tr}\,[\,W^{\mu\nu}_{3}(x)W^{3}_{\mu\nu}(x)\,]$$

$$\begin{aligned}
- &\quad \mathrm{tr}\,[\,W^{-}_{\mu\nu}(x)W^{\mu\nu}_{+}(x)\,] = W^{\,\mu}_{-}(x)\,(\,g_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)\,W^{\,\nu}_{+}(x)\\
+ &\quad i\,[\,g\cos\theta_W\,Z^{\mu}_{0}(x) + e\,A^{\mu}(x)\,]\,\big[\,w^{-}_{\mu\nu}(x)\,W^{\,\nu}_{+}(x) - w^{+}_{\mu\nu}(x)\,W^{\,\nu}_{-}(x)\,\big]\\
- &\quad W^{\,-}_{\nu}(x)W^{\,\nu}_{+}(x)\,\big[\,g\cos\theta_W\,Z^{0}_{\mu}(x) + eA_{\mu}(x)\,\big]\,[\,g\cos\theta_W\,Z^{\mu}_{0}(x) + eA^{\mu}(x)\,]\\
+ &\quad W^{\,-}_{\mu}(x)W^{\,+}_{\nu}(x)\,[\,g\cos\theta_W\,Z^{\mu}_{0}(x) + eA^{\mu}(x)\,]\,[\,g\cos\theta_W\,Z^{\nu}_{0}(x) + eA^{\nu}(x)\,]
\end{aligned}$$

$$\begin{aligned}
- &\quad \tfrac{1}{2}\,\mathrm{tr}\,[\,W^{\mu\nu}_{3}(x)W^{3}_{\mu\nu}(x)\,] - \tfrac{1}{4}\,B_{\mu\nu}(x)B^{\,\mu\nu}(x)\\
= &\quad \tfrac{1}{2}\,Z^{\mu}_{0}(x)\,(\,g_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)\,Z^{\nu}_{0}(x) + \tfrac{1}{2}\,A^{\mu}(x)\,(\,g_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)\,A^{\nu}(x)\\
- &\quad i\,W^{\,-}_{\mu}(x)W^{\,+}_{\nu}(x)\,[\,g\cos\theta_W\,z^{\mu\nu}_{0}(x) + eF^{\,\mu\nu}(x)\,]\\
- &\quad \tfrac{1}{2}\,g^{2}\,\big[\,W^{\,-}_{\mu}(x)W^{\,\mu}_{-}(x)\,\big]^{2} + \tfrac{1}{2}\,g^{2}\,\big[\,W^{\,-}_{\mu}(x)W^{\,+}_{\nu}(x)\,\big]\,\big[\,W^{\,\mu}_{-}(x)W^{\,\nu}_{+}(x)\,\big]
\end{aligned}$$

Lowest Order Free Parameters

- Vacuum expectation value and masses

$$\upsilon = \frac{\mu c^2}{\sqrt{\lambda}} = \frac{2m_W}{g} = \frac{2^{-1/4}}{\sqrt{G_F}} \cong 246\ \text{GeV}$$

$$m_H = \mu\,\sqrt{2} = 125.7 \pm 0.4\ \text{GeV}/c^2$$

$$m_W = \tfrac{1}{2}\,g\upsilon = 80.385\ \pm\ 0.015\ \text{GeV}/c^2$$

$$m_Z = \frac{m_W}{\cos\theta_W} = 91.1876\ \pm\ 0.0021\ \text{GeV}/c^2$$

- Coupling constants

$$\sqrt{\lambda}\ =\ m_H\,c^2/\upsilon\,\sqrt{2}$$

$$\cos\theta_W = \frac{m_W}{m_Z} = 0.881 \quad\Longrightarrow\quad \theta_W \simeq 28^\circ \quad\Rightarrow\quad \sin^2\theta_W = 0.22$$

$$\tan\theta_W = \frac{g'}{g} = 0.53 \qquad g^{\,2} = \frac{e^{\,2}}{\sin^2\theta_W} = \frac{4\pi\alpha}{\sin^2\theta_W} \simeq 0.42$$

$$y_a/\sqrt{2} = m_a/\upsilon \qquad a = e, \mu, \tau, d, u, s, c, b, t$$

$$g = 0.648 \qquad g' = 0.343 \qquad \lambda = 0.130$$

$$y_e = 2.36\times 10^{\,-6} \qquad y_t = 0.99$$

- Wolfenstein parameters for the Cabibbo-Kobayashi-Maskawa matrix

$$\lambda_W = 0.2253\ \pm\ 0.0007 \qquad A = 0.808\,^{+0.022}_{-0.015}$$

$$\bar\rho = 0.132\,^{+0.022}_{-0.014} \qquad \bar\eta = 0.341\ \pm\ 0.013$$

Bibliography

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1978.

[2] Ian J.R. Aitchison and Anthony J.G. Hey, *Gauge theories in particle physics. A practical introduction*, Adam Hilger, Bristol, 1982.

[3] A. Bassetto, G. Nardelli, R. Soldati, *Yang-Mills Theories in Algebraic Non-Covariant Gauges* (Canonical Quantization and Renormalization) World Scientific, Singapore, 1991.

[4] V.B. Berestetskij, E.M. Lifšits and L.P. Pitaevskij, *Teoria quantistica relativistica*, Editori Riuniti/Edizioni Mir, Roma, 1978.

[5] N.N. Bogolyubov and D.V. Širkov, *Introduction to the Theory of Quantized Fields*, Interscience Publishers, New York, 1959.

[6] Lowell S. Brown, *Quantum Field Theory*, Cambridge University Press, Cambridge, United Kingdom, 1992.

[7] Ta-Pei Cheng & Ling-Fong Li, *Gauge theory of elementary particle physics*, Clarendon Press, Oxford (UK) 1984.

[8] B. De Wit, J. Smith, *Field Theory in Particle Physics – Volume 1*, Elsevier Science Publisher, Amsterdam, 1986.

[9] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher trascendental functions*, Volume I, Bateman Manuscript Project, McGraw-Hill, New York, 1953.

[10] L.D. Faddeev & A.A. Slavnov, *Gauge fields. Introduction to quantum theory*, Benjamin Cumm. Pub. Co., Reading, Massachusetts, 1980.

[11] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, Fifth Edition, Alan Jeffrey Editor, Academic Press, San Diego, 1996.

[12] Claude Itzykson and Jean-Bernard Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980.

[13] Taichiro Kugo and Izumi Ojima, *Local Covariant Operator Formalism of nonabelian Gauge Theories and Quark Confinement Problem* Supplement of the Progress of Theoretical Physics, No. 66, pp. 1-130 (1979).

[14] L.D. Landau, E.M. Lifšits, *Teoria dei campi*, Editori Riuniti/Edizioni Mir, Roma, 1976.

[15] Michel Le Bellac, *Des phénomènes critiques aux champs de jauge. Une introduction aux méthodes et aux applications de la théorie quantique des champs*, InterEditions/Editions du CNRS, 1988.

[16] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Perseus Books, Reading, Massachusetts, 1995.

[17] Stefan Pokorski, *Gauge field theories*, Cambridge University Press, Cambridge (UK), 1987.

[18] Pierre Ramond, *Field Theory: A Modern Primer*, Benjamin, Reading, Massachusetts, 1981.

[19] R.J. Rivers, *Path integral methods in quantum field theory*, Cambridge University Press, Cambridge (UK), 1987.

[20] Roberto Soldati, *Meccanica Statistica*, appunti on line per il corso annuale, http://www.robertosoldati.com, 2007.

[21] Roberto Soldati, *Introduction to Quantum Field Theory. A Basic Course for Primary Education*, notes on line for the I Semester Course, http://www.robertosoldati.com.

[22] Roberto Soldati, *Intermediate Quantum Field Theory. A Next-to-Basic Course for Primary Education*, notes on line for the II Semester Course, http://www.robertosoldati.com.

[23] Mark Srednicki, *Quantum Field Theory*, Cambridge University Press, Cambridge (UK) 2007; http://web.physics.ucsb.edu/ mark/qft.html.

[24] G. 't Hooft and M.J.G. Veltman, *Regularization and renormalization of gauge fields*, Nucl. Phys. B**44**, 189 (1972);
P. Breitenlohner and D. Maison, *Dimensional renormalization and the action principle*, Comm. Math. Phys. **52**, 11 (1977);

Dimensionally renormalized Green's functions for theories with massless particles, 1., Comm. Math. Phys. **52**, 39 (1977);
J. Collins, *Renormalization*, Cambridge University Press, Cambridge, United Kingdom, 1984.

[25] Anthony Zee, *Quantum Field Theory in a Nutshell*, Second Edition, Priceton University Press, 2010.